QUOTIENTS OF $S^2 \times S^2$

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Abstract. We consider closed topological 4-manifolds M with universal cover $S^2 \times S^2$ and Euler characteristic $\chi(M) = 1$. All such manifolds with $\pi = \pi_1(M) \cong \mathbb{Z}/4$ are homotopy equivalent. In this case, we show that there are four homeomorphism types, and propose a candidate for a smooth example which is not homeomorphic to the geometric quotient. If $\pi \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, we show that there are three homotopy types (and between 6 and 24 homeomorphism types).

1. Introduction

The goal of this paper is to characterize 4-manifolds with universal cover $S^2 \times S^2$ up to homeomorphism in terms of standard invariants, continuing the program of [10, Chapter 12]. Our approach combines the analysis of Postnikov sections with recent results in surgery. The main new ingredient is the use of bordism calculations to study the difference between homotopy self-equivalences and homeomorphisms of these 4-manifolds.

A 4-manifold M has universal covering space $\widetilde{M} \cong S^2 \times S^2$ if and only if $\pi = \pi_1(M)$ is finite, $\chi(M)|\pi| = 4$ and its Wu class $v_2(M)$ is in the image of $H^2(\pi; \mathbb{F}_2)$. There are eight such manifolds which are geometric quotients, with π acting through a subgroup of $\mathrm{Isom}(\mathbb{S}^2 \times \mathbb{S}^2) = (O(3) \times O(3)) \rtimes \mathbb{Z}/2$.

We first recall that closed topological manifolds with $\pi_1(M) = 1$ or $\pi_1(M) = \mathbb{Z}/2$ have already been classified (without assumption on the universal convering):

(i) If $\pi = \mathbb{Z}/n$, and M is orientable, then M is classified up to homeomorphism by its intersection form on $H_2(M; \mathbb{Z})/Tors$, $w_2(M)$ and the Kirby-Siebenmann (KS) invariant (see Freedman [3] for $\pi = 1$, and [7, Theorem C] for $\pi = \mathbb{Z}/n$).

(ii) If $\pi = \mathbb{Z}/2$, and M is non-orientable, then M is classified up to homeomorphism by explicit invariants (see [9, Theorem 2]), and a complete list of such manifolds is given in [9, Theorem 3].

If we further impose the condition that $\widetilde{M} = S^2 \times S^2$, then there are two orientable geometric $\mathbb{Z}/2$-quotients, namely the 2-sphere bundles $S(\eta \oplus 2\epsilon)$ and $S(3\eta)$ over RP^2, where η is the canonical line bundle over RP^2. The second manifold is non-spin and has a non-smoothable homotopy equivalent "twin" $*M$ with KS $\neq 0$.

In the non-orientable case, there are two geometric quotients: $S^2 \times RP^2$ and $S^2 \tilde{\times} RP^2 = S(2\eta \oplus \epsilon)$, and one further smooth manifold $RP^4 \sharp_{S^1} RP^4$ obtained by removing a tubular neighbourhood of $RP^1 \subset RP^4$, and gluing two copies of the complement together along

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the boundary. Each of these has a homotopy equivalent twin $*M$ with KS $\neq 0$, so there are six such non-orientable manifolds.

We now assume that $|\pi| = 4$, which implies that any quotient M of $S^2 \times S^2$ is non-orientable and $\chi(M) = 1$. If $\pi = \mathbb{Z}/4$, there is just one geometric quotient $\mathbb{M}$ obtained from the free action generated by $(u, v) \mapsto (-v, u)$, for $(u, v) \in S^2 \times S^2$.

Theorem A. *Let N be a closed topological 4-manifold with $\widetilde{N} = S^2 \times S^2$ and $\pi_1(N) = \mathbb{Z}/4$.*

(i) Each N is homotopy equivalent to the unique geometric quotient $\mathbb{M}$.
(ii) Every self homotopy equivalence of $\mathbb{M}$ is homotopic to a self-homeomorphism.
(iii) There are four such manifolds up to homeomorphism, of which exactly two have non-trivial Kirby-Siebenmann invariant.

Remark 1.1. An analysis of one construction of the geometric example $\mathbb{M}$ leads to the construction of another smooth 4-manifold in this homotopy type, which may not be homeomorphic to the geometric manifold (see Section 11).

If π has order 2 or 4 then $Wh(\pi) = 0$ and the natural homomorphism from $L_4(1)$ to $L_4(\pi, -)$ is trivial (see Wall [26, §3.4]). Thus if M is non-orientable we may surger the normal map $M\#E_8 \to M\#S^4 = M$ to obtain a *twin*: that is a homotopy equivalent 4-manifold $*M$ with the opposite Kirby-Siebenmann invariant. Here E_8 denotes a closed, 1-connected, topological 4-manifold constructed by Freedman [3], whose intersection form is definite of rank 8.

Remark 1.2. When $|\pi| = 4$ the mod 2 Hurewicz homomorphism is trivial. Hence pinch maps have trivial normal invariants, so do not provide "fake" self homotopy equivalences, meaning a self equivalence not homotopic to a homeomorphism (see [1, p. 420]). We rule out other fake self equivalences for $\pi = \mathbb{Z}/4$ in Section 10.

In the remaining cases, where $\pi = \mathbb{Z}/2 \times \mathbb{Z}/2$, we classify the homotopy types of Poincaré 4-complexes, and determine the homotopy types of closed manifolds.

Theorem B. *There are two quadratic 2-types of PD_4-complexes X with $\chi(X) = 1$ and $\pi_1(X) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, and seven homotopy types in all.*

(i) All such complexes have universal cover homotopy equivalent to $S^2 \times S^2$.
(ii) The two quadratic 2-types are represented by the total spaces of the two RP^2-bundles over RP^2.
(iii) A third homotopy type includes a smooth manifold N with $RP^4\#_{S^1}RP^4$ as a double cover.
(iv) The remaining four homotopy types do not include closed manifolds.

The primary homotopy invariants of a finite PD_4-complex X are its fundamental group $\pi := \pi_1(X, x_0)$, and its second homotopy group $\pi_2(X)$ as a module over the integral group ring $\Lambda := \mathbb{Z}[\pi]$. The *quadratic 2-type* (introduced in [6]) is represented by the quadruple:

$$[\pi_1(X), \pi_2(X), k_X, s_X]$$

where s_X denotes the equivariant intersection form $s_X \colon \pi_2(X) \times \pi_2(X) \to \Lambda$, and

$$k_X \in H^3(\pi; \pi_2(X))$$

is the first k-invariant of the algebraic 2-type $P := P_2(X)$ as introduced by MacLane and Whitehead [18]. The space P is a fibration over $K(\pi, 1)$, classified by k_X, with fibre $K(\pi_2(X), 2)$ and there is a 3-connected reference map $\tilde{c}\colon X \to P$ lifting the classifying map $c\colon X \to K(\pi, 1)$ for the universal covering $\widetilde{X} \to X$. Equivalently, $P_2(X)$ is the second stage of a Postnikov tower for X.

An *isometry* of two such quadruples is an isomorphism on π_1, π_2 inducing an isometry of the equivariant intersection forms, and respecting the k-invariants.

The first statement in Theorem B about the quadratic 2-types was proved in [10, Chapter 12, §6], but the homotopy classification is new. We use the invariants of [4] and [13] to determine which homotopy types contain closed manifolds. The homeomorphism classification appears difficult: all we can say at this stage is that in each case the TOP structure set has 8 members, so that there are between 6 and 24 homeomorphism types of such manifolds, of which half are not stably smoothable. To resolve this ambiguity, more information is needed about self homotopy equivalences.

Here is an outline of the paper. After some preliminary material in Sections 2-3, we show that there are either two or four homeomorphism types with $\pi \cong \mathbb{Z}/4$. We then review the constructions of the non-orientable smoothable quotients of $S^2 \times S^2$ with $\pi = \mathbb{Z}/2$ (see Sections 4-6).

In Section 7 we construct a new smooth 4-manifold N in the quadratic homotopy type of the bundle space $RP^2 \tilde{\times} RP^2$, but distinguished from it by its non-orientable double covers (see Definition 7.1). In particular, N is not a geometric quotient. In Sections 8-9 we show that there are no other homotopy types of 4-manifolds with $\pi \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\chi = 1$. This completes the proof of Theorem B.

In Section 10 we complete the proof of Theorem A via a stable homeomorphism classification result. In Section 11 we construct a smooth manifold with $\pi = \mathbb{Z}/4$, which may not be diffeomorphic or even homeomorphic to the geometric quotient (see Definition 11.2). The same strategy does not seem to provide a candidate for a smooth fake $RP^2 \times RP^2$.

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2. The structure set

Classical surgery theory studies the *structure set* $S_{TOP}(M)$, which consists of pairs (N, f) of closed 4-manifolds N and a homotopy equivalence $f\colon N \to M$, modulo those homotopic to homeomorphisms. *Here and throughout the paper we will always work with pointed spaces and base-point preserving maps.*

If M is non-orientable, we fix a local coefficient system $\{\mathbb{Z}^w\}$ induced by the classifying map $M \to K(\pi, 1)$ of the orientation double cover, and use it to define the homology of M with "twisted" coefficients. A choice of generator $[M] \in H_4(M; \mathbb{Z}^w) \cong \mathbb{Z}$ gives a fundamental class for Poincaré duality (see Wall [25, Chapter 1] and Taylor [21, §5]).

The surgery exact sequence

$$\cdots \to L_5(\pi, w_1) \to S_{TOP}(M) \to [M, G/TOP] \to L_4(\pi, w_1)$$

leads to a computation of $S_{TOP}(M)$ in favourable circumstances. The general theory due to Browder, Kervaire, Milnor, Novikov, Sullivan and Wall for high-dimensional smooth or PL manifolds (see [27]) was extended to topological manifolds by Kirby and Siebenmann [16], and to 4-manifolds with *good* fundamental groups by Freedman [3]. In particular, surgery theory "works" for topological 4-manifolds with finite fundamental group.

In our situation, it is not difficult to compute the size of the structure set $S_{TOP}(M)$. The remaining obstacle to obtaining a homeomorphism classification is to understand the action of homotopy self equivalences on the structure set.

Since $\pi_i(G/TOP) = 0$ in all odd dimensions and the first significant k-invariant of G/TOP is 0, there is a 6-connected map $G/TOP \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 4)$ (see [16, §2]). Hence if X is a closed 4-manifold then

$$[X, G/TOP] \cong H^2(X; \mathbb{Z}/2) \oplus H^4(X; \mathbb{Z}).$$

In these low dimensions, Poincaré duality with $\mathbb{L}_0$-theory coefficients

$$[X, G/TOP] = H^0(X; \mathbb{L}_0) \cong H_4(X; \mathbb{L}_0^w)$$

on the left-hand side agrees with ordinary Poincaré duality on the right-hand side, induced by cap product with a (twisted) fundamental class

$$[X] \in H_4(X; \mathbb{Z}^w) \cong \mathbb{Z}$$

where $w = w_1(X)$. This gives $[X, G/TOP] \cong H_2(X; \mathbb{Z}/2) \oplus H_0(X; \mathbb{Z}^w)$. Note that the H-space structure on G/TOP in this formula is the one induced by the connective $\mathbb{L}_0$-theory spectrum.

We can now determine the size of $S_{TOP}(M)$ for manifolds with $\widetilde{M} = S^2 \times S^2$ and fundamental groups of order four.

Theorem 2.1. *Let M be a closed topological 4-manifold with $\pi_1(M) \cong \mathbb{Z}/4$ and $\chi(M) = 1$. The structure set $S_{TOP}(M)$ has four members, and there are either two or four homomeomorphism types of manifolds homotopy equivalent to M.*

Proof. The normal invariant map in the surgery exact sequence

$$S_{TOP}(M) \to [M, G/TOP] = H^2(M; \mathbb{Z}/2) \oplus H^4(M; \mathbb{Z})$$

is a bijection, since the groups $L_5(\mathbb{Z}/4, -)$ and $L_4(\mathbb{Z}/4, -)$ are both zero (see Wall [26, Theorem 3.4.5]). The cohomology groups $H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$ and $H^4(M; \mathbb{Z}) = \mathbb{Z}/2$ were computed in [10, Chapter 12, §4]. Hence $|S_{TOP}(M)| = 4$. As observed in the Introduction, every such manifold N has a fake twin $*N$. □

Remark 2.2. In particular, if $h\colon M' \to M$ is a homotopy equivalence with nontrivial normal invariant $\eta(h) \in H^2(M; \mathbb{Z}/2)$, then every closed 4-manifold with $\pi \cong \mathbb{Z}/4$ and $\chi = 1$ is homeomorphic to one of M, M', $*M$ or $*M'$. The normal invariant of $M \sharp E_8 \to M$ is non-trivial in $H^4(M; \mathbb{Z}) \cong \mathbb{Z}/2$. After surgery, this produces the twin manifold $*M$.

Similarly, we have the manifold $*M'$ whose normal invariant is non-trivial in both summands of $[M, G/TOP]$, and $KS(*M') = 0$ by the formula on [16, p. 398]. In contrast, both M' and $*M$ have non-trivial Kirby-Siebenmann invariant. We do not know whether $*M'$ admits a smooth structure (see Section 11 for a candidate).

In general, the normal invariant is an invariant of a map. However, in this case we will complete the proof of Theorem A by showing that the homotopy type and the Kirby-Siebenmann invariant distinguish homeomorphism types completely (see Section 10).

The cases where $\pi_1(M) = \mathbb{Z}/2 \times \mathbb{Z}/2$ are similar.

Theorem 2.3. *Let M be a closed topological 4-manifold with $\pi_1(M) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\chi(M) = 1$. The structure set $S_{TOP}(M)$ has eight members, consisting of up to four distinct twin pairs of homomeomorphism types $(N, *N)$ of manifolds homotopy equivalent to M.*

Proof. The cohomology groups $H^2(M; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$ and $H^4(M; \mathbb{Z}) = \mathbb{Z}/2$. Moreover, since $L_5((\mathbb{Z}/2)^2, -) = 0$ and the surgery obstruction map from $[M, G/TOP]$ to $L_4((\mathbb{Z}/2)^2, -) = \mathbb{Z}/2$ is onto. Hence, for each homotopy type $S_{TOP}(M)$ has 8 elements (see [10, Chapter 12, §7]). Half of these members (N, f) have domains with nontrivial Kirby-Siebenmann invariant, and so the image of $\mathrm{Homeo}(M)$ in the group of (free homotopy classes of) self homotopy equivalences of M has index at most 4. However, whether every self homotopy equivalence of M is homotopic to a homeomorphism remains open. To make further progress we need explicit representatives for the self homotopy equivalences. □

3. Homotopy type invariants for finite PD_4-complexes

Let $B = P_2(X)$ denote the algebraic 2-type of a finite Poincaré 4-complex X. A B-polarized PD_4-complex consists of a pair (X, f), where $f\colon X \to B$ is a 3-equivalence. Two such pairs (X, f) and $Y, g)$ are equivalent if there exists a homotopy equivalence $h\colon X \to Y$ such that $f \simeq g \circ h$. Following [6, §1], we let $S_4^{PD}(B)$ denote the set of B-polarized homotopy types.

For PD_4-complexes with finite fundamental group, the set $S_4^{PD}(B)$ is determined by the quadratic 2-type and a secondary invariant depending on $\pi_2(X)$ as a $\pi_1(X)$-module. In the statement, $\Gamma_W(\pi_2(X))$ denotes Whitehead's quadratic functor.

In the rest of the paper, we will always assume that a PD_4-complex X has one top cell (see Wall [25, Corollary 2.3.1])

Theorem 3.1. *Each homotopy type within the quadratic 2-type of a PD_4-complex X with π finite may be obtained by varying the attaching map of the top cell to the 3-skeleton $X^{(3)}$. The torsion subgroup of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\pi_2(X))$ acts transitively on the set of PD_4-polarizations of the quadratic 2-type.*

Proof. This result is due to Hambleton and Kreck [6, Theorem 1.1], Teichner [22, Chap. 2], and Kasprowski and Teichner [12, Theorem 1.5]. □

Remark 3.2. In particular, the cardinality of this torsion subgroup is an upper bound for the number of homotopy types within the quadratic 2-type.

To progress from PD_4-polarizations to a homotopy type classification, we need some information about the self equivalences of Postnikov towers.

More generally, let X be a connected cell complex with fundamental group π, and let $G_\#(X)$ be the group of based self homotopy equivalences of X which induce the identity on all homotopy groups. Let $P_n(X)$ be the nth stage of the Postnikov tower for X. This may be constructed by adjoining cells of dimension $\geq n+2$ to X. Then $G_\#(P_2(X)) \cong H^2(\pi; \pi_2(X))$, by Tsukiyama [24, Theorem 2.2]. Furthermore, by [24, Proposition 1.5] there are exact sequences

$$H^n(P_{n-1}(X); \pi_n(X)) \to G_\#(P_n(X)) \to G_\#(P_{n-1}(X)), \tag{3.3}$$

for $n > 2$, and the image on the right is the subgroup which stabilizes

$$k_n(X) \in H^{n+1}(P_{n-1}(X), \pi_n(X)).$$

In particular, if $H^k(P_{k-1}(X); \pi_k(X)) = 0$ for $2 \leq k \leq n$, then $G_\#(P_n(X)) = 0$ and hence the self homotopy equivalences of $P_n(X)$ are detected by their actions on the homotopy groups.

If X is a PD_4-complex such that π is finite, then any based self homotopy equivalence of X lifts to a based self homotopy equivalence of the 1-connected PD_4-complex $\widetilde{X}$. If $\pi_2(X) \neq 0$, then by [1, Theorem 3.1, p. 419]

$$HE_{id}(\widetilde{X}) \cong \ker(w_2(\widetilde{X})) \cong G_\#(\widetilde{X})$$

since the elements of $\ker(w_2(\widetilde{X}))$ are represented by "pinching maps" which induce the identity on all homotopy groups. Therefore any based self homotopy equivalence of X which induces the identity on π and $\pi_2(X)$ is in $G_\#(X)$ (compare [8, Theorem A]).

We now specialize to the cases where $\pi_1(X) = \mathbb{Z}/4$.

Lemma 3.4. *Every PD_4-complex X with $\pi_1(X) \cong \mathbb{Z}/4$ and $\chi(X) = 1$ is homotopy equivalent to the geometric quotient $\mathbb{M}$. Moreover, the image $c_*[X] \in H_4(\pi; \mathbb{Z}^w)$ of its fundamental class is non-zero.*

Proof. See [10, Chapter 12, §6]) for this application of Theorem 3.1. There is a unique quadratic 2-type and the torsion subgroup of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\pi_2(X))$ is zero.

For the last statement, note that the group $\pi = \mathbb{Z}/4$ acts on $\Pi := \pi_2(X) = \mathbb{Z}^2$ via $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and so $\Pi \cong \Lambda/(t^2+1) = \mathbb{Z}[i]$. The general result of [11, Theorem 1.10] is a stable exact sequence

$$\mathcal{E} : 0 \to H_2(K; \Lambda^w) \to \pi_2(X) \oplus \Lambda^r \to H^2(K; \Lambda^w) \to 0$$

where K is any finite 2-complex with $\pi_1(K) = \pi$, and the extension class

$$[\mathcal{E}] \in \mathrm{Ext}^1_\Lambda(H^2(K; \Lambda), H_2(K; \Lambda))$$

can be naturally identified with the image $c_*[X] \in H_4(\pi; \mathbb{Z}^w)$ of the fundamental class of X. Since $H_1(\pi; \mathbb{Z}[i]) = 0$ and $H_1(\pi; H_2(K; \Lambda^w)) = H_4(\pi; \mathbb{Z}^w) = \mathbb{Z}/2$, the extension $\mathcal{E}$ must be non-split. □

Finally, we recall two additional invariants which can be used to show that not every finite PD_4-complex is homotopy equivalent to a closed manifold.

Example 3.5. Kim, Kojima and Raymond [14] defined a $\mathbb{Z}/4$-valued quadratic function $q_{KKR}(M)$ on $\pi_2(M) \otimes \mathbb{Z}/2$, for M a closed non-orientable 4-manifold, by the rule

$$q_{KKR}(M)(x) = e(\nu(S_x)) + 2|\,\mathrm{Self}(S_x)|,$$

where $S_x \colon S^2 \to M$ is a self-transverse immersion representing x, $e(\nu(S_x))$ is the Euler number of the normal bundle and $\mathrm{Self}(S_x)$ is the set of double points of the image of S_x. This is an enhancement of the mod 2 equivariant intersection pairing on $\widetilde{M}$, and is a homotopy invariant for M.

The second invariant is an obstruction to the reducibility of the Spivak normal fibre space to a vector bundle.

Example 3.6. Let $X' \to X$ be a double cover of finite PD_{2n}-complexes, classified by a map $f \colon X \to RP^{k+1}$, for some $k >> n$. Following Hambleton and Milgram [4], we say that the double covering is *Poincaré splittable* if the homotopy class of the map f contains a representative which is Poincaré transverse to $RP^k \subset RP^{k+1}$. This always holds if $X' \to X$ is a double cover of closed manifolds, or more generally if the Spivak normal fibre space is reducible. There is a quadratic map

$$q \colon H^n(X'.\mathbb{Z}/2) \to \mathbb{Z}/2$$

refining the non-singular bilinear form

$$\ell(a, b) = \langle a \cup T^*b, [X'] \rangle,$$

where $a, b \in H^n(X'; \mathbb{Z}/2)$ and $T \colon X' \to X'$ is the free involution induced by the double cover. Let $A(X, f) \in \mathbb{Z}/2$ denote the Arf invariant of this quadratic form. Then $A(X, f)$ defines a homomorphism $\mathscr{N}_{2n}(RP^\infty) \to \mathbb{Z}/2$, which vanishes for double covers of manifolds (see [4, Proposition 2.1]). If X is orientable, then $A(X, f) = 0$ for any double cover (see [5]), but there exist non-orientable double covers in each even dimension ≥ 4 for which $A(X, f) \neq 0$ (see [4, Theorem 3.1]).

4. Non-orientable quotients of $S^2 \times S^2$ with $\pi = \mathbb{Z}/2$

We introduce some notation for later use. Let A be the antipodal involution of S^2, and let $\eta\colon S^3 \to S^2$ denote the Hopf fibration. Let $\bar{\eta}\colon S^3 \to RP^2$ be the composite of η with the projection $S^2 \to RP^2 = S^2/\{x \sim A(x)\}$. In this section we describe the homotopy types of non-orientable quotients of $S^2 \times S^2$ by a free involution.

Proposition 4.1. *Let X be a finite non-orientable PD_4-complex with $\pi_1(X) = \mathbb{Z}/2$. If $\widetilde{X} \simeq S^2 \times S^2$, then*

(i) X has the quadratic 2-type of $S^2 \times RP^2$.

(ii) Exactly three of the four distinct homotopy types in this quadratic 2-type are represented by closed manifolds.

The manifolds in this quadratic 2-type are $S^2 \times RP^2$, $S^2 \tilde{\times} RP^2$ and $RP^4 \#_{S^1} RP^4$.

Remark 4.2. The following observation will be used below and in Section 8. Since the PD_4-complex X has one top cell, we have $H^k(X, X^{(3)}; \mathbb{F}_2) = 0$, for $k \leq 3$. Hence the ring homomorphism $H^*(X; \mathbb{F}_2) \to H^*(X^{(3)}; \mathbb{F}_2)$ induced by the inclusion of the 3-skeleton is an isomorphism in degrees ≤ 3.

Proof. There are two quadratic 2-types of non-orientable PD_4-complexes X with $\pi = \mathbb{Z}/2$ and $\chi(X) = 2$. Moreover, all such quotients of $S^2 \times S^2$ have the quadratic 2-type of $S^2 \times RP^2$ (see [10, Chapter 12, §6]). We now apply Theorem 3.1 to analyse the homotopy types.

Let $K = \overline{S^2 \times RP^2 \setminus D^4}$ be the 3-skeleton of $S^2 \times RP^2$, let $I_1, I_2\colon S^2 \to \widetilde{K} = \overline{S^2 \times S^2 \setminus 2D^4}$ be the inclusions of the factors, and let $[J]$ be the homotopy class of a fixed lift $\widetilde{J}\colon S^3 \to \widetilde{K}$ of the natural inclusion $J\colon S^3 = \partial D^4 \to K$.

Since $\pi_2(S^2 \times RP^2) \cong \mathbb{Z}^2$ is generated by I_1 and I_2, the group $\Gamma_W(\pi_2(S^2 \times RP^2))$ has basis $[I_1, I_2]$, $\eta_1 = I_1 \circ \eta$, and $\eta_2 = I_2 \circ \bar{\eta}$. Since the nontrivial element of π fixes I_1 and changes the sign of I_2, it fixes η_1 and η_2 and changes the sign of $[I_1, I_2]$. Hence $\Gamma_W(\pi_2(S^2 \times RP^2)) \cong \mathbb{Z}^w \oplus \mathbb{Z}^2$, and so $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\pi_2(S^2 \times RP^2)) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$. In particular, the torsion subgroup of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(\pi_2(S^2 \times RP^2))$ is isomorphic to $(\mathbb{Z}/2)^2$, and is generated by the images of η_i, for $i = 1, 2$.

The four homotopy types represented by the PD_4-complexes $W_\alpha = K \cup_{[J]+\alpha} e^4$ corresponding to $\alpha = 0$, η_1, η_2 and $\eta_1 + \eta_2$ are in fact distinct, as we shall see. Clearly $W_0 = K \cup_{[J]} D^4 \simeq S^2 \times RP^2$. There are two other closed 4-manifolds, namely the bundle space $S^2 \tilde{\times} RP^2$ and the manifold $RP^4 \#_{S^1} RP^4$, which are described explicitly in the next section, and shown to have distinct homotopy types in §6. In [4] it is shown that the PD_4-complex $P_{HM} = W_{\eta_2}$ is not homotopy equivalent to a closed 4-manifold, but note that [4] writes the factors in the opposite order. The remaining two possibilities W_{η_1} and $W_{\eta_1+\eta_2}$ must give $S^2 \tilde{\times} RP^2$ and $RP^4 \#_{S^1} RP^4$. According to [14, p. 80], $W_{\eta_1} \simeq S^2 \tilde{\times} RP^2$ and $W_{\eta_1+\eta_2} \simeq RP^4 \#_{S^1} RP^4$.

Since the homomorphisms $H^*(W_\alpha; \mathbb{F}_2) \to H^*(K; \mathbb{F}_2)$ are isomorphisms in degrees ≤ 3, $H^1(W_\alpha; \mathbb{F}_2) = \langle x \rangle$, where $x^3 = 0$ in all cases. Let $p\colon K \to S^2$ denote the restriction of the projection map to the first factor of $S^2 \times RP^2$. The group $H^2(K; \mathbb{F}_2)$ is generated by x^2 and the class u pulled back by $p\colon K \to S^2$. Since $p \circ \eta_2$ is a constant map, it follows that $p \circ (J + \eta_2) = p \circ J$, which extends across D^4. Therefore the map p extends to a map from P_{HM} to S^2, and so $u^2 = 0$ in $H^4(P_{HM}; \mathbb{F}_2)$. Also since $x^4 = 0$, it follows that $v_2(P_{HM}) = 0$. On the other hand, this projection does not extend in this way when $\alpha = \eta_1$ or $\eta_1 + \eta_2$. $\square$

Remark 4.3. The only other quadratic 2-type with $\pi = \mathbb{Z}/2$, $w_1 \neq 1$ and $\chi = 2$ is that of $RP^4 \# CP^2$ (the nontrivial RP^2-bundle over S^2), which contains two homotopy types. One of these is not homotopy equivalent to a closed 4-manifold, by [22, §3.3.1]. These PD_4-complexes have universal cover $\widetilde{X} \simeq S^2 \tilde{\times} S^2$, and do not cover PD_4-complexes with $\chi = 1$ (see [10, Lemma 12.3]).

5. Explicit constructions for $S^2 \tilde{\times} RP^2$ and $RP^4 \#_{S^1} RP^4$

The goal of this section is to express these two smooth model manifolds in terms of explicit building blocks. The "coordinate" formulas will be used in later sections to compute homotopy type invariants, and to construct smooth model manifolds with $\chi(M) = 1$.

Let E be a regular neighbourhood of $RP^2 = \{[x{:}y{:}z{:}0{:}0] \mid x^2 + y^2 + z^2 = 1\}$ in RP^4, and note the following properties:

(i) $\nu = \overline{RP^4 \setminus E}$ is a regular neighbourhood of $RP^1 = \{[0{:}0{:}0{:}u{:}v] \mid u^2 + v^2 = 1\}$;
(ii) $\partial E = \partial \nu$ is both the total space of a non-trivial S^1-bundle over RP^2 and the mapping torus $S^2 \tilde{\times} S^1 = S^2 \times [0,1]/(s,0) \sim (A(s),1)$;
(iii) In particular, $\pi_1(\partial E) \cong \mathbb{Z}$, and so E is not the product $RP^2 \times D^2$;
(iv) On passing to the universal cover we see that $S^4 = \widetilde{E} \cup \widetilde{\nu}$;
(v) We may assume that $\widetilde{E} = \{(x, y, z, u, v) \in S^4 \mid u^2 + v^2 \leq \frac{1}{4}\}$.

Now let $h \colon \widetilde{E} \to S^2 \times D^2$ be the homeomorphism given by $h(\tilde{e}) = (x/r, y/r, z/r, 2u, 2v)$, where $r = \sqrt{x^2 + y^2 + z^2}$, for all $\tilde{e} = (x, y, z, u, v) \in \widetilde{E}$. It follows that we may write $E = S^2 \times D^2/(s,d) \sim (A(s), -d)$, and the projection $p \colon E \to RP^2$ is then given by $p([s,d]) = [s] \in RP^2$. The space E is also an orbifold bundle with general fibre S^2 over the marked disc $D(2)$, via the projection $p'([s,d]) = d^2$. Here we view D^2 as the unit disc in the complex plane.

We shall view S^2 henceforth as the purely imaginary quaternions of length 1. The antipodal map A is multiplication by -1, while conjugation by $\mathbf{k}$ induces rotation R_π through a half-turn about the $\mathbf{k}$-axis. The sphere is the union of two hemispheres $S^2 = D_- \cup D_+$ with boundary $S^1 = D_- \cap D_+$ in the $(\mathbf{i}, \mathbf{j})$-plane.

The orthogonal projection λ of the purely imaginary quaternions onto the $(\mathbf{i}, \mathbf{j})$-plane restricts to homeomorphisms from each of D_- and D_+ onto the unit disc in this plane, and $\lambda((R_\pi(s)) = A(\lambda(s)) = -\lambda(s)$, for all $s \in S^2$.

Definition 5.1 (Construction of $S^2 \tilde{\times} RP^2$)**.** Doubling E along its boundary gives the total space of an S^2-bundle over RP^2. This space DE is non-orientable and $v_2(DE) \neq 0$, since the core RP^2 in E has self-intersection 1 (mod 2). Thus DE is the nontrivial, non-orientable S^2-bundle space

$$S^2 \tilde{\times} RP^2 = S^2 \times S^2/(s,t) \sim (A(s), R_\pi t).$$

Composition of the double covering of RP^2 with the projection of $S^2 \times S^2$ onto its first factor induces the S^2-bundle projection $DE \to RP^2$.

The space DE is also the total space of an orbifold bundle with general fibre S^2 over the orbifold $S(2,2)$ (the double of $D(2)$).

We may construct a different 4-manifold by identifying two copies of E via a diffeomorphism of their boundaries which does not extend across E. The action of conjugation by $e^{\pi \mathbf{i} t}$ on S^2 inside the unit quaternions is rotation through $2\pi t$ radians about the $\mathbf{i}$-axis.

Definition 5.2 (Construction of $RP^4 \#_{S^1} RP^4$)**.** Let E_1 and E_2 be two copies of E, and let $\xi\colon \partial E_1 \to \partial E_2$ be the map given by

$$\xi([s, y^2\mathbf{i}]_1) = [y\mathbf{i}s(y\mathbf{i})^{-1}, y^2\mathbf{i}]_2,\ \forall\ s \in S^2,\ \forall\ y = e^{\pi \mathbf{k} t},\ 0 \le t \le 1.$$

We define $RP^4 \#_{S^1} RP^4 = E_1 \cup_\xi E_2$ (see [9, p. 651] for another description).

Note that $e^{\pi \mathbf{k} t}$ is a square root for $e^{2\pi \mathbf{k} t}$. This "twist map" ξ does not extend to a homeomorphism from E_1 to E_2 (see [13, Corollary 2.2]).

Remark 5.3. The complication in the formula for ξ in Definition 5.2 flows from the fact that this copy of S^1 is not closed under quaternionic multiplication, whereas its translate $S^1\mathbf{i}$ is the unit circle in $\mathbb{R} \oplus \mathbb{R}\mathbf{k} \cong \mathbb{C}$.

Remark 5.4. The manifold $RP^4 \#_{S^1} RP^4$ is the total space of an orbifold bundle with regular fibre S^2 over $S(2,2)$. The exceptional fibres are the cores RP^2 of the copies of E, and each has self-intersection 1. Hence $v_2(RP^4 \#_{S^1} RP^4) \neq 0$. We shall show in the next section that $RP^4 \#_{S^1} RP^4$ is *not* homotopy equivalent to a bundle space [14], and hence it is not geometric.

We conclude this section with an explicit identification of $\widetilde{X} \simeq S^2 \times S^2$ for the model manifold $X = RP^4 \#_{S^1} RP^4$.

The universal cover of $RP^4 \#_{S^1} RP^4$ is the union $\widetilde{E}_1 \cup_{\tilde{\xi}} \widetilde{E}_2$, where $\tilde{\xi}$ is the lift of ξ given by $\tilde{\xi}((s,x)_1) = (xsx^{-1}, x)_2$, for all $(s,x) \in S^2 \times S^1 = \partial \widetilde{E}_1$. Let $\mu_t(x) = \cos(\frac{\pi}{2}t)\mathbf{1} + \sin(\frac{\pi}{2}t)x$, for $x \in S^1$ and $0 \le t \le 1$. Then $\mu_0(x) = \mathbf{1}$ and $\mu_1(x) = x$, for all $x \in S^1$, and

$$\tilde{\xi}_t((s,x)_1) = (\mu_t(x)s\mu_t(x)^{-1}, x)_2$$

defines an isotopy from the identity to $\tilde{\xi}$. Hence $\widetilde{E}_1 \cup_{\tilde{\xi}} \widetilde{E}_2 \cong S^2 \times S^2$.

We may make this explicit as follows. Let $P(r,x) = \sin(\frac{\pi}{2}r)x + \cos(\frac{\pi}{2}r)\mathbf{k}$, for $0 \le r \le 1$ and $x \in S^1 = D_- \cap D_+$. Then $P(0,x) = \mathbf{k}$ and $P(1,x) = x$, for all $x \in S^1$. Let $V\colon D_+ \to S^3$ be the function defined by $V(d) = P(r,x)$ if $\lambda(d) = rx$, with $0 \le r \le 1$ and $x \in S^1$. Then the function $H\colon S^2 \times S^2 \to \widetilde{E}_1 \cup \widetilde{E}_2$, defined by

$$H(s,d) = (s,d)_1 \in \widetilde{E}_1,\ \forall\ (s,d) \in S^2 \times D_- \tag{5.5}$$

and

$$H(s,d) = (V(d)sV(d)^{-1}, d)_2 \in \widetilde{E}_2,\ \forall\ (s,d) \in S^2 \times D_+, \tag{5.6}$$

is a homeomorphism. Hence $RP^4 \#_{S^1} RP^4 \cong S^2 \times S^2/\langle\psi\rangle$, where ψ is the free involution given by

$$\psi(s,d) = (A(s), R_\pi(d)),\ \forall\ (s,d) \in S^2 \times D_-,$$

and

$$\psi(s,d) = (V(R_\pi(d))^{-1}V(d)A(s)V(d)^{-1}V(R_\pi(d)), R_\pi(d)),\ \forall\ (s,d) \in S^2 \times D_+.$$

It is clear from the formula that ψ is an involution, since $R_\pi^2 = Ix$.

If we set $x = \cos(2\pi t)\mathbf{i} + \sin(2\pi t)\mathbf{j}$ for some $0 \le t \le 1$ then we may write the factor $V(R_\pi(d))^{-1}V(d)$ more explicitly as

$$V(R_\pi(d))^{-1}V(d) = \cos(\pi r)\mathbf{1} + \sin(\pi r)\sin(2\pi t)\mathbf{i} - \sin(\pi r)\cos(2\pi t)\mathbf{j}.$$

Thus $V(R_\pi(d))^{-1}V(d) = \mathbf{1}$ when $r = 0$ and $V(R_\pi(d))^{-1}V(d) = -\mathbf{1}$ when $r = 1$. (This expression was found by solving the linear system

$$V(R_\pi(d))(u\mathbf{i} + v\mathbf{j} + w\mathbf{k} + z\mathbf{1}) = V(d),$$

for the unknowns $u, v, w, z \in \mathbb{R}$.)

6. Distinguishing the homotopy types

We shall follow [14] in using the mod 2 intersection pairing (in the guise of v_2) and the invariant q_{KKR} to show that $RP^4\#_{S^1}RP^4$ is not homotopy equivalent to either of the S^2-bundle spaces. As our construction of $RP^4\#_{S^1}RP^4$ differs slightly from that of [14], we shall give details of the geometric computation of q_{KKR} for these manifolds.

Theorem 6.1. *The model manifolds $S^2 \times RP^2$, $S^2\tilde{\times}RP^2$ or $RP^4\#_{S^1}RP^4$ represent distinct homototpy types, distinguished by the Wu class v_2 and the invariant q_{KKR}.*

Proof. Let $M = S^2 \times RP^2$, $S^2\tilde{\times}RP^2$ or $RP^4\#_{S^1}RP^4$, and let $x, y \in \pi_2(M)$ be the classes corresponding to the first and second factors of $S^2 \times S^2$. Then $x + y$ corresponds to the diagonal. In each case x is represented by the (general) fibres of the (orbifold) bundle projections to RP^2, $S(2,2)$ and $S(2,2)$, respectively, which are embedded with trivial normal bundle, and so $q_{KKR}(M)(x) = 0$, while the normal Euler number of the diagonal is ± 2.

Let $f : S^2 \to S^2$ be the map which folds one hemisphere onto another, and let $g : S^2 \to RP^2$ be the 2-fold cover. The 2-sphere $\{(f(s), s) | s \in S^2\}$ represents y, and has trivial normal bundle, since f is null homotopic. Its image in $S^2 \times RP^2$ has a single double point, and so $q_{KKR}(S^2 \times RP^2)(y) \equiv 2 \pmod 4$. The graph $\Gamma_g \subset S^2 \times RP^2$ is an embedded 2-sphere which lifts to the diagonal embedding in $S^2 \times S^2$. Since there are no self intersections, $q_{KKR}(S^2 \times RP^2)(x + y) \equiv 2 \pmod 4$ also. Hence $q_{KKR}(S^2 \times RP^2)$ is nontrivial for $S^2 \times RP^2$.

In $S^2\tilde{\times}RP^2$ the fibre of the bundle projection to RP^2 represents y. Hence

$$q_{KKR}(S^2\tilde{\times}RP^2)(x) = q_{KKR}(S^2\tilde{\times}RP^2)(y) = 0.$$

The image of the diagonal has a circle of self-intersections. However id_{S^2} is isotopic to a self-homeomorphism of S^2 which is the identity on one hemisphere and moves the equator off itself in the other hemisphere. Hence the diagonal embedding is isotopic to an embedding whose image has just one self-intersection. Hence $q_{KKR}(S^2\tilde{\times}RP^2)(x+y)=0$ also, and so $q_{KKR}(S^2\tilde{\times}RP^2)$ is identically 0 for $S^2\tilde{\times}RP^2$.

In $RP^4\#_{S^1}RP^4$ the class y is represented by the image of $\{\mathbf{j}\}\times S^2$. Double points in the image correspond to pairs $\{s,s'\}\subset S^2$ such that $\psi(\mathbf{j},s)=(\mathbf{j},s')$. If $\{s,s'\}$ is such a pair then $s,s'\in D_+$, $s'=R_\pi(s)$ and

$$\mathbf{j}V(R_\pi(s))^{-1}V(s)=-V(R_\pi(s))^{-1}V(s)\mathbf{j}.$$

On using the explicit formula for $V(R_\pi(d))^{-1}V(d)$ given at the end of Section 5, we see that we must have $\cos(\pi r)=0$ and $\cos(2\pi t)=0$. Thus there are just two possibilities for s, differing by the rotation R_π. We may check that the double point is transverse. Hence $|\operatorname{Self}(S_y)|=1$. Since $\{\mathbf{j}\}\times S^2$ has trivial normal bundle in $S^2\times S^2$, $q_{KKR}(RP^4\#_{S^1}RP^4)(y)\equiv 2 \pmod 4$, and so $RP^4\#_{S^1}RP^4$ is not homotopy equivalent to $S^2\tilde{\times}RP^2$. It is not homotopy equivalent to $S^2\times RP^2$ either, since $v_2(RP^4\#_{S^1}RP^4)\neq 0$. Thus these three manifolds may be distinguished by the invariants v_2 and q_{KKR}. □

7. PD_4-complexes with $\pi\cong(\mathbb{Z}/2)^2$ and $\chi=1$

We now consider the cases where $\pi\cong\mathbb{Z}/2\times\mathbb{Z}/2$. As mentioned in the Introduction, there are two geometric quotients, namely $RP^2\times RP^2$ and the non-trivial bundle $RP^2\tilde{\times}RP^2$. In this section, we will construct a third smooth manifold N with universal cover $S^2\times S^2$ and fundamental group π, which is *not* a geometric quotient.

Recall from Definition 5.2 that the manifold $RP^4\#_{S^1}RP^4=E_1\cup_\xi E_2$ was expressed in terms of the glueing map $\xi\colon\partial E_1\to\partial E_2$. We can define a smooth free involution θ of $\partial E_i=S^2\tilde{\times}S^1$, with quotient $RP^2\times S^1$, by the map $\theta([s,x])=[-s,x]$. Note that the maps θ and ξ commute.

Definition 7.1. Let N denote the quotient space of $RP^4\#_{S^1}RP^4$ by the smooth free involution F given by the formula $F([s,d]_i)=[-s,d]_{3-i}$ for all $[s,d]_i\in E_i$ and $i=1,2$.

By construction, the manifold N has $\pi_1(N)=\mathbb{Z}/2\times\mathbb{Z}/2$ and $\chi(N)=1$. We summarize some of its properties.

Proposition 7.2. *The smooth closed 4-manifold N has the following properties:*

- *(i) N is the quotient of $RP^4\#_{S^1}RP^4$ by a smooth free involution;*
- *(ii) the universal covering $\widetilde{N}=S^2\times S^2$;*
- *(iii) N is in the quadratic 2-type of $RP^2\tilde{\times}RP^2$;*
- *(iv) N is not a geometric quotient.*

Proof. Part (i) is immediate from Definition 7.1, and part (ii) follows from the construction of $RP^4\#_{S^1}RP^4$ given in Definition 5.2.

Part (iv) follows from Theorem 6.1: the manifold N is not homotopy equivalent to a geometric quotient (i,e, a bundle space over RP^2), since it is covered by $RP^4\sharp_{S^1}RP^4$, which is not homotopy equivalent to a bundle space.

In order to prove part (iii), we first collect some information about the quadratic 2-types in this setting. By [10, Chapter 12, §6], there are exactly two equivalence classes of quadratic 2-types realized by PD_4-complexes X with universal cover $\widetilde{X}\simeq S^2\times S^2$ and

$$\pi_1(X)\cong\pi=\langle t,u\mid t^2=u^2=(tu)^2=1\rangle.$$

Let $\{t^*, u^*\}$ be the dual basis for $H^1(\pi;\mathbb{F}_2)$. If X is a PD_4-complex with $\pi_1(X) \cong \pi$ and $\chi(X) = 1$, then we may assume that $v_1(X) = t^* + u^*$ and $v_2(X)$ is either t^*u^* or $t^*u^* + (u^*)^2$. This is an easy consequence of Poincaré duality with coefficients $\mathbb{F}_2$ and the Wu formulas.

Let X^+ denote the orientation double cover of X. If $v_2(X) = t^*u^*$ then $v_2(X^+) = t^{*2} \neq 0$ and both non-orientable double covers have $v_2 = 0$, while if $v_2(X) = t^*u^* + (u^*)^2$ then $v_2(X^+) = 0$ and just one of the non-orientable double covers has $v_2 = 0$.

The two possibilities for v_2 are realized respectively by $RP^2 \times RP^2$ (with orientation double cover $S(3\eta)$) and the nontrivial bundle space $RP^2 \tilde{\times} RP^2 = S^2 \times S^2/\pi$, where π acts by $t(s, s') = (-s, s')$ and $u(s, s') = (R_\pi(s), -s')$, for all $s, s' \in S^2$.

It now follows that N is in the quadratic 2-type of $RP^2 \tilde{\times} RP^2$, since its orientation double covering N^+ has $v_2(N^+) = 0$ (see Remark 5.4). In particular, $N^+ = S(\eta \oplus 2\epsilon)$. $\square$

Remark 7.3. In Section 9, we will show that there are exactly four distinct homotopy types in the quadratic 2-type of $RP^2 \tilde{\times} RP^2$, of which two are represented by manifolds.

8. The quadratic 2-type of $RP^2 \times RP^2$

In this section, we study the quadratic 2-type for the geometric quotient $RP^2 \times RP^2$, and show that it contains only one homotopy type represented by a closed manifold.

Proposition 8.1. *There are three homotopy types of PD_4-complexes X_α in the quadratic 2-type of $RP^2 \times RP^2$.*

Proof. Let $K = \overline{RP^2 \times RP^2 \setminus D^4}$, let $I_1, I_2\colon S^2 \to \widetilde{K} = \overline{S^2 \times S^2 \setminus 4D^4}$ be the inclusions of the factors, and let $[J]$ be the homotopy class of a fixed lift $\widetilde{J}\colon S^3 \to \widetilde{K}$ of the natural inclusion $J\colon S^3 = \partial D^4 \to K$. Then $\Pi = \pi_2(K) \cong \mathbb{Z}^2$ is generated by I_1 and I_2, and

$$\Pi \cong \Lambda/(t+1, u-1) \oplus \Lambda/(t-1, u+1)$$

as a Λ-module, $\Lambda = \mathbb{Z}[\pi]$ denotes the integral group ring. The Hurewicz homomorphism $h\colon \pi_3(K) \to H_3(\widetilde{K};\mathbb{Z}) \cong \mathbb{Z}^3$ is surjective, with kernel the image of $\Gamma_W(\Pi)$, generated by Whitehead products and composites with η. Then $h([J])$ generates $H_3(\widetilde{K};\mathbb{Z})$ as a Λ-module, and $H_3(\widetilde{K};\mathbb{Z}) \cong \Lambda/(1-t)(1-u)\Lambda$.

The elements $\eta_1 = I_1 \circ \bar{\eta}$, $\eta_2 = I_2 \circ \bar{\eta}$ and $\zeta = [I_1, I_2]$ are a basis for $\Gamma_W(\Pi) \cong \mathbb{Z}^3$. Since $\Gamma_W(\Pi)$ is torsion free and $2\eta_i = [I_i, I_i]$, we see that $t\eta_i = u\eta_i = \eta_i$ for $i = 1, 2$, while $t\zeta = u\zeta = -\zeta$. Hence $\mathbb{Z}^w \otimes_\Lambda \Gamma_W(\Pi) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, and the torsion subgroup is generated by the images of η_1 and η_2. Since the k-invariant is symmetric under the involution which interchanges the summands of Π, there are three homotopy types of PD_4-complexes $X_\alpha = K \cup_{[J]+\alpha} e^4$ in this quadratic 2-type, represented by $\alpha = 0$, η_1 and $\eta_1 + \eta_2$.

As above, let $\{t^*, u^*\}$ be the basis of $H^1(\pi;\mathbb{F}_2)$ dual to $\{t, u\}$. Let X^t_α and X^u_α be the covering spaces associated to the subgroups $\langle t\rangle = \mathrm{Ker}(u^*)$ and $\langle u\rangle = \mathrm{Ker}(t^*)$ of π, respectively. Since the homomorphisms $H^*(X_\alpha;\mathbb{F}_2) \to H^*(K;\mathbb{F}_2)$ are isomorphisms in degrees ≤ 3 (see Remark 4.2) and $(t^*)^3 = (u^*)^3 = 0$ in $H^3(RP^2 \times RP^2;\mathbb{F}_2)$, we see that $(t^*)^3 = (u^*)^3 = 0$ in $H^3(X_\alpha;\mathbb{F}_2)$, for all α. It follows easily from the nonsingularity of Poincaré duality that the rings $H^*(X_\alpha;\mathbb{F}_2)$ are all isomorphic. In particular, $w_1(X_\alpha) =$

t^*+u^*, $v_2(X_\alpha)=t^*u^*$ and $x^4=0$, for all $x\in H^1(X_\alpha;\mathbb{F}_2)$, in each case. Hence $X_\alpha^+\simeq S^2\times S^2/\langle\sigma^2\rangle$, while the non-orientable double covers X_α^t and X_α^u each have $v_2=0$. $\square$

We shall now adapt the argument of [4, §3] to show that if $\alpha\neq 0$ then X_α is not homotopy equivalent to a closed 4-manifold.

Theorem 8.2. *Let M be a closed 4-manifold with $\pi=\pi_1(M)\cong(\mathbb{Z}/2)^2$ and $\chi(M)=1$, and such that $x^4=0$ for all $x\in H^1(M;\mathbb{F}_2)$. Then M is homotopy equivalent to $RP^2\times RP^2$.*

Proof. Our hypotheses imply that M is in the quadratic 2-type of $RP^2\times RP^2$, and so $M\simeq X_\alpha=K\cup_{[J]+\alpha}e^4$, for some $\alpha=0$, η_1 or $\eta_1+\eta_2$. If M in the quadratic 2-type of $RP^2\tilde{\times}RP^2$ then there is a class $x\in H^1(M;\mathbb{F}_2)$ such that $x^3\neq 0$. Poincaré duality considerations then imply that $x^4\neq 0$ (see [10, Chapter 12, §§4-6]).

Suppose that $\alpha=\eta_1$ or $\eta_1+\eta_2$. Then the image of α in $\pi_3(RP^2)$ under composition with the projection pr_1 to the first factor is $\bar\eta$. Hence the composite of the inclusion $K\subset RP^2\times RP^2$ with pr_1 extends to a map $p\colon X_\alpha\to L=RP^2\cup_{\bar\eta}e^4$ (note that $\mathrm{Ker}(\pi_1(p)=\langle u\rangle)$. Let $\tilde p\colon X_\alpha^u\to\widetilde{L}$ be the induced map of double covers, and let $f\colon X_\alpha\to RP^{k+1}$ (for k large) be the classifying map for the double cover $X_\alpha^u\to X_\alpha$.

Let $a=\tilde p^*(c)$ be the image of the generator of $H^2(\widetilde{L};\mathbb{F}_2)=\mathbb{F}_2$, let $\bar b=(u^*)^2\in H^2(X_\alpha;\mathbb{F}_2)$, and let b be the image of $\bar b$ in $H^2(X_\alpha^u;\mathbb{F}_2)$. The 3-skeleton of X_α^u is K^u, and so the covering transformation t acts on $H^2(X_\alpha^u;\mathbb{F}_2)$ via the identity. Hence the quadratic form q defined in [4, §2], and used in computing the Arf invariant $A(X_\alpha,f)$ of the covering $X_\alpha^u\to X_\alpha$, is an enhancement of the ordinary cup product.

The pair $\{a,b\}$ is a symplectic basis with respect to the cup product, and $q(a)=1$ since $\alpha\neq 0$, by the argument of [4, p. 1325]. Since $(u^*)^3=(u^*)^4=0$ in $H^*(X_\alpha;\mathbb{F}_2)$, $Sq_i\bar b=Sq^{2-i}\bar b=0$ for $i=0$ or 1. Hence we also have $q(b)=1$, by [4, Proposition 1.5], and so $A(X_\alpha,f)$ is nonzero. But this contradicts the assumption that X_α is homotopy equivalent to a closed manifold, by [4, Proposition 2.2], since any double covering of manifolds is Poincaré splittable. Hence $\alpha=0$ and so M is homotopy equivalent to $RP^2\times RP^2$. $\square$

Corollary 8.3. *There is exactly one homotopy type for a closed manifold in the quadratic 2-type of $RP^2\times RP^2$.*

Remark 8.4. The inclusion $RP^2\to L=RP^2\cup_{\bar\eta}e^4$ induces isomorphisms on π_i for $i\leq 2$. Since L is covered by $S^2\cup_\eta e^4\cup_{A\eta}e^4\simeq S^2\cup_\eta e^4\vee S^4=CP^2\vee S^4$, $\pi_3(L)=0$. Hence we may view L as the 4-skeleton of $P_2(RP^2)$. (See [19].)

9. The quadratic 2-type of $RP^2\tilde{\times}RP^2$

In this section, we study the quadratic 2-type for the geometric quotient $RP^2\tilde{\times}RP^2$, and show that it contains exactly two homotopy type represented by the closed manifolds $RP^2\tilde{\times}RP^2$, and N (see Definition 7.1).

Proposition 9.1. *There are four homotopy types of PD_4-complexes Y_α in the quadratic 2-type of $RP^2\tilde{\times}RP^2$.*

Proof. Let $K' = \overline{RP^2 \tilde{\times} RP^2 \setminus D^4}$, and let $\Pi' = \pi_2(K')$. Let $J' \colon S^3 = \partial D^4 \to K'$ be the natural inclusion, and let $\eta_i' \colon S^3 \to K'$ (with $i = 1, 2$) be "Hopf maps" factoring through inclusions of the factors of the universal cover $S^2 \times S^2$. We again find that $\mathbb{Z}^w \otimes_\Lambda \Gamma_W(\Pi') \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, and the torsion subgroup is generated by the images of η_1' and η_2'. However there are four homotopy types of PD_4-complexes $Y_\alpha = K' \cup_{[J']+\alpha} e^4$ in this quadratic 2-type, represented by $\alpha = 0$, η_1', η_2' and $\eta_1' + \eta_2'$.

Let $\{t^*, u^*\}$ be the basis of $H^1(\pi; \mathbb{F}_2)$ dual to $\{t, u\}$, and let Y_α^t and Y_α^u be the covering spaces associated to the subgroups $\langle t \rangle = \mathrm{Ker}(u^*)$ and $\langle u \rangle = \mathrm{Ker}(t^*)$ of π, respectively. We may assume that u^* is induced from the base RP^2, so $(u^*)^3 = 0$, and then $(t^*)^3 \neq 0$, since $v_2(RP^2 \tilde{\times} RP^2) \neq 0$. We again find that the $\mathbb{F}_2$-cohomology rings of the Y_α are all isomorphic. In particular, $v_2(Y_\alpha) = t^*u^* + (u^*)^2$ in each case. Hence in each case Y_α^+ is homotopy equivalent to the S^2-bundle space over RP^2 which is a spin manifold. The covering space Y_α^t is homotopy equivalent to $S^2 \times RP^2$, since $v_2(Y_\alpha^t) = 0$, while Y_α^u is homotopy equivalent to one of either $RP^4 \#_{S^1} RP^4$ or $S^2 \tilde{\times} RP^2$, since $v_2(Y_\alpha^u) \neq 0$. □

Theorem 9.2. *Let M be a closed 4-manifold with $\pi = \pi_1(M) \cong (\mathbb{Z}/2)^2$ and $\chi(M) = 1$, and such that $x^4 \neq 0$ for some $x \in H^1(M; \mathbb{F}_2)$. Then M is homotopy equivalent to $RP^2 \tilde{\times} RP^2$ or N.*

Proof. We shall adapt the proof of Theorem 8.2, again based on the arguments of [4]. In this case M must be in the quadratic 2-type of $RP^2 \tilde{\times} RP^2$, and so $M \simeq Y_\alpha = K' \cup_{[J']+\alpha} e^4$ for some $\alpha = 0$, η_1', η_2' or $\eta_1' + \eta_2'$. The double covering space M^t is homotopy equivalent to $RP^2 \times S^2$. As in Theorem 8.2, the covering automorphism induces the identity on $H^2(M^u; \mathbb{F}_2)$.

Suppose that $\alpha = \eta_2'$ or $\eta_1' + \eta_2'$. The composite of the inclusion $K' \subset RP^2 \tilde{\times} RP^2$ with the bundle projection extends to a map $p \colon Y_\alpha \to L$. Let $\tilde{p} \colon Y_\alpha^u \to \widetilde{L}$ be the induced map of double covers, and let $a = \tilde{p}^*(c)$ be the image of the generator of $H^2(\widetilde{L}; \mathbb{F}_2)$. Let $\bar{b} = (t^*)^2 \in H^2(Y_\alpha; \mathbb{F}_2)$, and let b be the image of $\bar{b}$ in $H^2(Y_\alpha^t; \mathbb{F}_2)$. Then $\{a, b\}$ is a symplectic basis for the cup product pairing. We again find that $q(a) = q(b) = 1$, so the Arf invariant associated to the 2-fold covering $Y_\alpha^u \to Y_\alpha$ is nonzero, contradicting the hypothesis that M is a closed manifold. Therefore either $\alpha = 0$ or $\alpha = \eta_1'$. Since $Y_0 = RP^2 \tilde{\times} RP^2$ and N are manifolds in this quadratic 2-type (see Proposition 7.2), and are not homotopy equivalent, we must have $Y_{\eta_1'} \simeq N$ and M must be one of these two manifolds. □

The manifolds $RP^2 \tilde{\times} RP^2$ and N may be distinguished by their (non-orientable) double covers. However, in general we do not know which of the PD_4-complexes W_α of §4 are double covers of the PD_4-complexes X_β or Y_γ of §8 or §9.

10. Stable classification for $\pi = \mathbb{Z}/4$

Let $\xi \colon B \to \mathrm{BTop}$ denote the normal 1-type of the geometric quotient M of $S^2 \times S^2$ with fundamental group $\pi = \mathbb{Z}/4$. We may assume that $B = \mathrm{BTopSpin} \times K(\pi, 1)$, since $w_2(\widetilde{M}) = 0$ (see [22, Theorem 5.2.1 and §8.1]). Let $\gamma \colon B \to K(\pi, 1)$ be the projection onto the second factor.

There is a preferred local coefficient system $\{\mathbb{Z}^w\}$ on M pulled back from $K(\pi, 1)$, and its pullback by γ gives a preferred local coefficient system on B. There is a preferred Thom class $U(\nu_M)$ induced by the collapse map $\varphi\colon S^{k+4} \to T(\nu_M)$, for large k, and the cap product $\varphi_*[S^{k+4}] \cap U(\nu_M)$ which is determined by the fundamental class $[M] \in H_4(M; \mathbb{Z}^w)$ (see [21, §6]). This gives a preferred Thom class $U(\xi)$, and fixes a fundamental class $[N] \in H_4(N; \mathbb{Z}^w)$ by pull-back for each bordism element $[N, f] \in \Omega_4(B, \xi)$, since $f\colon N \to B$ is a lift of the classifying map $\nu_N\colon N \to BTOP$.

In order to compute the bordism group $\Omega_4(B, \xi)$ we use the Atiyah-Hirzebruch spectral sequence with $E^2_{p,q} = H_p(\pi; \Omega_q^{TopSpin})$ where the coefficients

$$\Omega_q^{TopSpin} = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, \qquad \text{for } 0 \leq 1 \leq 4,$$

are twisted by w_1 (and denoted $\mathbb{Z}^w$). We have $E^2_{p,0} = H_p(\pi; \mathbb{Z}^w) = \mathbb{Z}/2$, for p even, and $E^2_{p,0} = 0$ for p odd. Similarly, $E^2_{0,4} = \mathbb{Z}/2$. The first differential

$$d_2\colon E^2_{p,q} \to E^2_{p-2,q+1}$$

is dual to a map on mod 2 cohomology

$$\hat{d}\colon H^{p-2}(\pi; \mathbb{Z}/2) \to H^p(\pi; \mathbb{Z}/2)$$

for the cases $(4, 2)$ and $(3, 1)$.

Note that the cohomology ring $H^*(\pi; \mathbb{Z}/2) = P(u) \otimes E(x)$, where $|u| = 2$ and $|x| = 1$, with $Sq^1 u = 0$ and $x^2 = 0$. The classes $w_1(\nu_M) = x$ and $w_2(\nu_M) = u$.

The d_2 differentials starting at $E^2_{*,0} = H_*(\pi; \mathbb{Z}^w)$ factor through the reduction mod 2. According to Teichner [23, §2] the dual map $\hat{d}$ is given by the formula

$$\hat{d}(\alpha) = Sq^2\alpha + (Sq^1\alpha) \cdot w_1 + \alpha \cdot w_2.$$

We compute using this formula and obtain:

$$\begin{aligned}
\hat{d}\colon H^1(\pi; \mathbb{Z}/2) &\to H^3(\pi; \mathbb{Z}/2), & \hat{d}(x) &= xu \neq 0\\
\hat{d}\colon H^2(\pi; \mathbb{Z}/2) &\to H^4(\pi; \mathbb{Z}/2), & \hat{d}(u) &= 0\\
\hat{d}\colon H^3(\pi; \mathbb{Z}/2) &\to H^5(\pi; \mathbb{Z}/2), & \hat{d}(xu) &= 0\\
\hat{d}\colon H^4(\pi; \mathbb{Z}/2) &\to H^6(\pi; \mathbb{Z}/2), & \hat{d}(u^2) &= u^3 \neq 0.
\end{aligned}$$

After dualizing, we get $E^3_{0,4} = \mathbb{Z}/2$, $E^3_{3,1} = 0$, $E^3_{2,2} = H_2(\pi; \mathbb{Z}/2) = \mathbb{Z}/2$, and $E^3_{4,0} = \mathbb{Z}/2$. Moreover, the only nonzero entry on the line $p+q = 5$ of the E^3 page is $E^3_{3,2} = E^2_{3,2} = \mathbb{Z}/2$.

We remark that the non-zero element in $E^3_{0,4} = \mathbb{Z}/2$ is represented by the image of the E_8-manifold under the inclusion map

$$\Omega_4^{TopSpin}(*) \to \Omega_4(B, \xi).$$

However, we have a factorization:

$$\Omega_4^{TopSpin}(*) \to \Omega_4(B, \xi) \to \Omega_4^{TopSpin^c}(*),$$

and the E_8-manifold represents a non-trivial element in $\Omega_4^{TopSpin^c}(*)$, as noted in [9, p. 654]. Hence the $E^3_{0,4}$-term survives to $E^\infty_{0,4}$. The $E_{4,0}$-term is detected by the image of the twisted fundamental class. Let $[N, f] \in \Omega_4(B, \xi)$ represent an element with $0 \neq \gamma_* f_*[N] \in$

$H_4(\pi; \mathbb{Z}^w)$. Then N is non-orientable and $2[N, f] = 0$ from the null-bordism $f \circ p_1 \colon N \times I \to B$. Hence there are no extensions in passing from $E^\infty_{*,*}$ to the bordism group. The conclusion is that

$$\Omega_4(B, \xi) = \mathbb{Z}/2 \oplus H_2(\pi; \mathbb{Z}/2) \oplus \mathbb{Z}/2.$$

Let $c \colon M \to B$ denote the classifying map of the ξ-structure on M. To detect elements in this bordism group, we can define

$$\Omega_4(B, \xi)_M = \{[M', c'] \colon \gamma_* c'_*[M'] = \gamma_* c_*[M] \in H_4(\pi; \mathbb{Z}^w)\}.$$

By Lemma 3.4, the image $\gamma_* c_*[M] \in H_4(\pi; \mathbb{Z}^w)$ is non-zero. Therefore, $\Omega_4(B, \xi)_M$ is a coset of

$$\ker(\gamma_* \colon \Omega_4(B, \xi) \to H_4(\pi; \mathbb{Z}^w)) = \mathbb{Z}/2 \oplus H_2(\pi; \mathbb{Z}/2).$$

Hence $\Omega_4(B, \xi)_M$ consists of four distinct bordism classes.

We will now consider the natural map $\Omega_4(M, \xi) \to \Omega_4(B, \xi)$. These are the bordism group for the fibration $\xi \colon M(w_1, w_2) \to BTOP$ given by the pull-back diagram

(10.1)
$$\begin{tikzcd}
BTopSpin \arrow[r] \arrow[d, equal] & M(w_1, w_2) \arrow[r, "j"] \arrow[d, "\xi"] & M \arrow[d, "w_1(M) \times w_2(M)"] \\
BTopSpin \arrow[r] & BTOP \arrow[r, "w_1 \times w_2"] & K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2).
\end{tikzcd}$$

See Kirby and Siebenmann [15, p. 318] for the low-dimensional homotopy groups of $BTOP$ and related spaces. In particular, $\pi_4(BTOP) = \mathbb{Z} \oplus \mathbb{Z}/2$ and the map

$$\pi_4(BTOP) \to \pi_4(B(TOP/O)) = \pi_3(TOP/O) = \mathbb{Z}/2$$

is a split surjection. Topological bundles over S^4 are classified by the stable triangulation class $k \in H^4(BTOP; \mathbb{Z}/2)$ and the first Pontrjagin class. Let $\zeta_0 \colon S^4 \to BTOP$ be the topological bundle with $p_1(\zeta_0) = 0$, and $k(\zeta_0) \neq 0$.

Definition 10.2. We will define two reference maps for this bordism theory.

(i) We can define $\widehat{\mathrm{id}} \colon M \to M(w_1, w_2)$, since the map $(\mathrm{id} \times \nu_M) \colon M \to M \times BTOP$ factors through the pull-back $M(w_1, w_2) \subset M \times BTOP$.
(ii) Let $\zeta_M := p^*(\zeta_0)$ be the pull-back of the bundle ζ_0 over the pinch map $p \colon M \to S^4$.
(iii) Similarly, we can define $\widehat{\mathrm{id}}_{\zeta_M} \colon M \to M(w_1, w_2) \subset M \times BTOP$ by factoring $\mathrm{id} \times (\nu_M \oplus \zeta_M)$ through the pull-back (10.1).

Note that $w_1(\zeta) = w_2(\zeta) = 0$, and the bundle $\zeta_M \oplus \zeta_M$ is stably trivial. By construction, $\xi \circ \widehat{\mathrm{id}} = \nu_M$ and $\xi \circ \widehat{\mathrm{id}}_\zeta = \nu_M \oplus \zeta$.

A similar calculation to the one above shows that

$$\Omega_4(M, \xi) = \mathbb{Z}/2 \oplus H_2(M; \mathbb{Z}/2) \oplus H_3(M; \mathbb{Z}/2) \oplus H_4(M; \mathbb{Z}^w)$$

with the filtration quotients

(i) $\mathscr{F}_4/\mathscr{F}_3 \cong E^\infty_{4,0} = H_4(M; \mathbb{Z}^w) \cong \mathbb{Z}$;
(ii) $\mathscr{F}_3/\mathscr{F}_2 \cong E^\infty_{3,1} = H_3(M; \mathbb{Z}/2) = \mathbb{Z}/2$;
(iii) $\mathscr{F}_2/\mathscr{F}_0 \cong E^\infty_{2,2} = H_2(M; \mathbb{Z}/2) = \mathbb{Z}/2$;

(iv) $\mathscr{F}_0 \cong E^\infty_{0,4} = H_0(M;\mathbb{Z}^w) = \mathbb{Z}/2$.
(v) $\mathscr{F}_2 \cong H_2(M;\mathbb{Z}/2) \oplus \mathbb{Z}/2$, split by the KS-invariant.

An element $[N, \hat{f}, \hat{\nu}]$ of this bordism group is represented by triple consisting of a closed 4-manifold N together with a reference map $\hat{f}\colon N \to M(w_1, w_2)$, and a bundle map $\hat{\nu}\colon \nu_N \to \xi$ covering $\hat{f}$ (see Stong [20, p. 14], and Taylor [21, §6]).

As above, the local coefficient system and choice of fundamental class for N is determined by pull-back from M. By composition with the classifying map $c\colon M \to B$, we obtain an element $c_*[N, \hat{f}, \hat{\nu}] \in \Omega_4(B, \xi)$. To simplify the notation, we will write $[N, f]_\xi := [N, \hat{f}, \hat{\nu}]$ for short. Since B is the normal 1-type of M, we have the structure $[M, \mathrm{id}]_\xi \in \Omega_4(M, \xi)$ to serve as a base point.

Lemma 10.3. *Let M and N be closed non-orientable 4-manifolds with universal covering $S^2 \times S^2$. If $f\colon N \to M$ is a homotopy equivalence and $KS(M) = 0$, then $f^*(\nu_M) \cong \nu_N$ if $KS(N) = 0$, and $f^*(\nu_M \oplus \zeta_M) \cong \nu_N$ if $KS(N) \neq 0$,*

Proof. It follows from the assumptions that $f^*(\nu_M)$ and ν_N have the same Stiefel-Whitney classes. In particular, if $KS(N) = 0$ then $f^*\nu_M - \nu_N$ lifts to an orientable vector bundle λ with $w_i(\lambda) = 0$ for $i > 0$. By the Dold-Whitney classification [2, Theorem 2(c)], oriented vector bundles over a 4-complex are stably determined by p_1 and w_4. In our setting, the Pontrjagin class $p_1(\lambda)$ is divisible by 2, but $H^4(N;\mathbb{Z}) = \mathbb{Z}/2$ since N is non-orientable. Hence $p_1(\lambda) = 0$ and λ is (stably) trivial. If $KS(N) \neq 0$, then $f^*(\nu_M \oplus \zeta_M) - \nu_N$ lifts to an orientable vector bundle, which is again stably trivial. □

A homotopy equivalence $f\colon N \to M$ represents an element of $S_{TOP}(M)$. To define its normal invariant $\eta(f) \in [M, G/TOP]$, we can apply Lemma 10.3 to cover f by a bundle map to ν_M or to $(\nu_M \oplus \zeta)$, if $KS(N) \neq 0$. A choice of bundle isomorphism $f^*\nu_M \cong \nu_N$ (respectively, $f^*(\nu_M \oplus \zeta_M) \cong \nu_N$) fixes a ξ-structure and a fundamental class for N by pull-back, so that $f\colon N \to M$ composed with $\widehat{\mathrm{id}}$ (respectively, $\widehat{\mathrm{id}}_\zeta$) represents an element $[N, f]_\xi \in \Omega_4(M, \xi)$ with $f_*[N] = [M]$.

Define the subset

$$\Omega_4(M, \xi)_M = \{[N, g]_\xi : g_*[N] = [M] \in H_4(M;\mathbb{Z}^w)\}.$$

The next result is an application of topological surgery (see [3]).

Lemma 10.4. *Every element in $\Omega_4(M, \xi)_M$ has the form $[M, f']_\xi$, where $f'\colon M' \to M$ is a homotopy equivalence. If $[M', f']_\xi = [M'', f'']_\xi$, where both f' and f'' are homotopy equivalences, then there exists a homeomorphism $h\colon M' \to M''$ such that $f'' \circ h \simeq f'$.*

Proof. Let $[N, \hat{g}]_\xi$ be an element in $\Omega_4(M, \xi)_M$. Then $\hat{g}\colon N \to M(w_1, w_2)$ together with its bundle data gives a 2-connected map such that $j \circ \hat{g}\colon N \to M$ has degree one. Note that $K_2(\hat{g}) = K_2(j \circ \hat{g})$ since j is 3-connected. Since $L_4(\mathbb{Z}/4, -) = 0$, modified surgery can be performed to obtain a homotopy equivalence $f'\colon M' \to M$ in the same ξ-bordism class. Here we are doing surgery on the map $\hat{g}\colon N \to M(w_1, w_2)$ to eliminate the kernel group $K(\hat{g}) = K_2(j \circ \hat{g}) = \ker\{H_2(N;\Lambda) \to H_2(M;\Lambda)\}$ (compare [17, §5]).

If $[M', f']_\xi = [M'', f'']_\xi$, where both f' and f'' are homotopy equivalences, then a ξ-bordism (W, F) between these elements can be surgered (relative to the boundaries) to an s-cobordism since $L_5(\mathbb{Z}/4, -) = 0$. We then apply the topological s-cobordism theorem. $\square$

Corollary 10.5. *The map $c_*\colon \Omega_4(M,\xi)_M \to \Omega_4(B,\xi)_M$ is surjective. Every element in $\Omega_4(B,\xi)_M$ has the form $c_*[M,f']_\xi$, where $f'\colon M' \to M$ is a homotopy equivalence.*

Proof. By comparing the spectral sequences, we see that the filtration subgroup $\mathscr{F}_2 \subset \Omega_4(M,\xi)$ is mapped isomorphically into $\Omega_4(B,\xi)$. The term $E^\infty_{3,1}(M)$ is mapped to zero and the term $E^\infty_{4,0}(M) = \mathbb{Z}$ is mapped surjectively onto $E^\infty_{4,0}(B) = \mathbb{Z}/2$. $\square$

Remark 10.6. Since $\Omega_4(M,\xi)_M$ has 8 elements, and both $S_{TOP}(M)$ and $\Omega_4(B,\xi)_M$ have 4 elements, the uniqueness statement for the representatives of $\Omega_4(M,\xi)_M$ implies that M has some non-trivial self-homeomorphism. Indeed, the standard $\mathbb{Z}/4$-action on $S^2 \times S^2$ generated by $\tau(s,t) = (-t,s)$ extends to a smooth action of $D_8 = \langle \tau, \sigma \rangle$, where $\sigma(s,t) = (s,-t)$. Hence σ induces an involution on M, which is not homotopic to the identity since σ_* is non-trivial on homology.

The projection of the difference $[M', c\circ f] - [M,c]$ into $E^\infty_{2,2}(B) = H_2(\pi;\mathbb{Z}/2)$ is detected by the first component of the normal invariant $\eta(f) \in [M, G/TOP]$, with respect to the identification

$$S_{TOP}(M) = [M, G/TOP] = H^2(M;\mathbb{Z}/2) \oplus H^4(M;\mathbb{Z}) \cong H_2(M;\mathbb{Z}/2) \oplus \mathbb{Z}/2. \tag{10.7}$$

given by Poincaré duality. We will call this the *reduced normal invariant of* M, and denote by $\overline{\eta}(M') \in H_2(\pi;\mathbb{Z}/2)$ the equivalence class of $\eta(f)$ modulo the action on normal invariants by homotopy self equivalences of M. If this is zero, it follows that the difference $[M', c\circ f] - [M,c]$ is detected by the KS invariant.

Lemma 10.8. *Suppose that $f\colon M \to M$ is a self homotopy equivalence. Then the elements $(M, c\circ f)$ and (M,c) are ξ-bordant.*

Proof. By functoriality, the homotopy equivalence $f\colon M \to M$ induces a self homotopy equivalence $\phi\colon B \to B$, such that $c\circ f \simeq \phi\circ c$. However, since $B = BTopSpin \times K(\pi,1)$ has the homotopy type of $K(\mathbb{Z},4)\times K(\pi,1)$ through dimensions ≤ 5, the composition $\phi\circ c$ is determined by the map $\phi^*\colon H^4(B;\mathbb{Z}) \to H^4(B;\mathbb{Z})$. Either $\phi\circ c \simeq c$ or $\phi\circ c$ differs from c by a non-trivial map $K(\pi,1) \to K(\mathbb{Z},4)$. In this case, the normal invariant of f would have non-zero component in $H^2(\pi;\mathbb{Z}/2) \subset [M, G/TOP]$. But this would imply a change in the Kirby-Siebenmann invariant from domain to range of f, by the formula in [16, p. 398], which is impossible for a self homotopy equivalence. $\square$

Corollary 10.9. *Stably homeomorphic manifolds homotopy equivalent to M are homeomorphic. Such manifolds are distinguished by their reduced normal invariant and the KS invariant.*

Proof. According to the general theory of Kreck [17], to pass from bordism to the stable homeomorphism classification, we must consider the quotient of $\Omega_4(B,\xi)$ by the action of $\mathrm{Aut}(\xi)$. As pointed out by Kirby and Taylor [16, pp. 394-395], it suffices to divide out the natural action of $\mathrm{Out}(\pi, w_1, w_2)$. The calculations above shows that this action is trivial, and hence that the subset $\Omega_4(B,\xi)_M \subset \Omega_4(B,\xi)$ consists of 4 distinct stable homeomorphism classes, each represented by some homotopy equivalence $f\colon M' \to M$. However the structure set $S_{TOP}(M)$ has 4 elements (by Theorem 2.1), so there can be no non-trivial self homotopy equivalences. If follows that the choice of a homotopy equivalence $f\colon M' \to M$ is unique up to homotopy and composition with a homeomorphism. Hence the reduced normal invariant $\bar{\eta}(M') \in H_2(\pi; \mathbb{Z}/2)$ is a well-defined invariant of M'. □

11. A smooth fake version of $\mathbb{M}$?

In this section we construct another smooth manifold M'' with $\pi_1(M'') = \mathbb{Z}/4$, which is homotopy equivalent to the geometric quotient $\mathbb{M}$. At present we are not able to determine whether M'' is homeomorphic to $\mathbb{M}$.

Let $M^+ = S^2 \times S^2/\langle\sigma^2\rangle = S^2 \times S^2/(s,s') \sim (A(s), A(s'))$ be the orientable double cover of $M = S^2 \times S^2/\langle\sigma\rangle$. Let $\Delta = \{(s,s) \mid s \in S^2\}$ be the diagonal in $S^2 \times S^2$. We may isotope Δ to a nearby sphere which meets Δ transversely in two points, by rotating the first factor, and so Δ has self-intersection ± 2. The diagonal is invariant under σ^2, and so $\delta = \Delta/\langle\sigma^2\rangle \cong RP^2$ embeds in M^+ with an orientable regular neighbourhood. Since $\sigma(\Delta) \cap \Delta = \emptyset$ this also embeds in M. We shall see that the complementary region also has a simple description.

We shall identify S^3 with the unit quaternions $\mathbb{H}_1$, and view S^2 as the unit sphere in the space of purely imaginary quaternions. The standard inner product on the latter space is given by $v \bullet w = \mathfrak{Re}(v\bar{w})$, for v, w purely imaginary quaternions. Let

$$C_x = \{(s,t) \in S^2 \times S^2 \mid s \bullet t = x\},\ \forall\, x \in [-1,1].$$

Then $C_1 = \Delta$ and $C_{-1} = \sigma(\Delta)$, while $C_x \cong C_0$ for all $|x| < 1$. The map $f\colon S^3 \to C_0$ given by $f(q) = (q\mathbf{i}q^{-1}, q\mathbf{j}q^{-1})$ for all $q \in S^3$ is a 2-fold covering projection, and so $C_0 \cong RP^3$.

It is easily seen that $N = \cup_{x \geq \varepsilon} C_x$ and $\sigma(N)$ are regular neighbourhoods of Δ and $\sigma(\Delta)$, respectively, while $C = \cup_{x \in [-\varepsilon,\varepsilon]} C_x \cong C_0 \times [-\varepsilon, \varepsilon]$. In particular, N and $\sigma(N)$ are each homeomorphic to the total space of the unit disc bundle in T_{S^2}, and $\partial N \cong C_0 \cong RP^3$. The subsets C_x are invariant under σ^2. Hence $N(\delta) = N/\langle\sigma^2\rangle$ is the total space of the tangent disc bundle of RP^2. In particular, $\partial N(\delta) \cong L(4,1)$ and δ represents the nonzero element of $H_2(M; \mathbb{F}_2)$, since it has self-intersection 1 in $\mathbb{F}_2$.

Remark 11.1. It is not hard to show that any embedded surface representing the nonzero element of $H_2(M; \mathbb{F}_2)$ is non-orientable but lifts to M^+, and so has an orientable regular neighbourhood.

We also see that $C/\langle\sigma^2\rangle \cong L(4,1) \times [-\varepsilon, \varepsilon]$. Since $f(q.\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{k})) = \sigma(f(q))$, the map $\tilde{\sigma}\colon S^3 \to S^3$ defined by right multiplication by $\frac{1}{\sqrt{2}}(\mathbf{1}+\mathbf{k})$ lifts σ. Hence $C_0/\langle\sigma\rangle = S^3/\langle\tilde{\sigma}\rangle = L(8,1)$, and so $MC = C/\langle\sigma\rangle$ is the mapping cylinder of the double cover $L(4,1) \to L(8,1)$. Since $S^2 \times S^2 = N \cup C \cup \sigma(N)$ it follows that $M = N(\delta) \cup MC$.

This construction suggests a candidate for another smooth 4-manifold in the same (simple) homotopy type.

Definition 11.2. Let $M'' = N(\delta) \cup MC'$, where MC' is the mapping cylinder of the double cover $L(4,1) \to L(8,5)$. Then $\pi_1(M'') \cong \mathbb{Z}/4$ and $\chi(M'') = 1$, and so there is a homotopy equivalence $h\colon M'' \simeq M$.

Some questions for further investigation:

(i) Is there an easily analyzed explicit choice for $h\colon M'' \to M$, with computable codimension two Kervaire invariant?
(ii) Are M and M'' homeomorphic? diffeomorphic?
(iii) Is there a computable homeomorphism (or diffeomorphism) invariant that can be applied here?

We remark that most readily computable invariants are invariants of homotopy type.

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