TOPOLOGICAL 4-MANIFOLDS WITH RIGHT-ANGLED ARTIN FUNDAMENTAL GROUPS: CORRIGENDUM

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ABSTRACT. We correct the stability bound in our classification of closed, spin$^+$, topological 4-manifolds with fundamental group π of cohomological dimension ≤ 3 (up to s-cobordism), after stabilization by connected sum with copies of $S^2 \times S^2$. If π is a right-angled Artin group whose defining graphs have no 4-cliques, then the new stability bound is $r \geq \max(b_3(\pi), 6)$. The other results of the paper are not affected.

1. INTRODUCTION

We correct the stability bounds used in the statements of Theorem A and Theorem 11.2 in our paper [5] (see Section 3 for the corrected results, and [4] for subsequent developments). We are indebted to Daniel Kasprowski for pointing out a gap in the last step of our arguments, and to Diarmuid Crowley for a very useful conversation. To explain and repair the error we need to briefly describe the setting.

A standard approach to the classification of topological 4-manifolds uses the theory of "modified surgery" due to Matthias Kreck [7, §6]. We briefly recall some of the features of modified surgery in our setting (see [7, Theorem 4, p. 735] for the notation):

- Let M and N be closed, oriented topological 4-manifolds with the same Euler characteristic, which admit normal 1-smoothings in a fibration $B \to BSTOP$.
- If W is a normal B-bordism between these two 1-smoothings, with normal B-structure $\bar{\nu}$, then there exists an obstruction $\Theta(W, \bar{\nu}) \in \ell_5(\pi_1(B))$ which is *elementary* if and only if $(W, \bar{\nu})$ is B-bordant relative to the boundary to an s-cobordism.
- Let $\pi := \pi_1(B)$ and $\Lambda := \mathbb{Z}[\pi]$ denote the integral group ring of the fundamental group. The elements of $\ell_5(\pi)$ are represented by pairs $(H(\Lambda^r), V)$, where V is a half-rank direct summand of the hyperbolic form $H(\Lambda^r)$.
- In a pair $(H(\Lambda^r), V)$, if the quadratic form vanishes on V, then the element $\Theta(W, \bar{\nu})$ lies in the image of $L_5(\mathbb{Z}\pi) \to \ell_5(\pi)$ (see [7, Proposition 8, p. 739] or [7, p. 734] for criteria to ensure that this will happen).

In our applications, we assumed the following "assembly map" conditions.

Definition 1.1. A group π satisfies properties (W-A) whenever

(i) The Whitehead group $\mathrm{Wh}(\pi)$ vanishes.
(ii) The assembly map $A_5(\pi)\colon H_5(\pi; \mathbb{L}_0) \to L_5(\mathbb{Z}[\pi])$ is surjective.

If, in addition, the assembly map $A_4(\pi)\colon H_4(\pi; \mathbb{L}_0) \to L_4(\mathbb{Z}[\pi])$ is injective, we say that π satisfies properties (W-AA).

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These properties hold whenever the group π is torsion-free and satisfies the Farrell-Jones isomorphism conjectures in K-theory and L-theory (see [4, §3] for details). These conjectures have been verified for many classes of groups, and in particular for all right-angled Artin groups (see [2], [1]).

If M is a closed, oriented 4-manifold with $\pi_1(M, x_0) = \pi$, and π satisfies properties (W-A), we have the *partial assembly* map:

$$A_5(M)\colon H_5(M;\mathbb{L}_0) = H_1(M;\mathbb{Z}) \oplus H_3(M;\mathbb{Z}/2) \to L_5(\mathbb{Z}[\pi])$$

given by composing $A_5(\pi)$ with the induced map $H_5(M;\mathbb{L}_0) \to H_5(\pi;\mathbb{L}_0)$.

The action of elements in the image $\operatorname{Im} A_5(M) \subseteq L_5(\mathbb{Z}\pi)$ of the partial assembly map on $\Theta(W,\bar{\nu}) \in \ell_5(\pi_1(B))$ can be defined geometrically by the action of degree 1 normal maps on the B-bordism $(W,\bar{\nu})$.

More precisely, such elements in $L_5(\mathbb{Z}\pi)$ are represented by the surgery obstructions of *inertial* degree 1 normal maps

$$F\colon (U, \partial_0 U, \partial_1 U) \to (M \times I, M \times 0, M \times 1).$$

By definition, $\partial_0 U = \partial_1 U = M$, and F restricted to both boundary components is a homeomorphism. Such inertial normal cobordisms can be glued to $(W,\bar{\nu})$ to produce a new B-bordism $(W',\bar{\nu})$ between M and N, with surgery obstruction $\Theta(W',\bar{\nu}) = \Theta(W,\bar{\nu}) + \sigma(F)$ (see the proof of [6, Theorem 2.6]).

This is the argument we proposed for the final step to eliminate the obstruction $\Theta(W,\bar{\nu})$, and thus obtain an s-cobordism between M and N under the assumptions of Theorem 11.2 (and its application to Theorem A). Note that the fundamental groups π in these results all have $\operatorname{cd}\pi \leq 3$.

The Error: By assumption, the assembly map $A_5(\pi)$ in condition (W-A) is surjective, and if $\operatorname{cd}(\pi) \leq 4$ its domain:

$$A_5(\pi)\colon H_5(\pi;\mathbb{L}_0) = H_1(\pi;\mathbb{Z}) \oplus H_3(\pi;\mathbb{Z}/2) \to L_5(\mathbb{Z}[\pi]),$$

is expressed in terms of the low dimensional group homology of π. However, the above construction can only realize the action of elements in the image of the partial assembly map

$$H_5(M;\mathbb{L}_0) = H_1(M;\mathbb{Z}) \oplus H_3(M;\mathbb{Z}/2) \to H_5(\pi;\mathbb{L}_0) \to L_5(\mathbb{Z}[\pi])$$

from the homology of M. Since the reference map $M \to B$ is 2-connected, the summand $H_1(M;\mathbb{Z}) \cong H_1(\pi;\mathbb{Z})$. However, if the map $H_3(M;\mathbb{Z}/2) \to H_3(\pi;\mathbb{Z}/2)$ is not surjective, we will not be able to realize all possible obstructions by this construction.

Remark 1.2. The statements of [6, Theorems 2.2 & 2.6] are a bit misleading, since they appear (incorrectly) to be stated for arbitrary fundamental groups. However, the goal of [6] was to study fundamental groups π of geometric (and hence cohomological) dimension at most two. In these cases, $H_3(\pi;\mathbb{Z}/2) = 0$ so the domain of $A_5(\pi)$ is just $H_1(\pi;\mathbb{Z})$, and the problem above does not arise. In contrast, if $\operatorname{cd}\pi = 3$ and $\pi_1(M) = \pi$, then by Poincaré duality:

$$\begin{array}{ccc} H^1(M;\mathbb{Z}/2) & \xleftarrow{\ \cong\ } & H^1(\pi;\mathbb{Z}/2) \\ {\scriptstyle\cong}\downarrow{\scriptstyle \cap[M]} & & \downarrow{\scriptstyle c_*[M]} \\ H_3(M;\mathbb{Z}/2) & \longrightarrow & H_3(\pi;\mathbb{Z}/2) \end{array}$$

and the map $H_3(M;\mathbb{Z}/2) \to H_3(\pi;\mathbb{Z}/2)$ is zero since $0 = c_*[M] \in H_4(\pi;\mathbb{Z}/2)$.

2. A stable range for L-theory

For any finitely presented group π, the odd dimensional surgery obstruction groups are defined as $L_5(\mathbb{Z}[\pi]) = SU(\Lambda)/RU(\Lambda)$, in the notation of Wall [10, Chap. 6]. Here $SU(\Lambda)$ is the limit of the automorphism groups $SU_r(\Lambda)$ of the hyperbolic (quadratic) form $H(\Lambda^r)$ under certain injective maps

$$\dots SU_r(\Lambda) \to SU_{r+1}(\Lambda) \to \dots \to SU(\Lambda),$$

and $RU(\Lambda)$ is a suitable subgroup determined by the surgery data, so that $L_5(\mathbb{Z}[\pi])$ is an abelian group. To define a stable range for groups π with $\operatorname{cd}(\pi) \leq 3$, we will assume that π has *type* F_3, meaning that there is a model for the classifying space $B\pi$ with finite 3-skeleton. In particular, groups of type F_3 are finitely presented, but not conversely.

Definition 2.1. For an element $x \in L_5(\mathbb{Z}[\pi])$, we denote its *stable* L_5*-range* by:

$$\mathfrak{sr}(x) = \min\{r \geq 0 : x \text{ is represented by an matrix in } SU_r(\Lambda)\}.$$

The *stable range* of a finitely presented group π with $\operatorname{cd}(\pi) \leq 3$ is defined as:

$$\mathfrak{sr}_3(\pi) = \min_{\mathcal{B}}\{\max\{\mathfrak{sr}(A_5(\alpha)) : \text{where } \alpha \in \mathcal{B} \text{ varies over a } \mathbb{Z}/2\text{-basis } \mathcal{B} \text{ for } H_3(\pi;\mathbb{Z}/2)\}\}.$$

Remark 2.2. If π has type F_3 then the stable range will be finite. In general, $\mathfrak{sr}_3(\pi)$ could be infinite, since there are finitely presented groups with $H_3(\pi;\mathbb{Z}/2)$ of infinite rank (see Stallings [8]).

Lemma 2.3. *Let π be a right-angled Artin group with* $\operatorname{cd}(\pi) \leq 3$*. Then* $\mathfrak{sr}_3(\pi) \leq 6$*.*

Proof. Every right-angled Artin group has type F_3 since it is defined by a finite graph. The homology group $H_3(\pi;\mathbb{Z}/2)$ has $\mathbb{Z}/2$-rank $b_3(\pi)$, which is equal to the number of 3-cliques in the defining graph for π. Moreover, since each 3-clique determines a subgroup $\rho \subseteq \pi$, with $\rho \cong \mathbb{Z}^3$, the group $H_3(\pi;\mathbb{Z}/2)$ is generated by the images of the fundamental classes under all the induced maps $H_3(T^3;\mathbb{Z}/2) \to H_3(\pi;\mathbb{Z}/2)$. It is therefore enough to determine the stable range for $\rho = \mathbb{Z}^3$.

By definition of the assembly map, we need to determine the minimum representative in $SU_r(\Lambda)$ for the surgery obstruction of the degree one normal map

$$g := (\operatorname{id} \times f)\colon N \times T^2 \to N \times S^2$$

given by the the product of the Arf invariant one normal map $f\colon T^2 \to S^2$ with the identity on $N = T^3$. Let $\rho = \pi_1(N) = \mathbb{Z}^3$. After surgery on the generators of

$$K_1(g) = \ker\{H_1(N \times T^2;\Lambda) \to H_1(N \times S^2;\Lambda)\} = \mathbb{Z} \oplus \mathbb{Z}$$

to obtain (N', g'), we get a 2-connected normal map with $K_2(g') = I(\rho) \oplus I(\rho)$, where $I(\rho) := \ker\{\mathbb{Z}[\rho] \to \mathbb{Z}\}$ is the augmentation ideal of the group ring $\mathbb{Z}[\rho]$. According to the recipe provided by Wall [10, Chap. 6, pp. 58-59], the surgery obstruction is determined by representing generators of $K_2(g')$ by a collection of disjointly embedded framed 2-spheres in N'. The process starts by picking an epimorphism from a free Λ-module $\Lambda^r \to K_2(g')$, which leads to a surgery obstruction automorphism in $SU_r(\Lambda)$ of the hyperbolic form $H(\Lambda^r)$. Since $I(\rho)$ is minimally generated by an epimorphism $\Lambda^3 \to I(\rho)$, we conclude that an epimorphism $\Lambda^r \to K_2(g')$ requires $r \geq 6$. □

We will use a stable range condition to realize the action of $L_5(\mathbb{Z}[\pi])$ on a B-bordism, after a suitable stabilization. The following statement is an application of this result in the setting of Kreck [7, Theorem 4].

Proposition 2.4. *Let π be a discrete group with $\operatorname{cd}(\pi) \leq 3$ of type F_3 satisfying properties* (W-A)*. Let M and N be closed, oriented topological 4-manifolds with the same Euler characteristic, which admit normal 1-smoothings in a fibration $B \to BSTOP$. Suppose that $(W, \bar{\nu})$ is a normal B-bordism between these two 1-smoothings. If $r \geq \mathfrak{sr}_3(\pi)$, then for any $x \in L_5(\mathbb{Z}[\pi])$ there exists a B-bordism $(W', \bar{\nu})$ between the stabilized 1-smoothings $M' := M \# r(S^2 \times S^2)$ and $N' := N \# r(S^2 \times S^2)$, with $\Theta(W', \bar{\nu}) = \Theta(W, \bar{\nu}) + x \in \ell_5(\pi)$.*

Proof. By property (W-A), the assembly map $A_5(\pi)\colon H_1(\pi;\mathbb{Z}) \oplus H_3(\pi;\mathbb{Z}/2) \to L_5(\mathbb{Z}[\pi])$ is surjective. The elements $x \in L_5(\mathbb{Z}[\pi])$ in the image of $H_1(\pi;\mathbb{Z}) \cong H_1(M;\mathbb{Z})$ are realized as above (without stabilization). For the elements $x = A_5(\alpha) \in L_5(\mathbb{Z}[\pi])$ in the image of $\alpha \in H_3(\pi;\mathbb{Z}/2)$, we use the stabilized version of Wall realization due to Cappell and Shaneson [3, Theorem 3.1].

Any element is the image of a finite sum $\alpha = \sum \alpha_i$ of basis elements of $H_3(\pi;\mathbb{Z}/2)$, which all have stable L-range at most $\mathfrak{sr}_3(\pi)$. The realization construction can be done (for each term α_i of the finite sum) in small disjoint intervals

$$M' \times [t_{i-1}, t_i] \subset M' \times [0,1],$$

with $0 = t_0 < t_1 < \cdots < t_k = 1$, to produce degree one normal maps

$$F_i\colon (U_i, \partial_0 U_i, \partial_1 U_i) \to (M' \times [t_{i-1}, t_i], M' \times t_{i-1}, M \times t_i), \quad 1 \leq i \leq k,$$

such that $\partial_0 U_i = \partial_1 U_i = M' = M \# r(S^2 \times S^2)$. The restrictions of F_i to the boundary components have the property that $F_i\,|_{\,\partial_0 U} = \mathrm{id}$, and $F_i\,|_{\,\partial_1 U} := f_i$ is a simple homotopy equivalence. In other words, this construction produces elements of the structure set $\mathcal{S}(M')$ represented by self-equivalences of M'.

These normal bordisms can be glued (at disjoint levels) into a collar $M' \times [0,1]$ attached to the stablization $W \natural\, r(S^2 \times S^2 \times I)$ of the given B-bordism, and the reference map to B extended through M. After including all these bordisms, the induced homotopy equivalence with target $M' \times 1$ is the composite $f := f_1 \circ f_2 \circ \cdots \circ f_k$.The surgery obstruction over the collar $M' \times [0,1]$ is $x = A_5(\alpha) = \sum A_5(\alpha_i)$, and the result follows. ☐

The following application of the theory in Kreck [7, §6] may be useful in cases where a potentially harder bordism calculation is feasible.

Corollary 2.5. *If M and N are closed, oriented topological 4-manifolds with fundamental group π of type F_3 which admit B-bordant normal 2-smoothings in the same fibration $B \to BSTOP$, then they are s-cobordant after at most $\mathfrak{sr}_3(\pi)$ stabilizations, provided that $\operatorname{cd}(\pi) \leq 3$ and π satisfies properties* (W-A).

Proof. For normal 2-smoothings of M and N, the reference maps are 3-connected. In this case, Kreck [7, p. 734] shows that the surgery obstruction $\Theta(W, \bar{\nu})$ of a B-bordism $(W, \bar{\nu})$ lies in the image of $L_5(\mathbb{Z}[\pi]) \to \ell_5(\pi)$. The result now follows from Proposition 2.4. ☐

Remark 2.6. The definition of the stable range for elements in $L_5(\mathbb{Z}[\pi])$ and the results of this section have been generalized in [4, §4] to all discrete groups π which are *geometrically n-dimensional*, or $\text{g-dim}(\pi) \leq n$, meaning that there is a finite aspherical n-complex with fundamental group π. This leads to further applications of the stable range condition. Note that $\operatorname{cd}(\pi) \leq 3$ implies $\text{g-dim}(\pi) \leq 3$ (see [9, Theorem E]).

3. The main results corrected

To correct the statements of Theorem A and Theorem 11.2 in [5], we add the new stable range conditions to the previous stability estimate $r \geq b_3(\pi)$. Recall that $b_3(\pi)$ denotes the minimum number of generators for $H^3(\pi; \Lambda)$ as a Λ-module. For a right-angled Artin group, $b_3(\pi)$ equals the number of 3-cliques in the defining graph for π.

Theorem A. *Let π be a right-angled Artin group defined by a graph Γ with no 4-cliques. Suppose that M and N are closed, spin$^+$, topological 4-manifolds with fundamental group π. Then any isometry between the quadratic 2-types of M and N is stably realized by an s-cobordism between $M \# r(S^2 \times S^2)$ and $N \# r(S^2 \times S^2)$, whenever $r \geq \max\{6, b_3(\pi)\}$.*

As shown in the paper [5], this is a consequence of our main result:

Theorem 11.2. *Let π be a discrete group of type F_3 with $\operatorname{cd} \pi \leq 3$ satisfying the properties* (W-AA)*. If M and N are closed, oriented, spin$^+$, TOP 4-manifolds with fundamental group π, then any isometry between the quadratic 2-types of M and N is stably realized by an s-cobordism between $M \# r(S^2 \times S^2)$ and $N \# r(S^2 \times S^2)$, for $r \geq \max\{b_3(\pi), \mathfrak{s}\mathrm{r}_3(\pi)\}$.*

Remark 3.1. Note that we obtain s-cobordisms after connected sum with a *uniformly bounded* number of copies of $S^2 \times S^2$, where the bound depends only on the fundamental group. In contrast, "stable classification" results (such as [5, Theorem B]) might require an unbounded number of stabilizations as the manifolds M and N vary.

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