# CLOSED MANIFOLD SURGERY OBSTRUCTIONS AND THE OOZING CONJECTURE

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Abstract. We complete the description of surgery obstructions up to homotopy equivalence for closed oriented manifolds with finite fundamental group. New examples are presented of non-trivial obstructions for Arf invariant product formulas in codimensions $\geq 4$, which give counterexamples to the well-known "Oozing Conjecture" from the 1980's.

## 1. Introduction

The surgery exact sequence [23, Chapter 9] provides a framework for the classification of manifolds and their automorphisms. In this theory, the relationship between the topology of manifolds and homotopy theory is analysed by studying certain *assembly maps* in order to compute the surgery obstructions of degree 1 normal maps $(f, b)\colon N^n \to M^n$, where the domain and range are *closed manifolds* (see [16], [17], [7]). These obstructions take values in algebraically defined *surgery obstruction groups* $L_n(\mathbb{Z}\pi, w)$, depending only on the fundamental group $\pi := \pi_1(M)$, the orientation character $w\colon \pi \to \{\pm 1\}$, and $n = \dim M \pmod 4$. The image of the $L$-theory assembly map is the subgroup of $L_n(\mathbb{Z}\pi, w)$ consisting of the closed manifold surgery obstructions (see [23, Theorem 13B.3], [6, §2]).

In this paper we will restrict to surgery problems for oriented manifolds with finite fundamental group. We note that the celebrated Farrell-Jones conjectures [2] provide the best information to date about the $L$-theory assembly maps for infinite fundamental groups, but give no information for finite groups (see [10] for a recent survey on progress towards the Farrell-Jones conjectures, and [27] for a striking example).

The surgery obstructions up to simple homotopy equivalence for closed manifolds with finite fundamental group $\pi$ are determined by transfer to the 2-Sylow subgroup (see [4]). The cohomological formula of Wall [26] and Taylor-Williams [21] reduces the computation of the surgery obstruction map

$$[X, G/TOP] \to L^s_n(\mathbb{Z}\pi)$$

to some characteristic class information and the determination of two families of universal homomorphisms:

$$\mathscr{I}^s_j(\pi)\colon H_j(\pi; \mathbb{Z}_{(2)}) \to L^s_j(\mathbb{Z}\pi)_{(2)}$$

and

$$\kappa^s_j(\pi)\colon H_j(\pi; \mathbb{Z}/2) \to L^s_{j+2}(\mathbb{Z}\pi)_{(2)}$$

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defined for all $j \geq 0$. Note that the calculation of these homomorphisms is equivalent to determining the $L$-theory assembly map

$$A_\pi \colon B\pi^+ \wedge \mathbb{L}_0(\mathbb{Z}) \to \mathbb{L}_0^s(\mathbb{Z}\pi)$$

for oriented manifolds with finite fundamental group (see [4, §1]).

We recall that for oriented manifolds $\mathscr{I}_0^s$ may be identified with the ordinary signature, and $\kappa_0^s$ with the ordinary Arf-Kervaire invariant (defined by projection to $\pi = 1$). From now on, we assume that all $L$-groups are equipped with the standard oriented involution ($g \mapsto g^{-1}$) and localized at 2. By [24, Theorem 7.4] this is no loss of information, since the torsion in $L$-theory is all 2-primary for $\pi$ a finite group.

More generally, let $U \subseteq Wh(\mathbb{Z}\pi)$ be an involution invariant subgroup. Then one can consider the obstructions to surgery up to homotopy equivalence with torsions lying in $U$. The most important examples are $U = \{0\}$ (simple), $U = SK_1(\mathbb{Z}\pi)$ (weakly simple), and $U = Wh(\mathbb{Z}\pi)$, with corresponding obstruction groups $L^s$, $L'$, and $L^h$ respectively.

The closed manifold surgery obstructions in $L_*^U(\mathbb{Z}\pi)$, for surgery up to *weakly simple* homotopy equivalence, were substantially determined in [4, Theorem A], for any Whitehead torsion decoration

$$SK_1(\mathbb{Z}\pi) \subseteq U \subseteq \mathrm{Wh}(\mathbb{Z}\pi).$$

More precisely, all the weakly simple universal homomorphisms were computed except for $\kappa_4^U$. In particular, $\mathscr{I}_j^U = 0$, for $j > 0$ (see [4, Theorem 2.1] for a slightly sharper result).

For surgery up to simple homotopy equivalence, we do not yet have a general computation of the universal homomorphisms. Taylor and Williams [22, Theorem 4.1] generalized work of Stein [20] on product formulas for surgery obstructions to study the universal homomorphisms for finite fundamental groups with *special* 2-Sylow subgroups (cyclic, dihedral, semi-dihedral and quaternion), and obtained most of the information for abelian 2-Sylow subgroups.

We address some of the open questions concerning the universal homomorphisms. Our main results are:

(i) A complete determination of the surgery assembly map in terms of group homology, for surgery up to homotopy equivalence (Theorem E).
(ii) A complete description of simple closed manifold surgery obstructions for abelian or basic 2-groups (Theorem A).
(iii) A description of the image of simple closed manifold surgery obstructions under the change of coefficients map $L_*^s(\mathbb{Z}\pi) \to L_*^s(\widehat{\mathbb{Z}}_2\pi)$ (Theorem B and Theorem C).
(iv) Examples showing that $\ker \kappa_2^h \neq \ker \kappa_2^s$ (Corollary D).
(v) An example showing that the homomorphism $\kappa_4^h$ can be non-zero (see Corollary 5.8).

**Remark 1.1.** These results are obtained by further exploiting a key ingredient in our joint paper [4], namely a lifting of the $\{\kappa_*^s\}$ through the $L$-theory of a quadratic extension ring $R = \mathbb{Z}[\varepsilon]$, with $\varepsilon = (1+\sqrt{5})/2$. There is a natural factorization

$$\begin{tikzcd} H_j(\pi;\mathbb{Z}/2) \arrow[rr, "\kappa_j^s"] \arrow[dr, "\tilde\kappa_j^s"'] & & L^s_{j+2}(\mathbb{Z}\pi)_{(2)} \\ & L^s_j(R\pi) \arrow[ur, "trf"'] & \end{tikzcd}$$

where *trf* is the induced transfer map on $L$-theory (see [4, Theorem 1.16]).

Two important examples, due to Morgan-Pardon (unpublished, [11, Theorem 4.6]) and Cappell-Shaneson [1], showed the non-triviality of certain product formulas

$$(f\times 1, b\times 1)\colon K^{4n+2}\times M^k \to S^{4n+2}\times M^k$$

for the surgery obstructions in $L^h_{k+2}(\mathbb{Z}\pi)$. In these products, $(f,b)\colon K^{4n+2}\to S^{4n+2}$ denotes a degree 1 normal map with domain the closed topological Kervaire manifold with Arf-Kervaire invariant 1, and $M^k$ is a closed, oriented manifold with $\pi_1(M,x_0)=\pi$. We recall that these obstructions vanish if $k\equiv 0 \pmod 4$ and the Euler characteristic of $M$ is even (see [4, Theorem (0.2)(i)]).

More generally, one can also consider the oriented surgery obstructions of twisted products of the Arf invariant normal map $(f,b)\colon K^{4n+2}\to S^{4n+2}$ with non-orientable manifolds (see [6, §4]).

The surgery obstruction up to homotopy equivalence of such a (twisted) product normal map, modulo the image of the ordinary Arf-Kervaire invariant, is sometimes called a *codimension $k$ Arf invariant*. A well-known conjecture from the 1980's stated:

**Oozing Conjecture.** The *codimension* $k$ Arf invariants are trivial for $k\geq 4$.

The Morgan-Pardon example shows that $\kappa_2^h\neq 0$, and the Cappell-Shaneson example shows that $\kappa_3^h\neq 0$. In general, the oozing conjecture is equivalent (by the cohomological formula for surgery obstructions) to the vanishing of all the homomorphisms $\kappa_j^h$, for $j\geq 4$ (see [4, Theorem A]). However, our Theorem E below and the example in Corollary 5.8 shows that the oozing conjecture in this form is false.

There is a sharper version of this conjecture (also disproved by Corollary 5.8), concerning the oriented surgery obstructions up to *simple* homotopy equivalence of twisted products with the Arf invariant normal map $(f,b)\colon K^{4n+2}\to S^{4n+2}$. The vanishing of these obstructions in $L^s_{k+2}(\mathbb{Z}\pi)$ in codimension $k\geq 4$ is a stronger statement, equivalent to the vanishing of the homomorphisms $\kappa_j^s$, for $j\geq 4$. Here is a natural question.

**Question.** Does there exist a finite 2-group $\pi$ such that $\kappa_4^s(\pi)\neq 0$, but $\kappa_4^h(\pi)=0$ ?

In Section 6b we present an approach to answer this question positively through further GAP computations, and provide an example in Proposition 6.9.

1a. **Results for basic or abelian 2-groups.** Here are the results for finite abelian 2-groups and the *basic* 2-groups (cyclic, quaternion, dihedral, and semi-dihedral). These cases are important as building blocks for determining the surgery obstructions over more general finite 2-groups (compare [4, Theorem 0.3]).

**Theorem A** ([22], [4])**.** *Suppose that $\pi$ is an abelian or basic finite 2-group. Then*

*(i)* $\mathscr{I}^s_j = 0$, *for* $j > 0$;
*(ii)* $\kappa^s_0$ *and* $\kappa^s_1$ *are (split) injective;*
*(iii)* $\kappa^s_2 = 0$ *for the basic* 2*-groups;*
*(iv)* $\kappa^s_j = 0$, *for* $j > 2$, *unless* $\pi$ *is quaternion;*
*(v) If* $\pi$ *is quaternion,* $\kappa^s_3$ *is injective and* $\kappa^s_j = 0$, *for* $j \geq 4$.

*Moreover, if* $E \subseteq \pi$ *denotes the subgroup of an abelian* 2*-group* $\pi$ *consisting of elements of order* $\leq 2$*, then the sequence*

$$H_2(E; Z/2) \to H_2(\pi; \mathbb{Z}/2) \xrightarrow{\kappa^s_2} L^s_0(\mathbb{Z}\pi)$$

*is exact. In addition,* $\ker \kappa^s_2(\pi) = \ker \kappa^h_2(\pi)$ *and their images are isomorphic.*

**Notation**: we write $H^n(A) := \widehat{H}^n(\mathbb{Z}/2; A)$ for Tate cohomology with coefficients in a $\mathbb{Z}/2$-module $A$.

**Remark 1.2.** These results refine those proved in Taylor-Williams [22]: we will show how they can be derived from the information in [4]. The difference between $\kappa^s_*$ and $\kappa^h_*$ is partly measured by the intermediate comparison sequence

$$\cdots \to H^{n+1}(SK_1(\mathbb{Z}\pi)) \xrightarrow{\partial} L^s_n(\mathbb{Z}\pi) \to L'_n(\mathbb{Z}\pi) \to H^n(SK_1(\mathbb{Z}\pi)) \to \dots$$

where $L'_*(\mathbb{Z}\pi)$ denotes the *weakly simple* $L$-theory with torsion decorations in $\mathrm{Wh}'(\mathbb{Z}\pi) := \mathrm{Wh}(\mathbb{Z}\pi)/SK_1(\mathbb{Z}\pi)$. The passage from $L'_*$ to $L^h_*$ is given by the exact sequence

$$0 \to L'_{2k}(\mathbb{Z}\pi) \to L^h_{2k}(\mathbb{Z}\pi) \to \mathrm{Wh}'(\mathbb{Z}\pi) \otimes \mathbb{Z}/2 \to L'_{2k-1}(\mathbb{Z}\pi) \to L^h_{2k-1}(\mathbb{Z}\pi) \to 0$$

due to Wall [25, p. 78]. In particular, this shows that in even dimensions $\kappa^h_{2j}$ determines $\kappa'_{2j}$ for any finite group (meaning that $\ker \kappa'_{2j} = \ker \kappa^h_{2j}$ and their images are isomorphic).

The last part of Theorem A implies that a stronger statement holds for $\pi$ a finite abelian 2-group, namely that the image of $\kappa^s_2(\pi)$ has zero intersection with the image of the natural map $\partial\colon H^1(SK_1(\mathbb{Z}\pi)) \to L^s_0(\mathbb{Z}\pi)$.

1b. **Results for the 2-adic homomorphisms.** Next we concentrate on the compositions:

$$\bar{\kappa}^s_j\colon H_j(\pi; \mathbb{Z}/2) \to L^s_{j+2}(\mathbb{Z}\pi) \xrightarrow{\rho_2} L^s_{j+2}(\widehat{\mathbb{Z}}_2\pi)$$

of the universal homomorphisms under the map induced by $\rho\colon \mathbb{Z} \to \widehat{\mathbb{Z}}_2$ into the 2-adic $L$-groups. Recall that calculations of the surgery obstructions are based on separating the 2-adic information from the integral information via an exact sequence of the form

$$\cdots \to L_{j+1}(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi) \to L_j(\mathbb{Z}\pi) \to L_j(\widehat{\mathbb{Z}}_2\pi) \to \dots$$

with appropriate torsion decorations (see Wall [25, 1.2]). The relative $L$-groups have better induction and detection properties (see [25, 2.1], [5]), which we can exploit to obtain information about the integral $\kappa_*$-homomorphisms from their 2-adic images (see Lemma 3.4). In the next result let $s_r\colon H_{2^{r+2}}(\pi; \mathbb{Z}/2) \to H_4(\pi; \mathbb{Z}/2)$, for $r > 0$, denote the Hom-dual of the iterated squaring maps in cohomology.

**Theorem B.** *For any finite* 2*-group* $\pi$*, the* 2*-adic universal homomorphisms*

$$\bar{\kappa}^s_j\colon H_j(\pi; \mathbb{Z}/2) \to L^s_{j+2}(\widehat{\mathbb{Z}}_2\pi)$$

*for* $j \neq 2$ *are determined as follows:*

*(i)* $\bar{\kappa}^s_0$ *and* $\bar{\kappa}^s_1$ *are split injective.*
*(ii) If* $j > 0$*, then* $\bar{\kappa}^s_j = 0$ *provided* $j \neq 2^l$*, for* $l \geq 1$*.*
*(iii)* $\bar{\kappa}^s_{2r+2} = \bar{\kappa}^s_4 \circ s_r$ *for* $r \geq 0$*, and* $\bar{\kappa}^s_i$ *vanishes on the image of integral homology for* $i \geq 4$*.*

*Moreover, the image of* $\kappa'_j\colon H_j(\pi;\mathbb{Z}/2) \to L'_{j+2}(\mathbb{Z}\pi)$*, for* $j \equiv 0 \pmod 2$*, is detected by the natural map* $L'_{j+2}(\mathbb{Z}\pi) \to L'_{j+2}(\widehat{\mathbb{Z}}_2\pi) \oplus L'_{j+2}(\mathbb{Z}\pi^{\mathrm{ab}})$*.*

**Remark 1.3.** The homomorphism $\bar{\kappa}'_2\colon H_2(\pi;\mathbb{Z}/2) \to L'_0(\widehat{\mathbb{Z}}_2\pi)$ is computed in Proposition 3.5. In particular, $\kappa'_2(\pi)$ is non-zero if and only if $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \neq 0$. In Appendix A we list the 2-groups of order $\leq 128$ for which $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \neq 0$ (a necessary condition is that $\pi$ is non-abelian). See Theorem C for more information about $\bar{\kappa}^s_2$.

1c. **Results for surgery up to simple homotopy equivalence.** We now settle some open questions from [4]. Recall that there is a boundary map $\partial\colon H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L^s_0(\widehat{\mathbb{Z}}_2\pi)$ in the Ranicki-Rothenberg sequence comparing $L^s_*(\widehat{\mathbb{Z}}_2\pi))$ to $L'_*(\widehat{\mathbb{Z}}_2\pi)$ under the change of Whitehead torsion decorations.

**Theorem C.** *Let* $\pi$ *be a finite 2-group. Suppose that the following conditions hold:*
*(i) the boundary map* $\partial\colon H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L^s_0(\widehat{\mathbb{Z}}_2\pi)$ *is non-zero.*
*(ii)* $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$*.*

*Then* $\kappa^s_2(\pi)$ *is non-zero and* $\bar{\kappa}'_2(\pi) = 0$*. If* $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L^s_0(\widehat{\mathbb{Z}}_2\pi)$ *is injective, then* $\mathrm{Im}\{\bar{\kappa}^s_2(\pi)\colon H_2(\pi;\mathbb{Z}/2) \to L^s_0(\widehat{\mathbb{Z}}_2\pi)\} \cong H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$*.*

**Corollary D.** *There exist finite* 2*-groups* $\pi$ *such that* $\ker\kappa^s_2(\pi) \subsetneq \ker\kappa^h_2(\pi)$*.*

**Remark 1.4.** This answers a question of Kasprowski, Nicholson and Veselá [9, 1.5], where it is shown in [9, Theorem C] that the existence of such examples implies the existence of closed smooth 4-manifolds which are homotopy equivalent but not simple homotopy equivalent (even up to stabilisations).

In Section 4 we give a number of examples of groups $\pi$ for which the conditions in Theorem C are satisfied (see Examples 4.12). Among these groups, we have examples for Corollary D in which $\ker\kappa^s_2(\pi) \subsetneq \ker\kappa^h_2(\pi)$, contrasting with the Morgan-Pardon example: $\pi = \mathbb{Z}/2 \times \mathbb{Z}/4$, where $\kappa^h_2(\pi) \neq 0$. In Section 4b we give details for the group $\pi = SG[128, 1377]$ in the Small Groups Library, where the notation $SG[i,j]$ means the $j^{th}$ group of order $i$ (see the list in Examples 4.13).

1d. **Results on the Oozing Conjecture.** Finally, we reduce the original "oozing conjecture" for surgery up to homotopy equivalence to an explicit calculation in group homology, and present a counter-example.

In the following statement, let $s_*\colon H_4(\pi;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2)$ denote the dual homomorphism to $Sq^2\colon H^2(\pi;\mathbb{Z}/2) \to H^4(\pi;\mathbb{Z}/2)$, and $\beta\colon H_2(\pi;\mathbb{Z}/2) \to H^0(\pi^{\mathrm{ab}})$ the map induced by the Bockstein. In addition, the short exact sequence

$$0 \to \mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi) \to \overline{I(\widehat{\mathbb{Z}}_2\pi)} \to \pi^{\mathrm{ab}} \to 0$$

induces a (surjective) coboundary map $\delta\colon H^0(\pi^{\mathrm{ab}}) \to H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi))$. For the notation see [14, p. 163].

**Theorem E.** *Let $\pi$ be a finite 2-group, and let*

$$\lambda_4(\pi) := \delta \circ \beta \circ s_* \colon H_4(\pi; \mathbb{Z}/2) \to H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)).$$

*Then the homomorphism $\kappa'_4(\pi) \neq 0$ if and only if $\lambda_4(\pi) \neq 0$. Moreover, $\kappa'_4(\pi)$ is zero on the image of the integral homology $H_4(\pi;\mathbb{Z}) \to H_4(\pi;\mathbb{Z}/2)$.*

An example showing that there exists a finite 2-group of order 16384 with $\kappa'_4(\pi) \neq 0$ (and hence $\kappa^s_4(\pi) \neq 0$) is presented in Section 5c, Corollary 5.8. Computer investigations suggest that the smallest such examples would have orders $\geq 512$.

**Corollary 1.5.** *The oozing conjecture is false for surgery up to simple homotopy equivalence. There exist non-trivial codimension $k$ Arf invariant problems in codimensions $k = 2^l$, for $l \geq 2$.*

**Remark 1.6.** In Section 5, we derive the explicit formula in Theorem E via the commutative diagram (see Proposition 3.10):

$$\begin{tikzcd} H_4(\pi;\mathbb{Z}/2) \arrow[r, "\lambda_4(\pi)"] \arrow[d, "\kappa'_4(\pi)"'] & H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \arrow[d, "\partial"] \\ L'_2(\mathbb{Z}\pi) \arrow[r, "\rho_2"] & L'_2(\widehat{\mathbb{Z}}_2\pi) \end{tikzcd}$$

in which the boundary map $\partial$ is injective. This reduces the original oozing conjecture that $\kappa^h_j = 0$, for $j \geq 4$, to an explicit group homology calculation, namely computing the map $\lambda_4(\pi)$. Recall that there is an injection $L'_2(\mathbb{Z}\pi) \to L^h_2(\mathbb{Z}\pi)$ and hence $\kappa^h_4 \neq 0$ if and only if $\kappa'_4 \neq 0$, so it is enough to determine $\kappa'_4(\pi)$.

## Contents



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[†]August 17, 1945 - January 11, 2018

## 2. Abelian or basic 2-groups

We begin by summarizing some results about the $K$-theory of group rings (a convenient reference and guide to the literature is [14]). Recall that there is a short exact sequence

$$0 \to SK_1(\mathbb{Z}\pi) \to \mathrm{Wh}(\mathbb{Z}\pi) \to \mathrm{Wh}'(\mathbb{Z}\pi) \to 0$$

where $SK_1(\mathbb{Z}\pi) = \ker\{K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Q}\pi)\}$, and

$$\mathrm{Wh}(\mathbb{Z}\pi) = K_1(\mathbb{Z}\pi)/SK_1(\mathbb{Z}\pi) \oplus \{\pm 1\} \oplus \pi^{\mathrm{ab}}.$$

The quotient $\mathrm{Wh}'(\mathbb{Z}\pi)$ is finitely generated and torsion free, of $\mathbb{Z}$-rank equal to the difference between the number of irreducible real and rational representations of $\pi$. In addition, we have a short exact sequence

$$0 \to Cl_1(\mathbb{Z}\pi) \to SK_1(\mathbb{Z}\pi) \to SK_1(\widehat{\mathbb{Z}}_2\pi) \to 0$$

where $SK_1(\widehat{\mathbb{Z}}_2\pi) = \ker\{K_1(\widehat{\mathbb{Z}}_2\pi) \to K_1(\hat{\mathbb{Q}}_2\pi)\}$.

The Tate cohomology of these various $K$-groups appear in Rothenberg sequences relating the variant forms of $L$-theory, namely $L^s$, $L'$ and $L^h$. Here are some of the facts we need (for $\pi$ of 2-power order):

- [14, 14.2; 14.4 ] $SK_1(\mathbb{Z}\pi) = 0$ if $\pi$ is basic or elementary abelian.
- [14, 8.6] There is a natural isomorphism $\Theta_\pi \colon SK_1(\widehat{\mathbb{Z}}_2\pi) \cong H_2(\pi;\mathbb{Z})/H_2^{\mathrm{ab}}(\pi)$. The involution $g \mapsto g^{-1}$ induces multiplication by $-1$ on $SK_1(\widehat{\mathbb{Z}}_2\pi)$. If $\pi$ is abelian, $SK_1(\mathbb{Z}\pi) = Cl_1(\mathbb{Z}\pi)$.
- [13, Prop. 13] $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$ if $\pi$ is basic or abelian.
- [14, 8.7] The transfer $trf \colon SK_1(\widehat{R}_2\pi) \to SK_1(\widehat{\mathbb{Z}}_2\pi)$ is an isomorphism.

*The proof of Theorem A.* Since $SK_1(\mathbb{Z}\pi) = 0$ for any basic or abelian 2-group, we have $L^s_*(\mathbb{Z}\pi) = L'_*(\mathbb{Z}\pi)$ and $\kappa^s_*(\pi) = \kappa'_*(\pi)$. Then part (i) follows immediately from [4, Theorem 2.1]: this is essentially the argument in [22, Theorems 1.2 & 4.1].

Part (ii) follows from naturality and [4, Theorem 0.3(ii)], and the vanishing results of parts (iv) and (v) follow from [4, Theorem 6.8] by applying the $L$-theory transfer. For $\pi$ quaternion, the Cappell-Shaneson example shows that $\kappa^s_3(\pi)$ is injective.

Next we consider part (iii). From the calculations of Wall [25, §5.2], $L'_0(\mathbb{Z}\pi)$ is torsion free if $\pi$ is a basic 2-group, since $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$. This gives $\kappa^s_2(\pi) = \kappa'_2(\pi) = 0$.

It remains to consider $\kappa^s_2(\pi) = \kappa'_2(\pi)$ for $\pi$ abelian. We can write $\pi = C_1 \times C_2 \times \cdots \times C_r$, where $C_i \cong \mathbb{Z}/2^{k_i}$. Then there is a surjection $\bigoplus H_1(C_i;\mathbb{Z}/2) \to H_1(\pi;\mathbb{Z}/2)$, and

$$H_2(\pi;\mathbb{Z}/2) \cong \bigoplus_{i=1}^{r} H_2(C_i;\mathbb{Z}/2) \oplus \bigoplus_{i<j} H_1(C_i;\mathbb{Z}/2) \otimes H_1(C_j;\mathbb{Z}/2). \tag{2.1}$$

In general, if $\pi$ is an abelian 2-group of rank $r$ with $s$ summands of order two, then

$$L'_0(\mathbb{Z}\pi) = \Sigma \oplus \left(2^r - r - 1 - \binom{s}{2}\right) \cdot \mathbb{Z}/2 \tag{2.2}$$

by [25, Theorem 3.3.2]. In particular, $L'_0(\mathbb{Z}\pi)$ is torsion free for $\pi$ cyclic or $\pi = C_2 \times C_2$.

Suppose first that $\pi$ is elementary abelian of rank $\geq 2$, and let $\gamma = C_2 \times C_2$ be a subgroup of $\pi$, with $i_*\colon \gamma \to \pi$ the inclusion map. Then

$$\kappa'_2(i_*(x)) = i_*(\kappa'_2(x)) \in \operatorname{Im}\{L'_0(\mathbb{Z}\gamma) \xrightarrow{i_*} L'_0(\mathbb{Z}\pi)\}.$$

But $\kappa'_2(\gamma) = 0$ since $L'_0(\mathbb{Z}\gamma)$ is torsion free, hence $\kappa'_2(i_*(x)) = 0$. Since every class in $H_2(\pi;\mathbb{Z}/2)$ is a sum of classes induced from subgroups of rank $\leq 2$, and $\kappa^s_2 = 0$ for cyclic groups, it follows that $\kappa^s_2(\pi) = 0$ for elementary abelian 2-groups.

For $\pi$ any abelian group, we can again restrict to elements $x \in H_2(\pi;\mathbb{Z}/2)$ with $\kappa'_2(x) = 0$ which are a sum of classes induced from rank 2 subgroups. If $\pi$ has rank $r$ and $s$ summands of order two, we can write $\pi = \pi_1 \times \pi_2$, where $\pi_1 = C_\times \cdots \times C_s$ is elementary abelian, and $\pi_2 = C'_1 \times \dots C'_{r-s}$ consists of the factors with order $\geq 4$. We denote by $\{\alpha_i\}$ the generators of $H_1(C_i;\mathbb{Z}/2)$, and by $\{\beta_j\}$ those of $H_1(C'_j;\mathbb{Z}/2)$. By formula (2.1) we may assume that

$$x = \sum a_{ij}(\alpha_i \otimes \beta_j) + \sum b_{kl}(\beta_k \otimes \beta_l) \in H_2(\pi;\mathbb{Z}/2)$$

modulo the image of $H_2(E;\mathbb{Z}/2)$. Note that these classes are the reductions of integral homology classes in $H_2(\pi;\mathbb{Z})$. For each subgroup $\gamma = C_i \times C'_j$ or $\gamma = C'_k \times C'_l$, we have $H_2(\gamma;\mathbb{Z}) = \mathbb{Z}/2$, and the non-triviality of the Morgan-Pardon example in $L'_0(\mathbb{Z}/2 \times \mathbb{Z}/4) \cong \mathbb{Z}/2$ shows by projection that $\kappa'_2(\gamma)$ is an isomorphism for all such groups.

Since there exist independent projections $\{p_\gamma\colon \pi \to \mathbb{Z}/2 \times \mathbb{Z}/4\}$, such that $p_\gamma(\gamma') = 1$ for $\gamma' \neq \gamma$, it follows by naturality that $x \equiv 0 \pmod{H_2(E;\mathbb{Z}/2)}$ and exactness is proved.

To establish the last part of Theorem A, consider the diagram

$$\begin{tikzcd}
H_2(E;Z/2) \arrow[r, "\kappa^s_2"] \arrow[d, "i_*"'] & L^s_0(ZE) \arrow[r, "\cong"] \arrow[d] & L'_0(ZE) \arrow[d, "i_*"] \\
H_2(\pi;Z/2) \arrow[r, "\kappa^s_2"] \arrow[rr, bend right, "\kappa'_2"'] & L^s_0(\mathbb{Z}\pi) \arrow[r] & L'_0(\mathbb{Z}\pi)
\end{tikzcd}$$

If $\kappa^s_2(x) \neq 0$ for some $x \in H_2(\pi;\mathbb{Z}/2)$, but $\kappa'_2(x) = 0$, then there exists $y \in H_2(E;\mathbb{Z}/2)$ such that $i_*(y) = x \in H_2(\pi;\mathbb{Z}/2)$. But then $\kappa^s_2(x) = \kappa^s_2(i_*(y)) = i_*(\kappa^s_2(y)) = 0$, since $\kappa^s_2 = 0$ for elementary abelian 2-groups. If $\kappa'_2(x) \neq 0$, then the exact sequence

$$\cdots \to H^1(SK_1(\mathbb{Z}\pi)) \to L^s_0(\mathbb{Z}\pi) \to L'_0(\mathbb{Z}\pi) \to \dots$$

shows that the image of $\kappa^s_2(\pi)$ has zero intersection with the image of $H^1(SK_1(\mathbb{Z}\pi))$. This completes the proof. □

## 3. The 2-adic universal homomorphisms

Information about the 2-adic $\kappa$-homomorphisms is provided by [4, §7] and Milgram-Oliver [12]. We will recall these results below.

3a. **The 2-adic $\kappa_2^s$ homomorphisms.**

**Theorem 3.1** (Milgram-Oliver [12]). *Let $\pi$ be a finite 2-group. There is a commutative diagram:*

$$\begin{tikzcd} H_2(\pi;\mathbb{Z})] \arrow[r, "\theta"] \arrow[d, "\kappa_2^s"'] & H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \arrow[d, "\partial"] \\ L_0^s(\mathbb{Z}\pi) \arrow[r, "\rho_2"] & L_0^s(\widehat{\mathbb{Z}}_2\pi) \end{tikzcd}$$

*where $\rho_2$ is the natural map to the 2-adic L-group.*

**Corollary 3.2.** *If $\pi$ is abelian, then $\bar\kappa_2^s(\pi)\colon H_2(\pi;\mathbb{Z}/2) \to L_0^s(\widehat{\mathbb{Z}}_2\pi)$ is zero.*

Oliver showed in [13, Theorem 3] that there is a natural isomorphism

$$\theta\colon H_2(\pi;\mathbb{Z})/H_2^{\mathrm{ab}}(\pi) \xrightarrow{\cong} SK_1(\widehat{\mathbb{Z}}_2\pi)$$

where $H_2^{\mathrm{ab}}(\pi) \subset H_2(\pi;\mathbb{Z})$ denotes the image under induction from abelian subgroups of $\pi$. Furthermore, Oliver proved that the standard involution induced by $g \mapsto g^{-1}$ is multiplication by $-1$ on $SK_1(\widehat{\mathbb{Z}}_2\pi)$ (see [14, Theorem 8.6]), hence $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \cong SK_1(\widehat{\mathbb{Z}}_2\pi)\otimes\mathbb{Z}/2$.

**Lemma 3.3.** *$L_0'(\widehat{\mathbb{Z}}_2\pi) \cong H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi))$ and $L_2'(\widehat{\mathbb{Z}}_2\pi) \cong \mathbb{Z}/2 \oplus H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi))$.*

*Proof.* We have the Ranicki-Rothenberg sequence

$$\cdots \to L_{n+1}^h(\widehat{\mathbb{Z}}_2\pi) \to H^n(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \to L_n'(\widehat{\mathbb{Z}}_2\pi) \to L_n^h(\widehat{\mathbb{Z}}_2\pi) \to \dots$$

and the calculations $L_n^h(\widehat{\mathbb{Z}}_2\pi) = \mathbb{Z}/2$, for $n$ even, detected by the discriminant or the Arf invariant, and $L_n^h(\widehat{\mathbb{Z}}_2\pi) = 0$ for $n$ odd (see [25, §1.2] and [8, §2, Theorem 3.1] for details, background on the "round" $L$-groups, and their relation to the surgery obstruction groups). Hence we obtain an isomorphism $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \to L_0'(\widehat{\mathbb{Z}}_2\pi)$ and a split injection $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \to L_0'(\widehat{\mathbb{Z}}_2\pi)$, with cokernel $\mathbb{Z}/2$ detected by the Arf invariant.

Similarly, we have an exact sequence

$$0 \to \mathbb{Z}/2 \to H^0(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \to L_3'(\widehat{\mathbb{Z}}_2\pi) \to 0$$

and $H^0(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \xrightarrow{\cong} L_1'(\widehat{\mathbb{Z}}_2\pi)$. □

*The proof of Theorem C.* By assumption, we have $L_0'(\widehat{\mathbb{Z}}_2\pi) \cong H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$ and the boundary map $\partial$ in the sequence

$$\cdots \to L_1'(\widehat{\mathbb{Z}}_2\pi) \to H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \xrightarrow{\partial} L_0^s(\widehat{\mathbb{Z}}_2\pi) \to L_0'(\widehat{\mathbb{Z}}_2\pi) \to \dots$$

is non-zero (and surjective). Since $\bar\kappa_2^s(\pi) = \rho_2\circ\kappa_2^s(\pi)$ and $\bar\kappa_2^h(\pi)$ factors through $\bar\kappa_2'(\pi) = 0$, the conclusions of Theorem C follow directly from Theorem 3.1. □

The next observation about the torsion in $L$-theory will be used in the proof of Theorem B, Corollary D and Theorem E. Note that the map $L_*(\mathbb{Z}\pi) \to L_*(\mathbb{R}\pi)$ has finite kernel and cokernel, and $L_*(\mathbb{R}\pi)$ is detected by the multi-signature (see [24, §7], [25, §2.2]).

**Lemma 3.4.** *For $\pi$ a finite 2-group, the torsion subgroup of $L_{2k}'(\mathbb{Z}\pi)$ is detected by the natural map $L_{2k}'(\mathbb{Z}\pi) \to L_{2k}'(\widehat{\mathbb{Z}}_2\pi) \oplus L_{2k}'(\mathbb{Z}\pi^{\mathrm{ab}})$.*

*Proof.* We first consider the case $k$ even. Suppose that $x \in \ker\{L'_0(\mathbb{Z}\pi) \to L'_0(\widehat{\mathbb{Z}}_2\pi) \oplus L'_0(\mathbb{Z}\pi^{\mathrm{ab}})$ is a torsion element. In the long exact sequence

$$\cdots \to L'_1(\widehat{\mathbb{Z}}_2\pi) \xrightarrow{\psi_1} L'_1(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi) \xrightarrow{\partial} L'_0(\mathbb{Z}\pi) \to L'_0(\widehat{\mathbb{Z}}_2\pi) \to \dots$$

a torsion element $x \in \ker\{L'_0(\mathbb{Z}\pi) \to L'_0(\widehat{\mathbb{Z}}_2\pi)\}$ is the image $\partial(y) = x$ for some torsion element $y \in L'_1(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi)$. By Wall [25, 5.2.2], the torsion subgroup of the relative $L$-group is a direct sum of copies of $\mathbb{Z}/2$, one for each type $O$ simple factor of the rational group algebra $\mathbf{Q}\pi$. We can apply the Quadratic Generation Theorem [5, Theorem 1.B.7 & Example 1.B.8 (iii)] to show that the torsion subgroup of $L'_1(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi)$ is generated by the images of torsion elements in the relative $L$-groups of basic subquotients of $\pi$.

Since the linear type $O$ representations are determined by the map $\pi \to \pi^{\mathrm{ab}}$, we only need to consider the (non-abelian) dihedral subquotients. However, the Wall groups $L'_0(\mathbb{Z}D(2^r))$, $r \geq 3$, are torsion free [25, Theorem 5.2.3]. It follows that $y \in L'_1(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi)$ lies in the image of $\psi_1$ and hence $\partial(y) = x = 0$.

For $k$ odd, a similar argument works for $L'_2(\mathbb{Z}\pi)$ since the relative group $L'_3(\mathbb{Z}\pi \to \widehat{\mathbb{Z}}_2\pi)$ is a sum of copies of $\mathbb{Z}/2$ over representations of type $Sp$, and hence is generated by quaternion subquotients. However, $L'_2(\mathbb{Z}Q(2^r)) \cong \mathbb{Z}/2$, for $r \geq 3$, is detected by the trivial group [25, Theorem 5.2.3] and the result follows. ☐

### 3b. The 2-adic $\kappa'_2$ homomorphisms.

**Proposition 3.5** ([4, 7.2])**.** *Let $\pi$ be a finite 2-group. There is a commutative diagram:*

$$\begin{tikzcd}
H_2(\pi;\mathbb{Z}/2) \arrow[r, "\beta"] \arrow[d, "\kappa'_2"'] & H^0(\pi^{\mathrm{ab}}) \arrow[r, "\delta"] & H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \arrow[d, "\partial"] \\
L'_0(\mathbb{Z}\pi) \arrow[rr, "\rho_2"] & & L'_0(\widehat{\mathbb{Z}}_2\pi)
\end{tikzcd}$$

**Remark 3.6.** We will lift this diagram to $R\pi$ and use the factorization $\kappa^s_j = trf \circ \widetilde{\kappa}^s_j$, where $\widetilde{\kappa}^s_j\colon H_j(\pi;\mathbb{Z}/2) \to L^s_j(R\pi)$ and $trf\colon L^s_j(R\pi) \to L^s_{j+2}(\mathbb{Z}\pi)$. The quadratic extension ring $R = \mathbb{Z}[\varepsilon]$, with $\varepsilon = (1+\sqrt{5})/2$ has the important properties that $\varepsilon + \bar{\varepsilon} = 1$ and $\varepsilon\bar{\varepsilon} = -1$ (see [4, Theorem 1.16]).

The following calculations will be needed (particularly for $\pi = C_2$ of order two). The details can be found in [4, 3.3, 3.11, 4.5, 4.11]:

(i) $L^{\tilde{Y}}_i(\widehat{R}_2\pi) \cong L^{\tilde{Y}}_i(\widehat{R}_2) \oplus H^0(\pi^{\mathrm{ab}})$;
(ii) $L^{\tilde{Y}}_i(\widehat{R}_2) = \mathbb{Z}/2, 0, 0, \mathbb{Z}/2$;
(iii) $L^{\tilde{Y}}_3(R\pi \to \widehat{R}_2\pi) = (\mathbb{Z}/2)^{s_2}$, where $s_2$ counts the type $O$ factors in $\mathbf{Q}\pi$;
(iv) $L^{\tilde{Y}}_1(R\pi \to \widehat{R}_2\pi) = (\mathbb{Z}/2)^{s_0}$, where $s_0$ counts the type $Sp$ factors in $\mathbf{Q}\pi$;
(v) $L^{\tilde{Y}}_i(R) = \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/2$;
(vi) $L^{\tilde{Y}}_i(R[C_2]) = (\mathbb{Z}/2)^2, (\mathbb{Z}/2)^3, \mathbb{Z}/2, 0$.

The round torsion decoration $Y = R^\times \oplus \pi^{\mathrm{ab}} \oplus SK_1(A\pi) \subseteq K_1(A\pi)$, for $A = R$ or $\widehat{R}_2$, and $\widetilde{Y}$ denotes the quotient dividing out $K_1(\mathbb{Z}) = \{\pm 1\}$. In the proof of Theorem B, we will also need the round *simple* torsion decoration $V = \widehat{R}^\times_2 \oplus \pi^{\mathrm{ab}} \subseteq K_1(\widehat{R}_2\pi)$ and the surgery groups $L^s_*(\widehat{R}_2\pi) := L^{\tilde{V}}_*(\widehat{R}_2\pi)$.

*The proof of Proposition 3.5.* The argument will be divided into two parts. First we claim that the following diagram

$$\begin{tikzcd} H_2(\pi;\mathbb{Z}/2) \arrow[r, "\beta"] \arrow[d, "\tilde\kappa_2'"'] & H^0(\pi^{\mathrm{ab}}) \\ L_2^{\tilde Y}(R\pi) \arrow[r, "\rho_2"] & L_2^{\tilde Y}(\widehat{R}_2\pi) \arrow[u, "\tau"'] \end{tikzcd} \tag{3.7}$$

commutes, where $\beta\colon H_2(\pi;\mathbb{Z}/2) \to H^0(\pi^{\mathrm{ab}}) = \{[g] \in \pi^{\mathrm{ab}} : [g^2] = 1\}$ is induced by the Bockstein and $\tau$ by the discriminant. By the 2-adic Detection Theorem [4, Theorem 3.4] and naturality, it is enough to show that the diagram commutes for $\pi = C_2$ of order two. In that case, $L_2^{\tilde Y}(R\pi) \cong \mathbb{Z}/2$ maps isomorphically onto $L_2^{\tilde Y}(\widehat{R}_2\pi) = \mathbb{Z}/2$, since $\tilde\psi_3\colon L_3^{\tilde Y}(\widehat{R}_2\pi) \to L_3^{\tilde Y}(R\pi \to \widehat{R}_2\pi)$ is an isomorphism (see [4, Theorem 4.10]) and $\pi = C_2$ has two type $O$ representations. The map $\beta$ is an isomorphism, and the discriminant $\tau\colon L_2^{\tilde Y}(\widehat{R}_2\pi) \to H^0(\pi^{\mathrm{ab}})$ is also an isomorphism. Therefore the diagram commutes for all finite 2-groups.

Next we consider the following diagram (for any finite 2-group):

$$\begin{tikzcd} H^0(\pi^{\mathrm{ab}}) \arrow[drr, bend left, "\tilde\delta", "\cong"'] & & \\ L_2^{\tilde Y}(\widehat{R}_2\pi) \arrow[u, "\tau", "\cong"'] \arrow[d, "trf"] & & H^1(\mathrm{Wh}^{\tilde Y}(\widehat{R}_2\pi)) \arrow[ll, "\cong", "\partial"'] \arrow[d, "trf"] \\ L_0'(\widehat{\mathbb{Z}}_2\pi) & & H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \arrow[ll, "\partial"] \end{tikzcd} \tag{3.8}$$

where the map $\tilde\delta$ is the coboundary induced in Tate cohomology by the upper sequence in the diagram (see [13, Theorem 2]):

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Wh}^{\tilde Y}(\widehat{R}_2\pi) \arrow[r] \arrow[d] & \overline{I(\widehat{R}_2\pi)} \arrow[r] \arrow[d] & \pi^{\mathrm{ab}} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi) \arrow[r] & \overline{I(\widehat{\mathbb{Z}}_2\pi)} \arrow[r] & \pi^{\mathrm{ab}} \arrow[r] & 0 \end{tikzcd} \tag{3.9}$$

Since $\varepsilon + \bar\varepsilon = 1$, we have $H^*(\overline{I(\widehat{R}_2\pi)}) = 0$ and hence $\tilde\delta$ is an isomorphism. By naturality, $\delta = trf \circ \tilde\delta\colon H^0(\pi^{\mathrm{ab}}) \to H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi))$. By combining diagrams (3.7) amd (3.8) above, the proof is complete. □

*The proof of Theorem B.* The parts (ii) and (iii) follow directly from the properties of the lifted $\tilde\kappa_*^s$ listed in [4, Theorem 6.8], since $L_*^{\tilde Y}(\widehat{R}_2\pi)$ is detected by $\pi \to \pi^{\mathrm{ab}}$ (by [4, Theorem 3.4]) and $L_*^s(\widehat{R}_2\pi) = L_*^{\tilde Y}(\widehat{R}_2\pi)$ for $\pi$ abelian (since the transfer $trf\colon SK_1(\widehat{R}_2\pi) \to SK_1(\widehat{\mathbb{Z}}_2\pi)$ is an isomorphism by [14, 8.7] ). For part (i) note that $\kappa_1^s$ is generated/detected by cyclic groups, and [25, 3.3] shows that for cyclic 2-groups $L_3'(\mathbb{Z}\pi) \cong L_3'(\widehat{\mathbb{Z}}_2\pi)$. The last statement follows from Lemma 3.4. Some examples of groups $\pi$ satisfying all the required conditions are given in Corollary 4.9. $\square$

*The proof of Corollary D.* The first step is to find a group $\pi$ satisfying the conditions listed in Theorem C. This will show that $\kappa_2^s(\pi) \neq 0$ and that $\bar{\kappa}_2^h(\pi) = 0$. By Lemma 3.4, the homomorphism $\kappa_2'(\pi)$ is detected by the map $L_0'(\mathbb{Z}\pi) \to L_0'(\mathbb{Z}\pi^{\mathrm{ab}})$. However, by naturality the image of $\kappa_2'(\pi)$ in $L_0'(\mathbb{Z}\pi^{\mathrm{ab}})$ equals the image of $\kappa_2'(\pi^{\mathrm{ab}})$. By Theorem A, we can conclude that $\kappa_2'(\pi^{\mathrm{ab}}) = \kappa_2^h(\pi^{\mathrm{ab}}) = 0$ whenever $\pi^{\mathrm{ab}}$ is an elementary abelian group, and the result follows. See Examples 4.13 for a list of groups of order 128 satisfying Corollary D.

$\square$

3c. **The $\kappa_4'$ homomorphisms.** In the following diagram, $s_*\colon H_4(\pi;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2)$ denotes the Kronecker dual of the map $Sq^2$ in mod 2 cohomology.

**Proposition 3.10** ([4, 7.3])**.** *Let $\pi$ be a finite 2-group. There is a commutative diagram:*

(3.11)
$$\begin{tikzcd}
H_4(\pi;\mathbb{Z}/2) \arrow[r, "s_*"] \arrow[dr, "\kappa_4'"'] & H_2(\pi;\mathbb{Z}/2) \arrow[r, "\beta"] & H^0(\pi^{\mathrm{ab}}) \arrow[r, "\delta"] & H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi) \arrow[dl, "\partial"] \\
& L_2'(\mathbb{Z}\pi) \arrow[r, "\rho_2"] & L_2'(\widehat{\mathbb{Z}}_2\pi) &
\end{tikzcd}$$

*Proof.* The argument is similar to that for the $\kappa_2'$ diagram. We first lift to a diagram over $R$, where it is enough to check commutativity for $\pi = C_2$.

(3.12)
$$\begin{tikzcd}
H_4(\pi;\mathbb{Z}/2) \arrow[r, "s_*"] \arrow[dr, "\kappa_4'"'] & H_2(\pi;\mathbb{Z}/2) \arrow[r, "\beta"] & H^0(\pi^{\mathrm{ab}}) \\
& L_0^{\tilde{Y}}(R\pi) \arrow[r, "\rho_2"] & L_0^{\tilde{Y}}(\widehat{R}_2\pi) \arrow[u, "\tau"']
\end{tikzcd}$$

In this diagram for $\pi = C_2$, the map $L_0^{\tilde{Y}}(R\pi) \cong (\mathbb{Z}/2)^2$ to $L_0^{\tilde{Y}}(\widehat{R}_2\pi) = (\mathbb{Z}/2)^2$ is an isomorphism (since $\mathbf{Q}\pi$ has no type $Sp$ factors), with one summand detected by the the trivial group $L_0^{\tilde{Y}}(R) = \mathbb{Z}/2$. Note that the discriminant map $\tau\colon L_0^{\tilde{Y}}(\widehat{R}_2\pi) \to H^0(\pi^{\mathrm{ab}})$ is induced by the projection $\pi \to \pi^{\mathrm{ab}}$ (just the identity for $\pi = C_2$). Since the image of $\kappa_4'(\pi)$ is zero for the trivial group, and both maps $s_*$ and $\beta$ are isomorphisms, the diagram commutes.

The remaining part of the proof uses a similar diagram to (3.8):

(3.13)
$$\begin{tikzcd}
H^0(\pi^{\mathrm{ab}}) \arrow[dr, bend left, "\tilde{\delta}", "\cong"'] & \\
L_0^{\tilde{Y}}(\widehat{R}_2\pi) \arrow[u, "\tau"'] \arrow[d, "trf"] & H^1(\mathrm{Wh}^{\tilde{Y}}(\widehat{R}_2\pi)) \arrow[l, "\partial"] \arrow[d, "trf"] \\
L_2'(\widehat{\mathbb{Z}}_2\pi) & H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \arrow[l, "\partial"]
\end{tikzcd}$$

and diagram (3.9) together with naturality under *trf* to complete the argument. $\square$

## 4. Non-vanishing examples for $\kappa_2$

In this section we establish some conditions for verifying the statements of Theorem C and Corollary D. We should point out that when $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$, we get $\bar{\kappa}'_2(\pi) = 0$ and so the image of $\bar{\kappa}^s_2(\pi)$ lies in the image of $\partial\colon H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L^s_0(\widehat{\mathbb{Z}}_2\pi)$. Since $\theta\colon H_2(\pi;\mathbb{Z}) \to H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$ is surjective, we get $\bar{\kappa}^s_2(\pi) \neq 0$ if and only if $\operatorname{Im}\partial \neq 0$.

4a. **Group theoretical conditions to construct examples.** Here we define a set $\Lambda(\alpha)$ and two maps $u_\alpha$ and $\delta^\alpha$ for any central extension

$$\sigma \rightarrowtail \widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$$

where $\widetilde{\pi}$ is a 2-group. Given such an extension, take any surjective group homomorphism $f\colon F \to \widetilde{\pi}$ where $F$ is a free group. Let $R$ be the kernel of the composition $\alpha \circ f$. Since $f(R) \subseteq \sigma \subseteq Z(\widetilde{\pi})$, we have $f([F,R]) = 1$ and therefore we get a well-defined homomorphism from $F/[F,R]$ to $\widetilde{\pi}$. Let $u_\alpha$ denote the restriction of this map to $\pi \wedge \pi$ where $\pi \wedge \pi = [F,F]/[F,R]$ (see proof of Statement 11.4.16 in [19] ). The map

$$u_\alpha\colon \pi \wedge \pi \to \widetilde{\pi}$$

restricts to a group homomorphism

$$\delta^\alpha\colon H_2(\pi;\mathbb{Z}) = \frac{R \cap [F,F]}{[F,R]} \to \sigma\,.$$

Let $q_1, q_2$ be in $\pi$ and $\widehat{q}_1, \widehat{q}_2$ be their lifts in $F$. Then $[\widehat{q}_1, \widehat{q}_2]$ is in $[F,F]$ and defines an element $q_1 \wedge q_2$ in $\pi \wedge \pi$ that only depends on $q_1, q_2$. If we further assume that $q_1q_2 = q_2q_1$ then we have $q_1 \wedge q_2$ in $H_2(\pi;\mathbb{Z})$. Now we define a subset $\Lambda(\alpha)$ of $H_2(\pi;\mathbb{Z})$ as follows:

$$\Lambda(\alpha) = \{\, q_1 \wedge q_2 \mid q_1, q_2 \in \pi \text{ and } q_1q_2 = q_2q_1 \,\}\,.$$

We have

$$SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi}) \cong \frac{\ker(\delta^\alpha)}{\langle \ker(\delta^\alpha) \cap \Lambda(\alpha)\rangle}$$

by Part (i) of Lemma 22 in [13]. The following is a special case of a result of Oliver.

**Theorem 4.1** ([13, Proposition 16])**.** *Let $\widetilde{\pi}$ be a 2-group and $\sigma \rightarrowtail \widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$ a central group extension. Assume that $1 \neq \sigma \subseteq [\widetilde{\pi}, \widetilde{\pi}]$ and $\sigma \cap \Omega(\alpha) = 1$ where*

$$\Omega(\alpha) = \{\, [\widetilde{g}_1, \widetilde{g}_2] \;\mid \widetilde{g}_1, \widetilde{g}_2 \in \widetilde{\pi} \text{ and } \alpha([\widetilde{g}_1, \widetilde{g}_2]) = 1 \,\}\,.$$

*Then $SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi}) \overset{\alpha_*}{\to} SK_1(\widehat{\mathbb{Z}}_2\pi)$ is not surjective. In particular this means $SK_1(\widehat{\mathbb{Z}}_2\pi) \neq 0$.*

*Proof.* Let $\iota\colon \pi \to \pi$ denote the identity map. Then $1 \rightarrowtail \pi \overset{\iota}{\twoheadrightarrow} \pi$ is also a central group extension. Take any surjective group homomorphism $f\colon F \to \widetilde{\pi}$ where $F$ is a free group. Then the map $\alpha \circ f\colon F \to \pi$ is also surjective. Hence, by definition, we have $\Lambda(\iota) = \Lambda(\alpha)$ as subsets of $\pi \wedge \pi$. We also have $\alpha \circ u_\alpha = u_\iota$ and hence $\alpha \circ \delta^\alpha = \delta^\iota$. Let $q_1$, $q_2$ be two elements in $\pi$ such that $q_1q_2 = q_2q_1$ and let $\widehat{q}_1, \widehat{q}_2$ be their lifts in $F$. Then we have $u_\alpha(q_1 \wedge q_2) = u_\alpha([\widehat{q}_1, \widehat{q}_2]) = [u_\alpha(\widehat{q}_1), u_\alpha(\widehat{q}_2)] \in \Omega(\alpha)$. Hence $\Lambda(\alpha) \subseteq u_\alpha^{-1}(\Omega(\alpha))$. Moreover, $\sigma \subset [\widetilde{\pi}, \widetilde{\pi}]$, Hence $\sigma$ is in the image of $\delta^\alpha$. We also have $\sigma \neq 1$ and therefore $\ker(\delta^\alpha) \subsetneqq \ker(\delta^\iota)$. This means

$$\Lambda(\iota) \cap \ker(\delta^\iota) \subseteq u_\alpha^{-1}(\Omega(\alpha)) \cap u_{\widetilde{\alpha}}^{-1}(\sigma) \subseteq u_{\widetilde{\alpha}}^{-1}(\Omega_p \cap \sigma) = u_{\widetilde{\alpha}}^{-1}(1) = \ker(\delta^\alpha) \subsetneqq \ker(\delta^\iota)$$

Hence the map $\alpha_*$ given by the composition

$$SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi}) \cong \frac{\ker(\delta^\alpha)}{\langle \ker(\delta^\alpha)\cap\Lambda(\alpha)\rangle} \to \frac{\ker(\delta^\iota)}{\langle \ker(\delta^\alpha)\cap\Lambda(\alpha)\rangle} \to \frac{\ker(\delta^\iota)}{\langle \ker(\delta^\iota)\cap\Lambda(\iota)\rangle} \cong SK_1(\widehat{\mathbb{Z}}_2\pi)$$

is not surjective. $\square$

**Remark 4.2.** The "converse" of the last part of Theorem 4.1 is true: if $SK_1(\widehat{\mathbb{Z}}_2\pi) \neq 0$, then there exists a suitable central extension of $\pi$ so that the $SK_1$ is not surjective. To check this, let $H_2(\pi)\rightarrowtail\bar{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$ be a Schur cover of $\pi$. Notice that $SK_1(\widehat{\mathbb{Z}}_2\bar{\pi}) = 0$. Let $\sigma_0 = H_2(\pi)\cap[\bar{\pi},\bar{\pi}]$ and $\sigma_1 = \langle c \in H_2(\pi) \,|\, c \text{ is a commutator in } \bar{\pi}\,\rangle$. Notice we have $\sigma_0 = H_2(\pi)$ and by Proposition 16 in [13], $\sigma_0/\sigma_1$ is nontrivial. Since $\sigma_0$ is an abelian group, there exists an element $z$ in $\sigma_0$ and there exists a subgroup $T$ in $\sigma_0$ such that $z$ is not in $T$, $\sigma_1$ is a subgroup of $T$ and $\sigma_0/T$ is $\mathbb{Z}/2$. Now define $\sigma = H_2(\pi)/T = \mathbb{Z}/2$ and $\widetilde{\pi} = \bar{\pi}/T$. Then $\sigma\rightarrowtail\widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$ satisfies the hypothesis of the Theorem 4.1.

**Theorem 4.3.** *Let $\widetilde{\pi}$ be a 2-group and $\sigma\rightarrowtail\widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$ a central group extension such that $1 \neq \sigma \subseteq [\widetilde{\pi},\widetilde{\pi}]$. Assume that every element in $\pi$ which is conjugate to its inverse lifts to some element in $\widetilde{\pi}$ with the same property and $H^1(SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi})) \overset{\alpha_*}{\to} H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$ is not surjective. Then the map $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L_0^s(\widehat{\mathbb{Z}}_2\pi)$ is not trivial.*

*Proof.* In the proof of Proposition 3.1 in [12], it is shown that under the assumption above any element in the kernel of the map $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L_0^s(\widehat{\mathbb{Z}}_2\pi)$ is in the image of the map $H^1(SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi})) \overset{\alpha_*}{\to} H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$. Hence the map $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L_0^s(\widehat{\mathbb{Z}}_2\pi)$ is non-trivial when $\alpha_*$ is not surjective. $\square$

4b. **A non-vanishing $\kappa_2^s$ example.** All through this subsection, let $\widetilde{\pi}$ be the $8177^{th}$ group of order 256 and $\pi$ is the $1377^{th}$ group of order 128 in the Small Groups Library. The group $\widetilde{\pi}$ of order 256 given by the following presentation

$$\widetilde{\pi} = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \,|\, r \in R\rangle$$

where $R$ contains the relations $x_1^2 = x_5x_6x_8$, $x_2^2 = x_5x_6$, $x_3^2 = x_5$, $x_4^2 = x_8$, $x_i^2 = 1$ for $i \in \{5,6,7,8\}$, $[x_1,x_2] = x_5$, $[x_1,x_3] = x_6$, $[x_1,x_4] = x_8$, $[x_2,x_3] = x_7$, $[x_2,x_4] = 1$, and $[x_i,x_j] = 1$ when $i,j$ are both in $\{3,4,5,6,7,8\}$ or $i$ in $\{1,2\}$ and $j$ in $\{5,6,7,8\}$. Then the center $Z(\widetilde{\pi})$ of $\widetilde{\pi}$ is equal to the commutator group $[\widetilde{\pi},\widetilde{\pi}]$ and generated by $x_5, x_6, x_7, x_8$. Define

$$\sigma = \langle x_7x_8\rangle \cong \mathbb{Z}/2 \text{ and } \pi = \widetilde{\pi}/\sigma$$

Let $\alpha$ be the natural quotient from $\widetilde{\pi}$ to $\pi$: Then we have a central extension

$$\sigma\rightarrowtail\widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$$

such that $\sigma \subseteq [\widetilde{\pi},\widetilde{\pi}]$. Let $\Omega(\alpha)$ be the set

$$\Omega(\alpha) = \{\, [\widetilde{g}_1,\widetilde{g}_2] \;\mid \widetilde{g}_1,\widetilde{g}_2 \in \widetilde{\pi} \text{ and } \alpha([\widetilde{g}_1,\widetilde{g}_2]) = 1 \,\}$$

Then the following lemma shows that $\Omega(\alpha)$ does not contain $x_7x_8$.

**Lemma 4.4.** $\sigma \cap \Omega(\alpha) = 1$

*Proof.* Let $n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4 \in \{0,1,2,3\}$ Then we have

$$[\, x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} \, , \, x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} \,] = x_5^{q_5} x_6^{q_6} x_7^{q_7} x_8^{q_8}$$

where

$$\begin{aligned} q_5 &\equiv n_1 m_2 + n_2 m_1 \pmod 2, \\ q_6 &\equiv n_1 m_3 + n_3 m_1 \pmod 2, \\ q_7 &\equiv n_2 m_3 + n_3 m_2 \pmod 2, \\ q_8 &\equiv n_1 m_4 + n_4 m_1 \pmod 2. \end{aligned}$$

We just need to show that there exists no $n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4 \in \{0,1,2,3\}$ such that $q_5 \equiv 0 \,(\mathrm{mod})\, 4$, $q_6 \equiv 0 \pmod 2$, $q_7 \equiv 1 \pmod 2$, and $q_8 \equiv 1 \pmod 2$. Suppose otherwise. Assume $n_1$ is even. Then $m_1$ is odd since $q_8$ is odd. So $n_2$ is even since $q_5$ is even. Therefore $n_3$ is odd, since $q_7$ is odd. This is a contradiction because $q_6$ is even. Hence $n_1$ is odd. Similarly $m_1$ is odd. Then $m_2 \equiv n_2 \pmod 2$ because $q_5$ is even. We also have $m_3 \equiv n_3 \pmod 2$ because $q_6$ is even. Then $q_7 \equiv 2n_2 m_3 \pmod 2$. This is a contradiction. □

Therefore by Theorem 4.1, we know that the map $SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi}) \overset{\alpha_*}{\to} SK_1(\widehat{\mathbb{Z}}_2\pi)$ is not surjective.

**Lemma 4.5.** *$H^1(SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi})) \overset{\alpha_*}{\to} H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$ is not surjective.*

*Proof.* Considering the central extension $Z(\pi) \rightarrowtail \pi \overset{p}{\twoheadrightarrow} \pi/Z(\pi)$. We know that $SK_1(\widehat{\mathbb{Z}}_2\pi)$ is a quotient of $\ker(\delta^p)$. Since $\ker(\delta^p) \subseteq H_2(\pi/Z(\pi);\mathbb{Z}) \cong (\mathbb{Z}/2)^4$, we know that $SK_1(\widehat{\mathbb{Z}}_2\pi)$ has exponent 2. Hence $H^1(SK_1(\widehat{\mathbb{Z}}_2\widetilde{\pi})) \overset{\alpha_*}{\to} H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$ is not surjective. □

Hence by the following lemma we know that the central extension $\sigma \rightarrowtail \widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$ satisfies all the hypothesis in Theorem 4.3.

**Lemma 4.6.** *All elements in $\pi$ that are conjugate to their inverses have lifts in $\widetilde{\pi}$ that have the same property.*

*Proof.* Suppose $ghg^{-1} = x_7 x_8 h^{-1}$ for some $g$, $h$ in $\widetilde{\pi}$. Then $[g,h]h^2 = x_7 x_8$. To show that this is impossible, it is enough to consider the case where $g = x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$ and $h = x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}$ for some $n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4 \in \{0,1,2,3\}$. In this case, we have

$$[g,h]h^2 = [\, x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} \, , \, x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} \,]\, (x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4})^2 = x_5^{q_5} x_6^{q_6} x_7^{q_7} x_8^{q_8}$$

where

$$\begin{aligned} q_5 &\equiv n_1 m_2 + n_2 m_1 + m_1 m_2 + m_1 + m_2 + m_3 \pmod 2, \\ q_6 &\equiv n_1 m_3 + n_3 m_1 + m_1 m_3 + m_1 + m_2 \pmod 2, \\ q_7 &\equiv n_2 m_3 + n_3 m_2 + m_2 m_3 \pmod 2, \\ q_8 &\equiv n_1 m_4 + n_4 m_1 + m_1 m_4 + m_1 + m_4 \pmod 2. \end{aligned}$$

Hence it is enough to show that there exists no $n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4 \in \{0,1,2,3\}$ such that $q_5 \equiv 0 \,(\mathrm{mod})\, 4$, $q_6 \equiv 0 \pmod 2$, $q_7 \equiv 1 \pmod 2$, and $q_8 \equiv 1 \pmod 2$. Suppose otherwise. Assume $m_1$ is even. Then $m_4$ is odd and $n_1$ is even odd since $q_8$ is odd. So $m_2$ is even since $q_6$ is even.Therefore $m_3$ is even since $q_5$ is even. This is a contradiction to $q_7$ being odd. Hence $m_1$ is odd. Assume $n_1$ is even. Then $n_2$, $m_3$ must have distinct parity since $q_5$ is even. So $n_2m_3$ is even. This means $m_2$ and $n_3+m_3$ are odd. This is a contradiction since $q_6$ is even. Hence $n_1$ is odd. Then $n_3$ and $m_2$ have distinct parity since $q_6$ is even. So $n_3m_2$ is even. Therefore $n_2+m_2$ and $m_3$ are odd since $q_7$ is odd. This contradicts $q_5$ being even. Hence we are done. ☐

Hence by Theorem 4.3 the map $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L_0^s(\widehat{\mathbb{Z}}_2\pi)$ is not trivial. Now to finish the example we prove the following lemma.

**Lemma 4.7.** $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$

*Proof.* Note that

$$H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \cong \frac{\langle g \in \pi_{\mathrm{ab}} \,|\, g^2 = 1\rangle}{\langle g \,|\, g \sim g^{-1}\rangle} \cong \frac{\langle [x_1], [x_2], [x_3], [x_4]\rangle}{\langle [x_1], [x_1x_2], [x_3], [x_4]\rangle} \cong 0 \tag{4.8}$$

because we have the relations

(i) $(x_2x_3x_4)x_1(x_2x_3x_4)^{-1} = x_1^{-1}$
(ii) $(x_2x_4)(x_1x_2)(x_2x_4)^{-1} = (x_1x_2)^{-1}$
(iii) $(x_3x_4)(x_1x_2x_3)(x_3x_4)^{-1} = (x_1x_2x_3)^{-1}$, and
(iv) $x_1x_4x_1^{-1} = x_4^{-1}$ ☐

**Corollary 4.9.** *The group* $\pi = SG[128, 1377]$ *has the property that* $\kappa_2^s(\pi) \neq 0$ *and* $\bar{\kappa}_2'(\pi) = 0$. *Moreover,* $\pi^{\mathrm{ab}}$ *is elementary abelian and* $\kappa_2^h(\pi) = 0$.

*Proof.* The first property follows from Lemma 4.5, Lemma 4.7 and Theorem 4.3, which establish the conditions of Theorem C. By inspection, we see that $\pi^{\mathrm{ab}}$ is elementary abelian, and hence $\kappa_2'(\pi^{\mathrm{ab}}) = 0$. Since $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$, we have $\bar{\kappa}_2'(\pi) = 0$ and Theorem B implies that $\kappa_2'(\pi) = 0$. Therefore $\kappa_2^h(\pi) = 0$ and $\ker \kappa_2^s \subsetneq \ker \kappa_2^h$. ☐

**Remark 4.10.** Let $\pi$ be a 2-group of order less than or equal to 64 and $\sigma \rightarrowtail \widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$ a central group extension which satisfies the hypothesis of Theorem 4.1. Then $SK_1(\widehat{\mathbb{Z}}_2\pi)$ is non-zero, and by Remark 4.2, without loss of generality we can assume that $\sigma = \mathbb{Z}/2$ in this central extention. Then by a GAP computation (see Appendix B), we show that $\pi$ is of order 64 with group number 149, 150, 151, 170, 171, 172, 177, or 182. Moreover, $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$ for the groups in this list.

One can ask if these groups can fit into a central extension that satisfies both the hypotheses of Theorem 4.1 and Theorem 4.3.

**Remark 4.11.** Let $\pi$ be the $i$th group of order 64 where $i$ is one of numbers in the previous list (4.10). Then by a GAP computation (see Appendix B, Listing 6) we show that $SK_1(\widehat{\mathbb{Z}}_2\pi) \cong \mathbb{Z}/2$. Then there does not exist any central group extension $\sigma \rightarrowtail \widetilde{\pi} \overset{\alpha}{\twoheadrightarrow} \pi$ which satisfies the hypothesis of Theorem 4.1, and those of Theorem 4.3, where $\sigma \cong \mathbb{Z}/2$.

We expect that $0 = \bar{\kappa}_2^s(\pi)\colon H_2(\pi; \mathbb{Z}/2) \to L_0^s(\widehat{\mathbb{Z}}_2\pi)$ for all the groups in the list (4.10). If this holds, then there will not exist any groups $\pi$ of order $\leq 64$ for which $\bar{\kappa}_2^s(\pi) \neq 0$. Note that all these groups have $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$, hence $\bar{\kappa}_2'(\pi) = 0$.

**Example 4.12.** A complete list of extensions among groups of order 128 that satisfies the hypothesis of Theorem 4.1 and Theorem 4.3 can be given as follows:

$(287, 1327)$, $(288, 1328)$, $(290, 1330)$, $(693, 3996)$, $(692, 3998)$, $(704, 4010)$, $(703, 4012)$, $(668, 4054)$, $(667, 4055)$, $(670, 4056)$, $(568, 4255)$, $(579, 4312)$, $(676, 4316)$, $(725, 4372)$, $(1375, 7736)$, $(1376, 8129)$, $(1377, 8177)$, $(1547, 9240)$, $(1549, 9242)$, $(1576, 10060)$

where the pair $(i, j)$ being in this list means the $j$th group of order 256 has a quotient that is isomorphic to the $i$th group of order 128 and considered as a $\mathbb{Z}/2$ extension it satisfies the conditions in Theorem 4.1 and Theorem 4.3. All these pairs also satisfy the result of Lemma 4.7 except $(568, 4255)$. Hence from the above list we can obtain a complete list of the minimal examples (all of order 128) that satisfy the conditions of Theorem C. For all the groups in the above list, $SK_1(\widehat{\mathbb{Z}}_2\pi) \cong H_2(\pi;\mathbb{Z})/H_2^{\mathrm{ab}}(\pi) \neq 1$, by a GAP calculation (see Appendix B), and all except for $(1547, 9240)$, $(1549, 9242)$, $(1576, 10060)$ have $SK_1(\widehat{\mathbb{Z}}_2\pi) = \mathbb{Z}/2$. The last three have $SK_1(\widehat{\mathbb{Z}}_2\pi) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

**Example 4.13.** The groups of order 128 in the list (4.12) with elementary abelianization are the following:

$(1375, 7736)$, $(1376, 8129)$, $(1377, 8177)$, $(1547, 9240)$, $(1549, 9242)$, $(1576, 10060)$.

These groups all satisfy Corollary D.

**Remark 4.14.** Let $\pi$ be the the Sylow 2-subgroup of $U_3(4)$, which is $SG[64, 245]$ in the Small Groups Library. Then $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$. Since $\pi$ has no $\mathbb{Z}/2$ extension that satisfy the hypothesis of Theorem 4.1, we know that $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) = 0$ by Remark 4.2. Hence we cannot use our method for $\pi$ to decide if $\kappa_2^s(\pi) \neq 0$. It is therefore important to study $SK_1(\mathbb{Z}\pi)$ in more detail.

**Question.** At present we do not have enough information about the subgroup $Cl_1(\mathbb{Z}\pi) \subseteq SK_1(\mathbb{Z}\pi)$ to make effective calculations of the maps in the Ranicki-Rothenberg exact sequences

$$\cdots \to H^1(SK_1(\mathbb{Z}\pi)) \to L_0^s(\mathbb{Z}\pi) \to L_0'(\mathbb{Z}\pi) \to H^0(SK_1(\mathbb{Z}\pi)) \to \dots$$

or in the exact sequence in Tate cohomology

$$\cdots \to H^0(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to H^1(Cl_1(\mathbb{Z}\pi)) \to H^1(SK_1(\mathbb{Z}\pi)) \to H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to \dots$$

This remains a challenge for the future (see [14, Theorem 9.6]). Here are some sample questions:

(i) Let $\pi$ be the the Sylow 2-subgroup of $U_3(4)$. Is the map $H^1(SK_1(\mathbb{Z}\pi)) \to L_2^s(\mathbb{Z}\pi)$ non-trivial ? For this group $\bar{\kappa}_2^s(\pi) = 0$ since $SK_1(\widehat{\mathbb{Z}}_2\pi) = 0$.

(ii) Let $\pi$ be one of the groups in the list (4.12), but not in (4.13). Is it true that $\kappa_2'(\pi) \neq 0$ ? Does $\ker \kappa_2^s(\pi) = \ker \kappa_2^h(\pi)$ hold for these groups ?

(iii) More generally, do there exist examples where $\bar{\kappa}_2^s(\pi) = 0$, but $\kappa_2^s(\pi) \neq 0$ ?

(iv) Do there exist examples where $\ker \kappa_2^s(\pi) \subsetneq \ker \kappa_2^{Cl_1}(\pi)$ ?

## 5. The proof of Theorem E

We reduce the original "oozing conjecture" for surgery up to homotopy equivalence to an explicit calculation in group homology. As an application, we provide a counterexample to the conjecture (see Section 5c), but we are far from a systematic understanding about how to generate such examples. Are there infinitely many finite 2-groups with non-trivial $\kappa'_4$ ? For any finite 2-group $\pi$, we introduced the notation

$$\lambda_4(\pi) := \delta \circ \beta \circ s_* \colon H_4(\pi; \mathbb{Z}/2) \to H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi))$$

for the composite, where $s_* \colon H_4(\pi; \mathbb{Z}/2) \to H_2(\pi; \mathbb{Z}/2)$ denote the dual homomorphism to $Sq^2 \colon H^2(\pi; \mathbb{Z}/2) \to H^4(\pi; \mathbb{Z}/2)$, and $\beta \colon H_2(\pi; \mathbb{Z}/2) \to H^0(\pi^{\mathrm{ab}})$ induced by the Bockstein. In addition, the short exact sequence

$$0 \to \mathrm{Wh}'(\mathbb{Z}\pi) \to \overline{I(\widehat{\mathbb{Z}}_2\pi)} \to \pi^{\mathrm{ab}} \to 0$$

induces a (surjective) coboundary map $\delta \colon H^0(\pi^{\mathrm{ab}}) \to H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi))$. For the notation see [14, p. 163].

*The proof of Theorem E.* By Proposition 3.10, we have a commutative diagram:

$$\begin{tikzcd}
H_4(\pi; \mathbb{Z}/2) \arrow[r, "\lambda_4(\pi)"] \arrow[d, "\kappa'_4(\pi)"'] & H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) \arrow[d, "\partial"] \\
L'_2(\mathbb{Z}\pi) \arrow[r, "\rho_2"] & L'_2(\widehat{\mathbb{Z}}_2\pi)
\end{tikzcd}$$

in which the boundary map $\partial$ is injective. By Lemma 3.4, the torsion subgroup of $L'_2(\mathbb{Z}\pi)$ is detected by the natural map

$$\rho_2 \oplus p_* \colon L'_2(\mathbb{Z}\pi) \to L'_2(\widehat{\mathbb{Z}}_2\pi) \oplus L'_2(\mathbb{Z}\pi^{\mathrm{ab}})$$

induced by $\rho_2$ and the projection $p \colon \pi \to \pi^{\mathrm{ab}}$. Moveover, the composite $\rho_2 \circ \kappa'_4 = 0$ for abelian groups since $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$, and $\rho_2$ is injective for $\pi$ abelian (see [25, 5.2.2]).

Therefore the computation of $\kappa'_4(\pi)$ is reduced to determining $\lambda_4(\pi)$, and the fact that $\kappa'_4(\pi)$ vanishes on integral homology classes follows from Corollary 5.5. Finally, there is an injection $L'_2(\mathbb{Z}\pi) \to L^h_2(\mathbb{Z}\pi)$ and hence $\kappa^h_4 \neq 0$ if and only if $\kappa'_4 \neq 0$, so it is enough to determine $\kappa'_4(\pi)$. This reduces the original oozing conjecture that $\kappa^h_j = 0$, for $j \geq 4$, to an explicit group homology calculation, namely computing the map $\lambda_4(\pi)$. □

## 5a. Necessary conditions for $\beta \circ s_* \neq 0$.

**Definition 5.1.** For any internal direct sum $\pi^{\mathrm{ab}} = C_1 \oplus C_2 \oplus \cdots \oplus C_n$ of non-trivial cyclic subgroups, the *associated basis* $\mathcal{B} = \{v_1, v_2, \dots v_n\}$ for $H^0(\pi^{\mathrm{ab}})$ is the linearly independent set consisting of the elements $v_i \in C_i$ of order two, for $1 \leq i \leq n$.

In any such decomposition of $\pi^{\mathrm{ab}}$ the orders of the $C_i$, and the number of cyclic subgroups of the same order, are invariants of $\pi$. However, the factors $C_i \subseteq \pi^{\mathrm{ab}}$ are not unique as subgroups of $\pi$.

**Lemma 5.2.** *Suppose that $H^1(Wh'(\widehat{\mathbb{Z}}_2\pi)) \neq 0$. Then there exist cyclic subgroups $\{C_i\}$ of $\pi^{\mathrm{ab}}$, such that $\pi^{\mathrm{ab}} = C_1 \oplus C_2 \oplus \cdots \oplus C_n$, where the associated basis $\mathcal{B} = \{v_1, v_2, \dots v_n\}$ for $H^0(\pi^{\mathrm{ab}})$ satisfies:*

*(i) For some $k \leq n$, the set $\{\delta(v_1), \dots, \delta(v_k)\}$ is a basis for $H^1(Wh'(\widehat{\mathbb{Z}}_2\pi))$;*
*(ii) The set $\{v_{k+1}, \dots, v_n\} \subseteq \mathcal{B}$ is a basis for $K$.*

*Proof.* Let $\{C_i \,|\, 1 \leq i \leq n\}$ denote a collection of cyclic subgroups of $\pi^{\rm ab}$ giving an internal direct sum decomposition. For each factor, pick a generator so that $C_i = \langle g_i \rangle$. If $|C_i| = m_i$, for $1 \leq i \leq n$, we can choose the ordering so that $m_1 \geq m_2 \geq \cdots \geq m_n > 1$.

Let $\{w_1, w_2, \dots, w_m\}$, with $0 < m \leq n$, be a basis for $W := H^1(Wh'(\widehat{\mathbb{Z}}_2\pi) \cong (\mathbb{Z}/2)^m$ over $\mathbf{F}_2$. In terms of the basis $\{v_1, v_2, \dots, v_n\}$ for $V := H^0(\pi^{\rm ab})$, we can express the linear map $\delta\colon V \to W$ as an $m \times n$ matrix $M = (a_{ij})$ of rank $m$ such that

$$w_i = \sum_{i=1}^{m} a_{ij} v_j \ .$$

By using row operations, we can assume that $M$ is in reduced row echelon form. Moreover, we are also allowed to use elementary column operations of the form $\mathrm{col}_j \to \mathrm{col}_j + \mathrm{col}_i$, if $j > i$. For example, if $\mathrm{row}_i$ has its first non-zero entry in column $k$, and $a_{ij} = 1$ for some $j > k$, we can replace the factor $C_j = \langle g_j \rangle$ by $C'_j = \langle g_j g_k^t \rangle$, where $t = m_k/m_j$. After repeating these column operations, we may assume that $M$ is only a single non-zero entry in each row. By re-ordering the cyclic factors, we obtain an associated basis $\mathcal{B}$ of the required form. □

Suppose that $\pi$ is a 2-group such that $\beta \circ s_*(\widehat{x}) \neq 0$ for some $\widehat{x} \in H_4(\pi; \mathbb{Z}/2)$. Let $\pi^{\rm ab} = C_1 \oplus C_2 \oplus \cdots \oplus C_n$ be a decomposition of $\pi^{\rm ab}$ with the properties given in Lemma 5.2. By re-ordering the cyclic factors if necessary, we may suppose that $p_*(\beta \circ s_*(\widehat{x})) \neq 0 \in H^0(C_1; \mathbb{Z}/2)$, where $p\colon \pi \to C_1$ denotes projection onto the first factor. In particular, the preimage $N \leq \pi$ of $C_2 \oplus \cdots \oplus C_n$ under the natural projection from $\pi$ to $\pi^{\rm ab}$ contains the preimage of $K$.

Let $g_1$ be an element in $\pi$ such that $[g_1]$ generates $C_1$ in $\pi^{\rm ab}$. Let $\widehat{C}_1$ denote the subgroup of $\pi$ generated by $g_1$. Since $v_1$ is in $C_1$, there exists a positive integer $r$ such that $[g_1]^r = v_1$. Hence $g_1^r$ is not conjugate to its inverse in $\pi$. Therefore the order of $g_1$ is strictly greater than $2r$. This means $|\widehat{C}_1| > |C_1|$. Also notice that $\pi/N$ is a cyclic group and $|\pi/N| = |C_1|$. Now let $p\colon \pi \to \pi/N$ denote the natural projection and consider the diagram

$$\begin{tikzcd}
H_4(\pi;\mathbb{Z}/2) \arrow[r, "s_*"] \arrow[d, "p_*"'] & H_2(\pi;\mathbb{Z}/2) \arrow[r, "\beta"] \arrow[d, "p_*"'] & H^0(\pi^{\rm ab}) \arrow[d, "p_*"'] \\
H_4(\pi/N;\mathbb{Z}/2) \arrow[r, "s_*"] & H_2(\pi/N;\mathbb{Z}/2) \arrow[r, "\beta"] & H^0(\pi/N)
\end{tikzcd} \tag{5.3}$$

Since $\pi/N$ is cyclic, the maps $s_*$ and $\beta$ in the lower row are both isomorphisms. Then we have $p_* \circ \beta \circ s_*(\widehat{x}) \neq 0$, and hence $p_*\colon H_4(\pi; \mathbb{Z}/2) \to H_4(\pi/N; \mathbb{Z}/2)$ is non-zero (and conversely). Therefore $p_*\colon H_2(\pi; \mathbb{Z}/2) \to H_2(\pi/N; \mathbb{Z}/2)$ is also nonzero. Dually this means $p^*\colon H^2(\pi/N; \mathbb{Z}/2) \to H^2(\pi; \mathbb{Z}/2)$ is nonzero.

We can summarize this last point as follows:

**Lemma 5.4.** *If $(\beta \circ s_*)(\widehat{x}) \neq 0$, for some $\widehat{x} \in H_4(\pi; \mathbb{Z}/2)$, then there exists a cyclic quotient $p\colon \pi \to C_1$ such that $0 \neq p_*(\widehat{x}) \in H_4(C_1; \mathbb{Z}/2)$. Moreover,*

*(i) $0 \neq \widehat{x} \cap \theta \in H_2(\pi; \mathbb{Z}/2)$, where $\theta = p^*(\bar{\theta}) \in H^2(\pi; \mathbb{Z}/2)$ is the pullback of the generator of $H^2(C_1; \mathbb{Z}/2)$.*

*(ii) If $\pi/N = C_1 \cong \mathbb{Z}/2$, we have $\theta = \phi^2$ where $\phi = p^*(\phi_1) \in H^1(\pi;\mathbb{Z}/2)$ is the pullback of the generator $\phi_1 \in H^1(C_1;\mathbb{Z}/2)$.*

**Corollary 5.5.** *If $\widehat{x} \in \operatorname{Im}\{H_4(\pi;\mathbb{Z}) \to H_4(\pi;\mathbb{Z}/2)\}$, then $(\beta \circ s_*)(\widehat{x}) = 0$.*

*Proof.* If $\widehat{x} \in \operatorname{Im}\{i_*\colon H_4(\pi;\mathbb{Z}) \to H_4(\pi;\mathbb{Z}/2)\}$, then the composite $p_* \circ i_*$ factors through $H_4(\pi/N;\mathbb{Z}) = 0$, for any cyclic quotient $\pi/N$. Hence $p_*(\widehat{x}) = 0$ and $(\beta \circ s_*)(\widehat{x}) = 0$ by Lemma 5.4. □

5b. **A sufficient condition for $\delta \circ \beta \circ s_* \neq 0$.** We assume that $H^1(Wh'(\widehat{\mathbb{Z}}_2\pi)) \neq 0$ and that

$$\pi^{\mathrm{ab}} = C_1 \oplus C_2 \oplus \cdots \oplus C_n$$

is an internal direct sum satisfying the conditions of Lemma 5.2. We will call this an *adapted decomposition of* $\pi^{\mathrm{ab}}$. Let $p\colon \pi \to C_1$ be the first factor projection with respect to an adapted decomposition of $\pi^{\mathrm{ab}}$, and let $N = \ker p$. Since $[\pi,\pi] \le N$. we have a natural map $N/[\pi,\pi] \to \pi^{\mathrm{ab}}$. If $H^1(Wh'(\widehat{\mathbb{Z}}_2\pi)) \neq 0$, then $\delta(v_1) \neq 0$, and $K = \ker\delta$ is contained in the image of the induced map $H^0(N/[\pi,\pi]) \to H^0(\pi^{\mathrm{ab}})$. We have the following lemma that gives a sufficient condition for $\delta \circ \beta \circ s_* \neq 0$.

**Lemma 5.6.** *Let $p\colon \pi \to C_1$ be the first factor projection with respect to an adapted decomposition of $\pi^{\mathrm{ab}}$, such that $\delta(v_1) \neq 0$, and let $N = \ker p$. If $p^*(x) \neq 0$, for some $x \in H^4(\pi/N;\mathbb{Z}/2)$, then $(\delta \circ \beta \circ s_*)(\widehat{x}) \neq 0$, for some $\widehat{x} \in H_4(\pi;\mathbb{Z}/2)$.*

*Proof.* First, notice that $H^4(\pi/N;\mathbb{Z}/2) \cong \mathbb{Z}/2$, hence $p_*\colon H_4(\pi;\mathbb{Z}/2) \to H_4(\pi/N;\mathbb{Z}/2)$ is nonzero if and only if $p^*\colon H^4(\pi/N;\mathbb{Z}/2) \to H^4(\pi;\mathbb{Z}/2)$ is nonzero. Hence the existence of $x \in H^4(\pi/N;\mathbb{Z}/2)$ implies that there exists $\widehat{x} \in H_4(\pi;\mathbb{Z}/2)$ such that $0 \neq p_*(\widehat{x}) \in H_4(\pi/N;\mathbb{Z}/2)$.

Second, notice that for all $\widehat{z} \in H^0(\pi^{\mathrm{ab}})$ we have $\delta(\widehat{z}) \neq 0$ whenever $p_*(\widehat{z}) \neq 0$. To see this, let $L := \ker p_* \subset H^0(\pi^{\mathrm{ab}})$, $\bar{L} := L/K \cap L$, and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \cap L \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \bar{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & H^0(\pi^{\mathrm{ab}}) \arrow[r, "\delta"] \arrow[d, "p_*"] & H^1(Wh(\widehat{\mathbb{Z}}_2\pi)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K/K \cap L \arrow[r] \arrow[d] & H^0(\pi/N) \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{5.7}$$

Note that $H^0(N/[\pi,\pi]) = L$, so that $K \subseteq L$ by construction. Hence $K \cap L = K$, $K/K \cap L = 0$, and $H^0(\pi/N) \cong V$, so that $L \cong \bar{L}$. Now we claim that $p_*(\widehat{z}) \neq 0$, for $\widehat{z} = \beta \circ s_*(\widehat{x})$. Since $\pi/N = C_1$ is cyclic, by assumption, this follows by the diagram (5.3) together with the fact that the composite $\beta \circ s_*\colon H_4(\pi/N;\mathbb{Z}/2) \to H^0(\pi/N)$ is an isomorphism. Therefore $(\delta \circ \beta \circ s_*)(\widehat{x}) \neq 0$. □

5c. **An example with $\delta \circ \beta \circ s_* \neq 0$.** Let $V \overset{i}{\rightarrowtail} \pi \overset{\alpha}{\twoheadrightarrow} W$ be a group extension and $M$ a $\pi$-module. Then we have the Lyndon-Hochschild-Serre spectral sequence $\{E_r^{n,m}(\alpha, M), d_r\}$ which converges to $H^*(\pi; M)$ and has second page given by:

$$E_2^{n,m}(\alpha, M) = H^n(W; H^m(V; M)).$$

Let $\pi := G(16384)$ be the group of order $2^{14} = 16384$ given by the following presentation

$$\pi = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, z_1, z_2, z_3, z_4, z_5, z_6, z_7 \,|\, r \in R\rangle$$

where $R$ contains the relations $x_i^2 = z_i$, $z_i^2 = 1$, $[x_i, z_j] = 1$, $[z_i, z_j] = 1$, $[x_i, x_1] = 1$ for all $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$, $[x_i, x_j] = 1$ for all $i \in \{2, 3\}$, $j \in \{4, 5, 6, 7\}$, $[x_i, x_j] = 1$ for all $i \in \{4, 5\}$, $j \in \{6, 7\}$, $[x_2, x_3] = z_1$, $[x_4, x_5] = z_2$, $[x_6, x_7] = z_3$. Notice that we have a central group extension $V \overset{i}{\rightarrowtail} \pi \overset{\alpha}{\twoheadrightarrow} W$ where both

$$V = \langle z_1, z_2, z_3, z_4, z_5, z_6, z_7\rangle \cong (\mathbb{Z}/2)^7$$

and

$$W = \pi/V = \langle [x_1], [x_2], [x_3], [x_4], [x_5], [x_6], [x_7]\rangle \cong (\mathbb{Z}/2)^7$$

are elementary abelian (see [15] for information about the spectral sequence calculations in this setting).

**Corollary 5.8.** *The group $\pi = G(16384)$ has $\kappa_4'(\pi) \neq 0$.*

*Proof.* We start by recording a GAP computation, showing that

$$H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2\pi)) \cong \frac{\{g \in \pi_{\mathrm{ab}} \,|\, g^2 = 1\}}{\langle g \,|\, g \sim g^{-1}\rangle} \cong \frac{\langle [x_1], [x_2], [x_3], [z_4], [z_5], [z_6], [z_7]\rangle}{\langle [z_4], [z_5], [z_6], [z_7]\rangle} \cong (\mathbb{Z}/2)^3$$

The Lyndon-Hochschild-Serre spectral sequence for the group extension $V \overset{i}{\rightarrowtail} \pi \overset{\alpha}{\twoheadrightarrow} W$ implies that

$$E_2^{0,1}(\alpha, \mathbb{Z}/2) = H^1(V; \mathbb{Z}/2) = V^* = \mathrm{Hom}(V, \mathbb{Z}/2) = \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\rangle$$

where $\zeta_i = z_i^*$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Considering

$$H^1(W; \mathbb{Z}/2) = W^* = \langle X_1, X_2, X_3, X_4, X_5, X_6, X_7\rangle$$

where $X_i = [x_i]^*$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$. We have

$$\begin{aligned}
d_2(\zeta_1) &= X_1^2 + X_2X_3 \\
d_2(\zeta_2) &= X_2^2 + X_4X_5 \\
d_2(\zeta_3) &= X_3^2 + X_6X_7 \\
d_2(\zeta_4) &= X_4^2 \\
d_2(\zeta_5) &= X_5^2 \\
d_2(\zeta_6) &= X_6^2 \\
d_2(\zeta_7) &= X_7^2
\end{aligned}$$

Let $I_2$ be the ideal generated by the image of $d_2$. Then we have

$$\begin{aligned}
d_3(\zeta_1^2) &= X_2^2X_3 + X_2X_3^2 + I_2 \\
d_3(\zeta_i^2) &= 0 + I_2
\end{aligned}$$

for all $i$ in $\{2,3,4,5,6,7\}$. Let $I$ be the kernel of the edge homomorphism from $H^*(W;\mathbb{Z}/2) = E_2^{*,0}(\alpha,\mathbb{Z}/2)$ to $H^*(\pi;\mathbb{Z}/2)$. By Lemma 5.9 below, it follows that $X_1^4$ is not in $I$. We have

$$X_2^4 = (d_2(\zeta_2))^2 + X_5^2 d_2(\zeta_4)$$

$$X_3^4 = (d_2(\zeta_3))^2 + X_6^2 d_2(\zeta_7)$$

Hence $X_i^4$ is in $I$ for all $i$ in $\{2,3,4,5,6,7\}$.

As in Section 5b, let $N = \langle z_1, x_2, x_3, x_4, x_5, x_6, x_7\rangle$, and let $p\colon \pi \to \pi/N$ denote the quotient map. Since $d_2$ is injective, $H^1(W;\mathbb{Z}/2) \xrightarrow{\alpha^*} H^1(\pi;\mathbb{Z}/2)$ is an isomorphism, and there is a unique class $\bar{X}^1 \in H^1(\pi;\mathbb{Z}/2)$ such that $p^*(\bar{X}_1) = \alpha^*(X_1) \in H^1(\pi;\mathbb{Z}/2)$. We let $x := \bar{X}_1^4 \in H^4(\pi/N;\mathbb{Z}/2)$ for use in Lemma 5.6. The following result checks that $X_1^4$ is not in the ideal $I$, and hence $0 \neq p^*(x) \in H^4(\pi;\mathbb{Z}/2)$, implying that $\delta \circ \beta \circ s_*(\widehat{x}) \neq 0$ for some $\widehat{x} \in H_4(\pi;\mathbb{Z}/2)$. $\square$

**Lemma 5.9.** *Let*

$$\alpha = X_1^2 + X_2X_3,$$

$$\beta = X_2^2 + X_4X_5,$$

$$\gamma = X_3^2 + X_6X_7,$$

$$\theta = X_2^2X_3 + X_2X_3^2$$

*be in* $R = \mathbb{F}_2[X_1, X_2, \ldots, X_7]$. *Then* $X_1^4$ *is not in the ideal generated by* $\alpha$, $\beta$, $\gamma$, $\theta$, $X_4^2$, $X_5^2$, $X_6^2$, $X_7^2$, $X_4^2X_5 + X_4X_5^2$, $X_6^2X_7 + X_6X_7^2$.

*Proof.* Suppose otherwise. Then

$$X_1^4 = f\alpha + g\beta + h\gamma + k\theta + z \tag{5.10}$$

for some $f$, $g$, $h$, $k$ in $R$ and for some $z$ in the ideal generated by the elements $X_4^2$, $X_5^2$, $X_6^2$, $X_7^2$, $X_4^2X_5 + X_4X_5^2$, $X_6^2X_7 + X_6X_7^2$. Assume that $m_1$, $m_2$, ..., $m_k$ are distinct monomials in $X_1$, $X_2$, ..., $X_7$. In the rest of the proof, when we write

$$w = c_1m_1 + c_2m_2 + \cdots + c_km_k + \bigstar$$

it will mean that

$$w = c_1m_1 + c_2m_2 + \cdots + c_km_k + \widetilde{w}$$

for some $\widetilde{w}$ which has no $m_1$, $m_2$, ... $m_k$ terms. For the rest of the proof, the coefficients $\{c_i\}$ are elements in $\mathbb{F}_2$. Assume that

$$f = c_1X_1^2 + c_2X_2X_3 + c_3X_2^2 + c_4X_3^2 + c_5X_4X_5 + c_6X_6X_7 + \bigstar,$$

and

$$g = c_7X_1^2 + c_8X_3^2 + c_9X_2X_3 + c_{10}X_6X_7 + \bigstar$$

and

$$h = c_{11}X_1^2 + c_{12}X_2^2 + c_{13}X_2X_3 + c_{14}X_4X_5 + \bigstar$$

and

$$k = c_{15}X_2 + c_{16}X_3 + \bigstar$$

Let $m$ be a monomial in $X_1$, $X_2$, ..., $X_7$. We will write $(m)$ to denote the equation on $c_i$'s that we get by considering the coefficient of the monomial $m$ in the Equation 5.10.

$$\begin{array}{rcll}
1 &=& c_1 & (\,X_1^4\,)\\
0 &=& c_2+c_8+c_{12}+c_{15}+c_{16} & (\,X_2^2X_3^2\,)\\
0 &=& c_3+c_9+c_{15} & (\,X_2^3X_3\,)\\
0 &=& c_4+c_{13}+c_{16} & (\,X_2X_3^3\,)\\
0 &=& c_5+c_7 & (\,X_1^2X_4X_5\,)\\
0 &=& c_5+c_9 & (\,X_2X_3X_4X_5\,)\\
0 &=& c_1+c_2 & (\,X_1^2X_2X_3\,)\\
0 &=& c_8+c_{14} & (\,X_3^2X_4X_5\,)\\
0 &=& c_3+c_7 & (\,X_1^2X_2^2\,)\\
0 &=& c_4+c_{11} & (\,X_1^2X_3^2\,)\\
0 &=& c_6+c_{11} & (\,X_1^2X_6X_7\,)\\
0 &=& c_6+c_{13} & (\,X_2X_3X_6X_7\,)\\
0 &=& c_{10}+c_{12} & (\,X_2^2X_6X_7\,)\\
0 &=& c_{10}+c_{14} & (\,X_4X_5X_6X_7\,)
\end{array}$$

By the last 10 equations above we have $c_1 = c_2$ and $c_3 = c_5 = c_7 = c_9$ and $c_4 = c_6 = c_{11} = c_{13}$ and $c_8 = c_{10} = c_{12} = c_{14}$. By the first equation we get $c_1 = c_2 = 1$. Now by the second equation we get $0 = 1 + c_{15} + c_{16}$ because $c_8 = c_{12}$. By third equation we get $c_{15} = 0$ because $c_3 = c_9$. By fourth equation we get $c_{16} = 0$ because $c_4 = c_{13}$. Hence $0 = 1 + c_{15} + c_{16} = 1 + 0 + 0 = 1$. This is a contradiction. □

**Remark 5.11.** By GAP computations we have verified that $\kappa_4'(\pi) = 0$, for all elementary by elementary 2-groups of order $\leq 256$, and for about a third of such groups of order 512.

5d. **An example with $\beta \circ s_* \neq 0$ and $\delta \circ \beta \neq 0$.** Here we give an example to show that both $\beta \circ s_*$ and $\delta \circ \beta$ could be nonzero, but the composite $\delta \circ \beta \circ s_* = 0$. Such examples illustrate the difficulty of finding groups with $\kappa_4'(\pi) \neq 0$. The spectral sequence calculations follow the same methods as above

Let $\pi$ be the group of order 256, identified as $SG[256, 9039]$ in the Small Groups Library. It is given by the following presentation

$$\pi = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \,|\, r \in R\rangle$$

where $R$ contains the relations $x_1^2 = x_5x_7x_8$, $x_2^2 = x_6$, $x_3^2 = x_5x_6$, $x_4^2 = x_5x_6$, $x_5^2 = x_6^2 = x_7^2 = x_8^2 = 1$, $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_8$, $[x_2, x_3] = x_7$, $[x_2, x_4] = x_5$, $[x_3, x_4] = 1$, and $[x_i, x_5] = [x_i, x_6] = [x_i, x_7] = [x_i, x_8]$ for every $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Notice that we have a central group extension $V \overset{i}{\rightarrowtail} \pi \overset{\alpha}{\twoheadrightarrow} W$ where

$$V = Z(\pi) = [\pi, \pi] = \langle x_5, x_6, x_7, x_8\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

and

$$W = \pi^{\mathrm{ab}} = \pi/[\pi, \pi] = \langle [x_1], [x_2], [x_3], [x_4]\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Also note that the GAP code in Appendix A shows that

$$H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2\pi)) \cong \frac{\{g \in \pi^{\mathrm{ab}} \mid g^2 = 1\}}{\langle g \mid g \sim g^{-1}\rangle} \cong \frac{\langle [x_1], [x_2], [x_3], [x_4]\rangle}{\langle [x_1] + [x_2], [x_2] + [x_3], [x_3] + [x_4]\rangle} \cong \mathbb{Z}/2.$$

By considering the Lyndon-Hochschild-Serre spectral sequence for the group extension $V \overset{i}{\rightarrowtail} \pi \overset{\alpha}{\twoheadrightarrow} W$ we get

$$E_2^{0,1}(\alpha, \mathbb{Z}/2) = H^1(V; \mathbb{Z}/2) = V^* = \mathrm{Hom}(V, \mathbb{Z}/2) = \langle \zeta_5, \zeta_6, \zeta_7, \zeta_8\rangle$$

where $\zeta_i = x_i^*$ for $i \in \{5, 6, 7, 8\}$. Considering

$$H^1(W; \mathbb{Z}/2) = W^* = \langle X_1, X_2, X_3, X_4\rangle$$

where $X_i = [x_i]^*$ for $i \in \{1, 2, 3, 4\}$. We use a GAP program to obtain the following results: we have

$$\begin{aligned}
d_2(\zeta_5) &= X_1^2 + X_1X_2 + X_2X_4 + X_3^2 + X_4^2 \\
d_2(\zeta_6) &= X_1X_3 + X_2^2 + X_3^2 + X_4^2 \\
d_2(\zeta_7) &= X_1^2 + X_2X_3 \\
d_2(\zeta_8) &= X_1^2 + +X_1X_4.
\end{aligned}$$

Let $I_2$ be the image of $d_2$. Then we have

$$\begin{aligned}
d_3(\zeta_5^2) &= X_1^2X_2 + X_1X_2^2 + X_2^2X_4 + X_2X_4^2 + I_2 \\
d_3(\zeta_6^2) &= X_1^2X_3 + X_1X_3^2 + I_2 \\
d_3(\zeta_7^2) &= X_2^2X_3 + X_2X_3^2 + I_2 \\
d_3(\zeta_8^2) &= X_1^2X_4 + X_1X_4^2 + I_2
\end{aligned}$$

Let $I$ be the kernel of the edge homomorphism from $H^*(W; \mathbb{Z}/2) = E_2^{*,0}(\alpha, \mathbb{Z}/2)$ to $H^*(\pi; \mathbb{Z}/2)$. Then it is straight forward to check $X_2^4 + X_3^4$ and $X_3^4 + X_4^4$ are not in $I$. It is also straight forward to check that $X_1^4$ and $X_2^4 + X_3^4 + X_4^4$ are in $I$. Hence the image of $s_*$ in $H_4(\pi; \mathbb{Z}/2)$ is generated by the duals of the cohomology classes $X_2^2 + X_3^2 + I$ and $X_3^2 + X_4^2 + I$.

It follows that the image of $\beta \circ s_*$ is generated by $[x_2] + [x_3]$ and $[x_3] + [x_4]$, and hence nonzero in $H^0(\pi^{ab})$. Moreover $\delta \circ \beta([x_i]) \neq 0$ for all $i$ in $\{1, 2, 3, 4\}$, implying that $\delta \circ \beta \circ s_* = 0$.

## 6. More information about $\kappa_4(\pi)$

In Theorem E we have provided an explicit algebraic condition to check for non-vanishing examples for which $\kappa_4'(\pi) \neq 0$, and Corollary 1.5 provides one such example, in which the group $\pi$ is elementary abelian by elementary abelian. We first present a conjecture for the vanishing of $\kappa_4'(\pi)$ in case $\pi$ does not satisfy this assumption. In the last sections we show how to obtain further non-vanishing $\kappa_4^s$ examples (see Proposition 6.7, Proposition 6.9, and and Remark 6.10).

6a. **A conjecture about $\kappa_4'(\pi)$.** Note that if $\kappa_4'(\pi) \neq 0$, then by Theorem E the composite $\delta \circ \beta \circ s_* \neq 0$ for the group $\pi$. In particular, if $\delta \circ \beta \circ s_*(\widehat{x}) \neq 0$, for some $\widehat{x} \in H_4(\pi; \mathbb{Z}/2)$, then by Lemma 5.4 there must exist a cyclic quotient $p\colon \pi \to \pi/N = C_1$ such that $0 \neq p_*(\widehat{x}) \in H_4(C_1; \mathbb{Z}/2)$. The following conjecture is motivated by trying to show that $C_1$ must have order two if $\kappa_4'(\widehat{x}) \neq 0$.

**Question.** If $\kappa_4'(\pi) \neq 0$, does it follow that $\pi^{\mathrm{ab}}$ is elementary abelian ?

If $\pi^{\mathrm{ab}}$ has cyclic factors of orders $\geq 4$ in an internal direct sum decomposition, then we can use an filtration approach to study the map $\delta \circ \beta \circ s_*$.

**Conjecture 6.1.** *Let $N \lhd T \trianglelefteq W \lhd \pi$ be an increasing sequence of normal subgroups of a 2-group $\pi$ such that $\pi/N$ is a cyclic group, $T/N \cong \mathbb{Z}/2$ and $\pi/W \cong \mathbb{Z}/2$. Let $i\colon W \to \pi$ be the inclusion and $p\colon \pi \to \pi/N$ be the natural quotient map. Assume that the maps*

- $p_*\colon H_4(\pi; \mathbb{Z}/2) \to H_4(\pi/N; \mathbb{Z}/2) \cong \mathbb{Z}/2$, *and*
- $(p \circ i)_*\colon H_2(W; \mathbb{Z}/2) \to H_2(\pi/N; \mathbb{Z}/2) \cong \mathbb{Z}/2$

*are both non-zero, but $(p \circ i)_*\colon H_4(W; \mathbb{Z}/2) \to H_4(\pi/N; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is zero. Then the number of $\pi$-conjugacy classes of elements in $T - N$ is odd.*

**Remark 6.2.** In Appendix C we provide some GAP code to test this conjecture. The smallest groups that contains such an increasing sequence are $SG[32, 8]$, $SG[64, 13]$ and $SG[64, 14]$. In these groups, each such increasing sequence of subgroups $T - N$ has an odd number of $\pi$-conjugacy classes, verifying the conjecture.

Recall the formula:

$$H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2\pi)) \cong \frac{\{[g] \in \pi^{\mathrm{ab}} : [g]^2 = 1\}}{\langle [g] : g \sim g^{-1} \rangle} \cong S/C,$$

where $S = \{g \in \pi \,|\, g^2 \in [\pi, \pi]\}$ and $C = \langle g \in S \,|\, g \sim g^{-1} \text{ or } g \in [\pi, \pi] \rangle$. Now let

$$T = \langle g \in S \,|\, g \in C, \text{ or } t^2 = g \text{ for some } \ t \in \pi \rangle,$$

and notice that $T/C$ is a subgroup of $H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2\pi))$. For example, when $\pi = SG(32, 4)$ we have $S = T$.

**Theorem 6.3.** *Let $\pi$ be a finite 2-group. Assume that Conjecture 6.1 is true and $\kappa_4'(H) = 0$, for every proper subgroup $H$ of $\pi$. Then the intersection of the image of $\kappa_4'(\pi)$ with $T/C$ is trivial.*

*Proof.* Suppose otherwise. Then there exists an element $\widehat{x} \in H_4(\pi; \mathbb{Z}/2)$, and an adapted decomposition of $\pi^{\mathrm{ab}}$ such that $0 \neq p_*(\beta \circ s_*(\widehat{x}))$ under the first factor projection $p\colon \pi \to C_1$, for which $\delta(v_1) \neq 0$ is in $T/C$. Let $N$ be as in Section 5b. Since $v_1$ is in $T$, $\pi/N$ is cyclic group of order at least 4. In fact there exists an increasing sequence $N \lhd T \trianglelefteq W \lhd \pi$ of normal subgroups of the 2-group $\pi$. Notice that we have $p_*\colon H_4(\pi; \mathbb{Z}/2) \to H_4(\pi/N; \mathbb{Z}/2) \cong \mathbb{Z}/2$ non-zero by Lemma 5.4 because $\delta(v_1) \neq 0$.

Moreover, $(p \circ i)_*\colon H_2(W; \mathbb{Z}/2) \to H_2(\pi/N; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is non-zero, since the composite

$$H_2(T; \mathbb{Z}/2) \to H_2(\pi; \mathbb{Z}/2) \to H^0(\pi^{\mathrm{ab}}) \to H^0(\pi/N)$$

is surjective, and $H^0(T/N) \cong H^0(\pi/N)$. This uses the fact that $H_2(T; \mathbb{Z}/2) \to H_2(\pi; \mathbb{Z}/2)$ factors through the map $H_2(W; \mathbb{Z}/2) \to H_2(\pi; \mathbb{Z}/2)$, and the isomorphisms

$$H_2(T/N; \mathbb{Z}/2) \cong H_2(W/N; \mathbb{Z}/2) \cong H_2(\pi/N; \mathbb{Z}/2)$$

induced by the inclusions.

But $(p \circ i)_* \colon H_4(W; \mathbb{Z}/2) \to H_4(\pi/N; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is zero because $\kappa'_4(W) = 0$. Then the number of $\pi$-conjugacy classes of elements in $T - N$ is odd. Consider the action on the set $\pi$-conjugacy classes of elements in $T - N$ given by taking inverses. This action must have a fixed point and hence contradicting $\delta(v_1) \neq 0$. ☐

**Corollary 6.4.** *Assume that Conjecture 6.1 is true, and that $\pi^{\rm ab}$ has no direct factors of order two. Then $\kappa'_4(\pi) = 0$.*

6b. **Codimension two twisting diagrams.** Here is an approach to finding an example such that $\kappa_4^s \neq 0$, but $\kappa'_4 = 0$. We look for a finite 2-group $\pi$ equipped with certain properties. For simplicity, we restrict to groups $\pi$ such that $\pi^{\rm ab}$ has exponent two. Recall from [13, Theorem 3] that there is a natural isomorphism

$$\omega \colon H_2(\pi; \mathbb{Z})/H_2^{\rm ab}(\pi) \xrightarrow{\cong} SK_1(\widehat{\mathbb{Z}}_2\pi)$$

where $H_2^{\rm ab}(\pi) \subseteq H_2(\pi; \mathbb{Z})$ denotes the image under induction from abelian subgroups of $\pi$, and a boundary map $\partial \colon H^1(SK_(\widehat{\mathbb{Z}}_2\pi)) \to L_{2i}(\widehat{\mathbb{Z}}_2\pi)$, for $i = 0, 1 \pmod 4$.

**Definition 6.5.** Let $\pi$ be a non-abelian finite 2-group and $\theta \in H^2(\pi; \mathbb{Z}/2)$ a cohomology class. We say that the pair $(\pi, \theta)$ is *compatible* provided that the following conditions hold.

(i) The abelianization $\pi^{\rm ab}$ has exponent two.
(ii) A central extension $1 \to \sigma \to \tilde{\pi} \to \pi \to 1$, classified by $\theta \in H^2(\pi; \mathbb{Z}/2)$, with $\sigma = \langle t \rangle \cong \mathbb{Z}/2$, defines a 2-cover $\alpha \colon \tilde{\pi} \to \pi$. We require that $t \in [\tilde{\pi}, \tilde{\pi}]$.
(iii) $H^1(Wh'(\widehat{\mathbb{Z}}_2\tilde{\pi})) = 0$.
(iv) $SK_1(\widehat{\mathbb{Z}}_2\pi) \neq 0$, but the map $SK_1(\widehat{\mathbb{Z}}_2\tilde{\pi}) \to SK_1(\widehat{\mathbb{Z}}_2\pi)$ is zero.
(v) The boundary map $\partial \colon H^1(SK_1(\widehat{\mathbb{Z}}_2\pi)) \to L_0(\widehat{\mathbb{Z}}_2\pi)$ is injective.
(vi) There exists a class $\widehat{x} \in H_4(\pi; \mathbb{Z}/2)$ such that $0 \neq \widehat{z} = \widehat{x} \cap \theta \in H_2(\pi; \mathbb{Z}/2)$ is the image $\widehat{z} = i_*(\widehat{z}_1)$ of an integral class $\widehat{z}_1 \in H_2(\pi; \mathbb{Z})$.
(vii) For the image $[\widehat{z}_1] \in H_2(\pi; \mathbb{Z})/H_2^{\rm ab}(\pi)$, we have $1 \neq \omega([\widehat{z}_1]) \in SK_1(\widehat{\mathbb{Z}}_2\pi)$.

**Remark 6.6.** Alternately, by duality in terms of the cohomology Bockstein, for condition (vi) we need a class $z \in H^2(\pi; \mathbb{Z}/2)$ such that $\beta^*(z) \neq 0$ and $z \cup \theta \neq 0$.

**Proposition 6.7.** *If $(\pi, \theta)$ is a compatible pair, then $\kappa_4^s(\pi) \neq 0$ and $\kappa'_4(\pi) = 0$.*

*Proof.* Note first that $H^1(Wh'(\widehat{\mathbb{Z}}_2\pi)) = 0$ by condition (iii). Hence $\kappa'_4(\pi) = 0$ by Theorem E and condition (i). Furthermore, $\kappa_2^s(\widehat{z}) \neq 0$ by conditions (iv), (v), and Theorem C applied to $\widehat{z} = \widehat{x} \cap \theta \in H_2(\pi; \mathbb{Z}/2)$.

We have a central extension $1 \to \sigma \to \tilde{\pi} \to \pi \to 1$, such that $\sigma \subset [\tilde{\pi}, \tilde{\pi}]$, and by condition (ii) we have $H^0(\tilde{\pi}^{\rm ab}) \cong H^0(\pi^{\rm ab})$ under the projection map. The class $\theta \in H^2(\pi; \mathbb{Z}/2)$ defines a circle bundle

$$S^1 \to Y \to K(\pi, 1)$$

inducing the given central extension on fundamental groups (see [23, pp128-131]). This circle bundle is orientable if and only if $\theta$ is the image of an integral class, or equivalently $\beta^*(\theta) = 0$, where $\beta^* \colon H^2(\pi; \mathbb{Z}/2) \to H^3(\pi; \mathbb{Z})$ is the Bockstein on cohomology.

Hence we have a codimension 2 twisting diagram:

(6.8)
$$\begin{tikzcd} H_4(\pi;\mathbb{Z}/2) \arrow[r, "\bar\kappa_4^s"] \arrow[d, "\cap\theta"] & L_2^s(\widehat{\mathbb{Z}}_2\pi) \arrow[dr, "a"] & \\ H_2(\pi;\mathbb{Z}/2) \arrow[r, "\bar\kappa_2^s"] & L_0^s(\widehat{\mathbb{Z}}_2\pi) \arrow[r, "b"] & L_2^s(\widehat{\mathbb{Z}}_2\tilde\pi \to \widehat{\mathbb{Z}}_2\pi) \end{tikzcd}$$

We claim that the map $b$ in this diagram is injective on the image of $\kappa_2^s$ restricted to $H_2(\pi;\mathbb{Z}) \subseteq H_2(\pi;\mathbb{Z}/2)$ by the Ranicki-Rothenberg comparison sequences. We have an exact sequence

$$\cdots \to L^s_{n+3}(\tilde\pi \to \pi) \to LS_n(\Phi) \to L^s_n(\widehat{\mathbb{Z}}_2\pi) \to L^s_{n+2}(\tilde\pi \to \pi) \to \dots$$

arising from the circle bundle (see [23, p127]). In our setting,

$$LS_n(\Phi) = LN_n(\tilde\pi \to \pi) \cong L^s_n(\widehat{\mathbb{Z}}_2\tilde\pi, \beta, t),$$

and this long exact sequence can be identified with the sequence

$$\cdots \to L_3^s(\widehat{\mathbb{Z}}_2\tilde\pi \to \widehat{\mathbb{Z}}_2\pi) \xrightarrow{\partial} L_0^s(\widehat{\mathbb{Z}}_2\tilde\pi, \beta, t) \xrightarrow{\alpha_*} L_0^s(\widehat{\mathbb{Z}}_2\pi) \xrightarrow{b} L_2^s(\widehat{\mathbb{Z}}_2\tilde\pi \to \widehat{\mathbb{Z}}_2\pi)$$

where $(\widehat{\mathbb{Z}}_2\tilde\pi, \beta, t)$ is the induced twisted antistructure and the middle map is induced by the projection $\alpha\colon \tilde\pi \to \pi$ (see Ranicki [18, p805,808]). Under our assumptions, $H^1(Wh'(\widehat{\mathbb{Z}}_2\tilde\pi)) = 0$ by condition (iii), and $\widetilde{\pi}^{\mathrm{ab}} \cong \pi^{\mathrm{ab}}$ implies that $H^1(Wh'(\widehat{\mathbb{Z}}_2\pi)) = 0$. It follows that $L_0^s(\widehat{\mathbb{Z}}_2\pi) \cong H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$ and $L_0^s(\widehat{\mathbb{Z}}_2\tilde\pi, \beta, t) \cong H^1(SK_1(\widehat{\mathbb{Z}}_2\tilde\pi))$ (see [8, §8]). Since the map $SK_1(\widehat{\mathbb{Z}}_2\tilde\pi) \to SK_1(\widehat{\mathbb{Z}}_2\pi)$ is zero, we conclude that the $\alpha_* = 0$ and $b$ is injective.

Since $0 \neq \kappa_2^s(\widehat{z}) = \kappa_2^s(i_*(\widehat{z}_1)) \in L_0^s(\widehat{\mathbb{Z}}_2\pi)$, it follows that $\kappa_4^s(\widehat{x}) \neq 0$ by commutativity of the twisting diagram and the injectivity of the map $b$. □

6c. **An example which satisfies the hypothesis of Proposition 6.7.** We can use GAP to search for examples satisfying the conditions in Definition 6.5. For finite 2-groups of orders up to 128, we will use the cohomology calculations of Green and King [3] (see the webpage `https://users.fmi.uni-jena.de/cohomology/`). The spectral sequence calculations follow the same methods as in previous sections.

**Proposition 6.9.** *There exists a compatible pair $(\pi, \theta)$ for $\pi = SG[128, 1376]$.*

*Proof.* Let $\tilde\pi$ be the group of order 256, identified as $SG[256, 8129]$ in the Small Groups Library. We will use the following presentation of this group

$$\tilde\pi = \left\langle \tilde x_1, \tilde x_2, \tilde x_3, \tilde x_4, \tilde x_5, \tilde x_6, \tilde x_7, \tilde x_8 \,|\, r \in \tilde R \right\rangle$$

where $\tilde R$ contains the relations $\tilde x_1^2 = \tilde x_6 \tilde x_7$, $\tilde x_2^2 = \tilde x_6$, $\tilde x_4^2 = \tilde x_5$, $\tilde x_3^2 = \tilde x_5^2 = \tilde x_6^2 = \tilde x_7^2 = \tilde x_8^2 = 1$, $[\tilde x_1, \tilde x_2] = \tilde x_8$, $[\tilde x_1, \tilde x_3] = \tilde x_5 \tilde x_6 \tilde x_8$, $[\tilde x_1, \tilde x_4] = 1$, $[\tilde x_2, \tilde x_3] = \tilde x_7$, $[\tilde x_2, \tilde x_4] = 1$, $[\tilde x_3, \tilde x_4] = \tilde x_5$, and $[\tilde x_i, \tilde x_5] = [\tilde x_i, \tilde x_6] = [\tilde x_i, \tilde x_7] = [\tilde x_i, \tilde x_8] = 1$ for every $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

Let $\pi$ be the group of order 128, identified as $SG[128, 1376]$ in the Small Groups Library. This group has the following presentation

$$\pi = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \,|\, r \in R \rangle$$

where $R$ contains the relations $x_1^2 = x_6x_7$, $x_2^2 = x_6$ $x_4^2 = x_5$, $x_3^2 = x_5^2 = x_6^2 = x_7^2 = 1$, $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = 1$, $[x_2, x_3] = x_7$, $[x_2, x_4] = 1$, $[x_3, x_4] = x_5$, and $[x_i, x_5] = [x_i, x_6] = [x_i, x_7] = 1$ for every $i \in \{1, 2, 3, 4, 5, 6, 7\}$. Now we define a surjective group homomorphism $\alpha$ from $\tilde{\pi}$ to $\pi$ by sending $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8$ to $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_5$ respectively. The kernel of this group homomorphism is

$$\sigma = \langle \tilde{x}_5\tilde{x}_8 \rangle \cong \mathbb{Z}/2.$$

Hence we have a central extension

$$1 \to \sigma \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1.$$

Let

$$\tilde{V} = Z(\tilde{\pi}) = [\tilde{\pi}, \tilde{\pi}] = \langle \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

and

$$V = Z(\pi) = [\pi, \pi] = \langle x_5, x_6, x_7 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

and

$$\tilde{W} = \tilde{\pi}^{\mathrm{ab}} = \tilde{\pi}/[\tilde{\pi}, \tilde{\pi}] = \langle [\tilde{x}_1], [\tilde{x}_2], [\tilde{x}_3], [\tilde{x}_4] \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

and

$$W = \pi^{\mathrm{ab}} = \pi/[\pi, \pi] = \langle [x_1], [x_2], [x_3], [x_4] \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

where $\alpha$ induces an isomorphism from $\tilde{\pi}^{\mathrm{ab}}$ to $\pi^{\mathrm{ab}}$ which sends $[\tilde{x}_i]$ to $[x_i]$ for all $i$ in $\{1, 2, 3, 4\}$. Now we have $gxg^{-1} = x^{-1}$ when

$$(g, x) \in \{(1, \tilde{x}_3), (\tilde{x}_3\tilde{x}_4, \tilde{x}_4), (\tilde{x}_1\tilde{x}_2\tilde{x}_4, \tilde{x}_2\tilde{x}_3), (\tilde{x}_2\tilde{x}_4, \tilde{x}_1\tilde{x}_3), \}.$$

Hence $[\tilde{x}_3]$, $[\tilde{x}_4]$, $[\tilde{x}_2\tilde{x}_3]$, and $[\tilde{x}_1\tilde{x}_3]$ are all zero in $H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2\tilde{\pi}))$, and we have

$$H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2\tilde{\pi})) \cong H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2\pi)) \cong 0.$$

Therefore condition (iii) holds, and condition (iv) is immediate from the GAP computations in Appendix B, which show that $SK_1(\widehat{\mathbb{Z}}_2\pi) \cong \mathbb{Z}/2$. Condition (v) follows by applying Theorem 4.1 and Theorem 4.3 (see List L2).

Now we get the corresponding Lyndon-Hochschild-Serre spectral sequences with

$$\begin{aligned}
d_2(\tilde{X}_5) &= \tilde{X}_1\tilde{X}_3 + \tilde{X}_3\tilde{X}_4 + \tilde{X}_4^2 \\
d_2(\tilde{X}_6) &= \tilde{X}_1^2 + \tilde{X}_1\tilde{X}_3 + \tilde{X}_2^2 \\
d_2(\tilde{X}_7) &= \tilde{X}_1^2 + \tilde{X}_2\tilde{X}_3 \\
d_2(\tilde{X}_8) &= \tilde{X}_1\tilde{X}_2 + \tilde{X}_1\tilde{X}_3.
\end{aligned}$$

and

$$\begin{aligned}
d_2(X_5) &= X_1X_2 + X_3X_4 + X_4^2 \\
d_2(X_6) &= X_1^2 + X_1X_3 + X_2^2 \\
d_2(X_7) &= X_1^2 + X_2X_3
\end{aligned}$$

Assume that $I$ denotes the ideal generated by $d_2(X_5), d_2(X_6), d_2(X_7)$ that is closed under Steenrod operations. Notice that $\alpha^*$ sends $d_2(X_5)$ to $d_2(\tilde{X}_5) + d_2(\tilde{X}_8)$. Hence

$$\theta = [X_1X_2 + X_1X_3] \in H^2(\pi; \mathbb{Z}/2)$$

is the cohomology class that corresponds to the central extension

$$1 \to \sigma \to \widetilde{\pi} \to \pi \to 1.$$

Now take

$$z = [X_3X_4] \in H^2(\pi; \mathbb{Z}/2).$$

Then $\theta \cup z \neq 0$ in $H^4(\pi; \mathbb{Z}/2)$, and pick $\hat{x} \in H_4(\pi; \mathbb{Z}/2)$ such that $\hat{x} \cap (\theta \cup z) \neq 0$. Then $(\hat{x} \cap \theta) \cap z = 1$, and for the class $\hat{z} = \hat{x} \cap \theta$ we have $\hat{z} \cap z = \hat{z} \cap [X_3X_4] = 1$.

Now $H_2(\pi; \mathbb{Z})$ has exponent two (by a GAP computation) and $\mathrm{Sq}^1(z) \neq 0$ (by using the GAP package SINGULAR), so

$$0 \neq \beta^*(z) \in H^3(\pi; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(\pi, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}).$$

By duality, there exists a unique class $\hat{z}_1 \in H_2(\pi; \mathbb{Z})$ such that $i_*(\hat{z}_1) = \hat{z}$.

. Next we claim that the class $\hat{z}_1$ is not in the subgroup $H_2^{\mathrm{ab}}(\pi) \subseteq H_2(\pi, \mathbb{Z})$. Let $q_*\colon H_2(\pi; \mathbb{Z}) \to H_2(\pi^{\mathrm{ab}}; \mathbb{Z})$ denote the natural map induced by the quotient $q\colon \pi \to \pi^{\mathrm{ab}}$. Then we have the formula

$$H_2(\pi^{\mathrm{ab}}; \mathbb{Z}) = \bigoplus_{1 \leq i < j \leq 4} \mathbb{Z}/2 \cdot e_{ij} \cong (\mathbb{Z}/2)^6.$$

The image of $q_*$ is generated by $\{e_{12}+e_{34}, e_{14}, e_{24}\}$, and only the basis elements $e_{14}$ and $e_{24}$ are in the image under $q_*$ of $H_2^{\mathrm{ab}}(\pi)$, since $\langle x_1, x_4\rangle \cong \mathbb{Z}/4 \times \mathbb{Z}/4$ and $\langle x_2, x_4\rangle \cong \mathbb{Z}/4 \times \mathbb{Z}/4$, but $[x_1, x_2] = [x_3, x_4] = x_5$.

Since $z = [X_3X_4] \in H^2(\pi; \mathbb{Z}/2)$, we have $z = q^*(X_3X_4)$ and hence

$$q_*(\hat{z}) \cap (X_3X_4) = \hat{z} \cap q^*(X_3X_4) = \hat{z} \cap [X_3X_4] = 1.$$

We have $(e_{12} + e_{34}) \cap (X_3X_4) = 1$, $e_{14} \cap (X_3X_4) = 0$, $e_{24} \cap (X_3X_4) = 0$. Hence $q_*(\hat{z}_1) = (e_{12} + e_{34}) + c_1e_{14} + c_2e_{24}$ for some constants $c_1$, $c_2$. Therefore, $\hat{z}_1 \notin H_2^{\mathrm{ab}}(\pi)$ and we have $\omega([\hat{z}_1]) \neq 0$ as required for condition (vii). We refer to the webpage:

https://users.fmi.uni-jena.de/cohomology/128web/128gp1376.html

for information about the cohomology ring of $\pi = SG[128, 1376]$. □

**Remark 6.10.** The list of groups in Example 4.12 is a possible source for further examples. Conditions (iii), (iv) and (v) in Definition 6.5 were checked explicitly in Section 4b for the group $\pi = SG[128, 1377]$ and its 2-extension $\widetilde{\pi} = SG[256, 8177]$. Using the methods above, it is not hard to verify the remaining conditions to conclude that $\kappa_4^s(\pi) \neq 0$, but $\kappa_4'(\pi) = 0$.

## Appendix A. Some GAP computations for $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2G))$

The following code takes a group $G$ as input and it outputs $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2G))$. It uses the fact that

$$H^1(\mathrm{Wh}'(\hat{\mathbb{Z}}_2G)) \cong \frac{\{g \in G_{\mathrm{ab}} \,|\, g^2 = 1\}}{\langle g \,|\, g \sim g^{-1}\rangle} \cong \frac{S}{C}$$

where $S = \{g \in G \,|\, g^2 \in [G, G]\}$ and $C = \langle g \in S \,|\, g \sim g^{-1}$ or $g \in [G, G]\rangle$ Here the variable `Sq_in_Comm` is loaded with a list of elements that generate $S$ and the variable `Conj_to_inverse` is loaded with elements that generate the group $C$.

```
H1_Wh_prime:=function(G)
local Comm_G,g,Sq_in_Comm,Conj_to_inverse;
      Comm_G:=CommutatorSubgroup(G,G);
      Sq_in_Comm:=[];
      Conj_to_inverse:=List(Comm_G);
      for g in G do
          if not(g^2 in Comm_G) then continue; fi;
          Add(Sq_in_Comm,g);
          if (g^(-1) in g^G) then Add(Conj_to_inverse,g); fi;
      od;
      return Group(Sq_in_Comm)/Group(Conj_to_inverse);
end;;
```

LISTING 1. Function for computing $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2 G))$

Here we can obtain a list of groups $G$ with nonzero $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2 G))$ as follows:

```
for n in [2,4,8,16,32,64,128] do
   for i in [1..NumberSmallGroups( n )] do
      G:=SmallGroup(n,i);
      H:=H1_Wh_prime(G);
      if Size(H)>1 then Print("[",n,",",i,",",Rank(H1_Wh_prime(G)),"], "); fi;
   od;
od;
```

LISTING 2. Code for listing groups with nonzero $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2 G))$

The result of the above listing is given as below. Note that `[s,i,r]` being in the list means $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2 SG[s,i])) \cong (\mathbb{Z}/2)^{\oplus r}$.

```
[32,4,1], [32,13,1], [64,3,1], [64,15,1], [64,27,1], [64,28,1], [64,46,1],
[64,48,1], [64,57,1], [64,62,1], [64,63,1], [64,64,2], [64,78,1], [64,81,1],
[64,82,3], [64,84,1], [64,106,1], [64,113,1], [64,162,1], [64,164,1],
[128,27,1], [128,43,1], [128,44,1], [128,45,1], [128,49,1], [128,56,1],
[128,76,1], [128,85,1], [128,99,1], [128,100,1], [128,101,1], [128,104,1],
[128,129,1], [128,130,1], [128,156,1], [128,158,1], [128,166,1], [128,172,1],
[128,174,1], [128,175,2], [128,177,1], [128,180,1], [128,183,1], [128,184,1],
[128,294,1], [128,298,1], [128,300,1], [128,302,1], [128,340,1], [128,348,1],
[128,457,1], [128,458,1], [128,476,1], [128,477,2], [128,478,2], [128,479,1],
[128,481,1], [128,484,1], [128,499,1], [128,502,1], [128,506,1], [128,539,1],
[128,549,1], [128,550,1], [128,551,1], [128,552,1], [128,553,1], [128,554,1],
[128,555,1], [128,556,1], [128,557,1], [128,558,1], [128,559,1], [128,560,1],
[128,561,1], [128,562,1], [128,563,1], [128,564,1], [128,565,1], [128,566,1],
[128,567,2], [128,568,1], [128,569,1], [128,570,1], [128,571,2], [128,572,3],
[128,573,2], [128,576,1], [128,578,1], [128,585,1], [128,649,1], [128,651,1],
[128,652,1], [128,656,1], [128,658,1], [128,659,1], [128,680,1], [128,762,1],
[128,766,1], [128,767,1], [128,772,1], [128,803,1], [128,804,1], [128,805,2],
[128,806,2], [128,807,1], [128,808,1], [128,809,2], [128,810,1], [128,811,2],
```

```
[128,812,1], [128,813,1], [128,814,1], [128,816,1], [128,817,1], [128,824,1],
[128,826,1], [128,828,1], [128,829,1], [128,831,1], [128,832,1], [128,833,2],
[128,834,1], [128,835,1], [128,836,3], [128,889,1], [128,895,1], [128,897,1],
[128,898,1], [128,965,1], [128,967,1], [128,970,1], [128,971,1], [128,999,1],
[128,1011,1], [128,1013,1], [128,1014,2], [128,1015,1], [128,1122,1],
[128,1126,1], [128,1132,3], [128,1133,1], [128,1215,1], [128,1235,1],
[128,1236,1], [128,1303,1], [128,1602,1], [128,1639,1], [128,1650,1],
[128,1818,1], [128,1821,1]
```

Listing 3. The output of above listing

## Appendix B. Some GAP computations for $H^1(SK_1(\widehat{\mathbb{Z}}_2\pi))$

Given a group 2-group $G$, the following function find elements $g$ in the commutator group $[G,G]$ such that the order of $g$ is 2 and $\langle g\rangle \rightarrowtail G \overset{\alpha}{\twoheadrightarrow} G/\langle g\rangle$ is a central group extension and $g$ is not in $\Omega(\alpha) = \{\, [\widetilde{g}_1, \widetilde{g}_2] \mid \widetilde{g}_1, \widetilde{g}_2 \in G \text{ and } \alpha([\widetilde{g}_1, \widetilde{g}_2]) = 1 \,\}$. Hence by Theorem 4.1, if we set $\pi := G/\langle g\rangle$, we obtain examples of finite 2-groups of order $\leq 128$ with $SK_1(\widehat{\mathbb{Z}}_2\pi) \neq 0$. The function also determines if the extra condition in Theorem 4.3 is satisfied.

```
Some_quotients_with_nonzero_SK1:=function(G)
local List_of_commutators_in_G, ZG_cap_CommG, g, h, List_of_subgroups,
List_with_extra_cond, extra_condition_holds, id_gr, h_tilde, h_tilde_set,
hom_from_cover, G_quotient_g, there_exist_lift;
List_of_commutators_in_G:=[];
for g in G do
 for h in G do
List_of_commutators_in_G:=Union(List_of_commutators_in_G,[g^(-1)*h^(-1)*g*h]);
 od;
od;
List_of_subgroups:=[];
List_with_extra_cond:=[];
ZG_cap_CommG:=Intersection(Center(G),CommutatorSubgroup(G,G));
for g in ZG_cap_CommG do
 if not(Order(g)=2) then continue; fi;
 if g in List_of_commutators_in_G then continue; fi;
 id_gr:=IdGroup(G/Group(g));
 if id_gr in List_of_subgroups then continue; fi;
 Add(List_of_subgroups,IdGroup(G/Group(g)));
 hom_from_cover:=NaturalHomomorphismByNormalSubgroup(G,Group(g));
 G_quotient_g:=Image(hom_from_cover);
 extra_condition_holds:=true;
 for h in G_quotient_g do
  if h^(-1) in h^G_quotient_g then
   h_tilde_set:=PreImages(hom_from_cover,h);
       there_exist_lift:=false;
       for h_tilde in h_tilde_set do
        if h_tilde^(-1) in h_tilde^G then
```

```
         there_exist_lift:=true;
         break;
        fi;
       od;
       extra_condition_holds := extra_condition_holds and there_exist_lift;
      fi;
  od;
  if extra_condition_holds then
      List_with_extra_cond:=Union(List_with_extra_cond,[id_gr]);
  fi;
od;
return [List_of_subgroups,List_with_extra_cond];
end;;
```

LISTING 4. Function listing quotients $\pi$ of a group $G$ such that $SK_1(\widehat{\mathbb{Z}}_2\pi) \neq 0$

Then one can find the list L1 of all 2-groups $\pi$ of order up to 128, which satisfy the criterion of Theorem 4.1, and hence have $SK_1(\widehat{\mathbb{Z}}_2\pi)$ is nonzero, The list L2 contains those groups in L1 which also satisfy the extra condition in Theorem 4.3.

```
L1:=[];
L2:=[];
List_of_2_covers:=[];
for size_G in [8,16,32,64,128,256] do
for index_G in [1..NumberSmallGroups( size_G )] do
 G:=SmallGroup(size_G,index_G);
 Result_of_func:=Some_quotients_with_nonzero_SK1(G);
 List_with_SK1_nonzero:=Result_of_func[1];
 List_which_satisfy_extra_condition:=Result_of_func[2];
 if Size(List_with_SK1_nonzero)=0 then continue; fi;
 Add(List_of_2_covers,[size_G,index_G,List_with_SK1_nonzero]);
 L1:=Union(L1,List_with_SK1_nonzero);
 L2:=Union(L2,List_which_satisfy_extra_condition);
od;
od;
```

LISTING 5. Code for generating the lists L1 and L2 as above

Here we display the results of the above code.

```
gap> L1;
[ [ 64, 149 ], [ 64, 150 ], [ 64, 151 ], [ 64, 170 ], [ 64, 171 ], [ 64, 172 ],
[ 64, 177 ], [ 64, 178 ], [ 64, 182 ], [ 128, 36 ], [ 128, 37 ], [ 128, 38 ],
[ 128, 39 ], [ 128, 40 ], [ 128, 41 ], [ 128, 138 ], [ 128, 139 ], [ 128, 144 ],
[128, 145 ], [ 128, 227 ], [ 128, 228 ], [ 128, 229 ], [ 128, 242 ], [128, 243 ],
[ 128, 244 ], [ 128, 245 ], [ 128, 246 ],[ 128, 247 ], [ 128, 265 ], [ 128, 266 ],
[ 128, 267 ], [ 128, 268 ], [128, 269 ], [ 128, 287 ], [ 128, 288 ], [ 128, 289 ],
[ 128, 290 ], [ 128, 291 ], [ 128, 292 ], [ 128, 293 ], [ 128, 301 ], [ 128, 324 ],
[ 128, 325 ], [ 128, 326 ], [ 128, 417 ], [ 128, 418 ], [ 128, 419 ], [ 128, 420 ],
[ 128, 421 ], [ 128, 422 ], [ 128, 423 ], [ 128, 424 ], [ 128, 425 ], [ 128, 426 ],
[ 128, 427 ], [ 128, 428 ], [ 128, 429 ], [ 128, 430 ], [ 128, 431 ], [ 128, 432 ],
[ 128, 433 ], [ 128, 434 ], [ 128, 435 ], [ 128, 436 ], [ 128, 446 ], [ 128, 447 ],
```

```
[ 128, 448 ], [ 128, 449 ], [ 128, 450 ], [ 128, 451 ], [ 128, 452 ], [ 128, 453 ],
[ 128, 454 ], [ 128, 455 ], [ 128, 541 ], [ 128, 543 ], [ 128, 568 ], [ 128, 570 ],
[ 128, 579 ], [ 128, 581 ], [ 128, 626 ], [ 128, 627 ], [ 128, 629 ], [ 128, 667 ],
[ 128, 668 ], [ 128, 670 ], [ 128, 675 ], [ 128, 676 ], [ 128, 678 ], [ 128, 691 ],
[ 128, 692 ], [ 128, 693 ], [ 128, 695 ], [ 128, 703 ], [ 128, 704 ], [ 128, 705 ],
[ 128, 707 ], [ 128, 724 ], [ 128, 725 ], [ 128, 727 ], [ 128, 950 ], [ 128, 951 ],
[ 128, 952 ], [ 128, 975 ], [ 128, 976 ], [ 128, 977 ], [ 128, 982 ], [ 128, 983 ],
[ 128, 987 ], [ 128, 1345 ], [ 128, 1346 ], [ 128, 1347 ], [ 128, 1348 ],
[ 128, 1349 ], [ 128, 1350 ], [ 128, 1351 ], [ 128, 1352 ], [ 128, 1353 ],
[ 128, 1354 ], [ 128, 1355 ], [ 128, 1356 ], [ 128, 1357 ], [ 128, 1358 ],
[ 128, 1359 ], [ 128, 1360 ], [ 128, 1361 ], [ 128, 1362 ], [ 128, 1363 ],
[ 128, 1364 ], [ 128, 1365 ], [ 128, 1366 ], [ 128, 1367 ], [ 128, 1368 ],
[ 128, 1369 ], [ 128, 1370 ], [ 128, 1371 ], [ 128, 1372 ], [ 128, 1373 ],
[ 128, 1374 ], [ 128, 1375 ], [ 128, 1376 ], [ 128, 1377 ], [ 128, 1378 ],
[ 128, 1379 ], [ 128, 1380 ], [ 128, 1381 ], [ 128, 1382 ], [ 128, 1383 ],
[ 128, 1384 ], [ 128, 1385 ], [ 128, 1386 ], [ 128, 1387 ], [ 128, 1388 ],
[ 128, 1389 ], [ 128, 1390 ], [ 128, 1391 ], [ 128, 1392 ], [ 128, 1393 ],
[ 128, 1394 ], [ 128, 1395 ], [ 128, 1396 ], [ 128, 1397 ], [ 128, 1398 ],
[ 128, 1399 ], [ 128, 1544 ], [ 128, 1545 ], [ 128, 1546 ], [ 128, 1547 ],
[ 128, 1548 ], [ 128, 1549 ], [ 128, 1550 ], [ 128, 1551 ], [ 128, 1552 ],
[ 128, 1553 ], [ 128, 1554 ], [ 128, 1555 ], [ 128, 1556 ], [ 128, 1557 ],
[ 128, 1558 ], [ 128, 1559 ], [ 128, 1560 ], [ 128, 1561 ], [ 128, 1562 ],
[ 128, 1563 ], [ 128, 1564 ], [ 128, 1565 ], [ 128, 1566 ], [ 128, 1567 ],
[ 128, 1568 ], [ 128, 1569 ], [ 128, 1570 ], [ 128, 1571 ], [ 128, 1572 ],
[ 128, 1573 ], [ 128, 1574 ], [ 128, 1575 ], [ 128, 1576 ], [ 128, 1577 ],
[ 128, 1783 ], [ 128, 1784 ], [ 128, 1785 ], [ 128, 1786 ], [ 128, 1864 ],
[ 128, 1865 ], [ 128, 1866 ], [ 128, 1867 ], [ 128, 1880 ], [ 128, 1881 ],
[ 128, 1882 ], [ 128, 1893 ], [ 128, 1894 ], [ 128, 1903 ], [ 128, 1904 ],
[ 128, 1924 ], [ 128, 1925 ], [ 128, 1926 ], [ 128, 1927 ], [ 128, 1928 ],
[ 128, 1929 ], [ 128, 1945 ], [ 128, 1946 ], [ 128, 1947 ], [ 128, 1948 ],
[ 128, 1949 ], [ 128, 1950 ], [ 128, 1951 ], [ 128, 1966 ], [ 128, 1967 ],
[ 128, 1968 ], [ 128, 1969 ], [ 128, 1970 ], [ 128, 1971 ], [ 128, 1972 ],
[ 128, 1983 ], [ 128, 1984 ], [ 128, 1985 ], [ 128, 1986 ], [ 128, 1987 ],
[ 128, 1988 ] ]
gap> L2;
[ [ 128, 287 ], [ 128, 288 ], [ 128, 290 ], [ 128, 568 ], [ 128, 579 ], [ 128, 667 ],
[ 128, 668 ], [ 128, 670 ], [ 128, 676 ], [ 128, 692 ], [128, 693 ], [ 128, 703 ],
[ 128, 704 ], [ 128, 725 ], [ 128, 1375 ], [ 128, 1376 ], [ 128, 1377 ],
[ 128, 1547 ], [ 128, 1549 ], [ 128, 1576 ] ]
```

Listing 6. The output of above listing for L1 and L2

Then one can compute the actual $SK_1(\widehat{\mathbb{Z}}_2\pi)$ by the following function.

```
SK1:=function(G)
local hom_from_SC_to_G,SC,K,Comm_SC,List_of_comm,s1,s2,h,SC_b,K_b,iso;
hom_from_SC_to_G:=EpimorphismSchurCover(G);
SC_b:=Source(hom_from_SC_to_G);
K_b:=Kernel(hom_from_SC_to_G);
iso := IsomorphismPermGroup( SC_b );
SC := Image( iso );
K:=Image( iso , K_b);
```

```
Comm_SC:=CommutatorSubgroup(SC,SC);
List_of_comm:=[];
for s1 in SC do
for s2 in SC do
h:=s1*s2*(s1^(-1))*(s2^(-1));
if not(h in List_of_comm) and (h in K) then
Add(List_of_comm,h);
fi;
od;
od;
return IdGroup(Intersection(Comm_SC,K)/Group(List_of_comm));
end;;
```

Listing 7. Function that computes $SK_1(\widehat{\mathbb{Z}}_2\pi)$ for a given group $\pi$

## Appendix C. Some GAP code to test Conjecture 6.1

```
LoadPackage("HAP");
for n in [2,4,8,16,32] do
 for i in [1..NumberSmallGroups( n )] do
  G:=SmallGroup(n,i);
  if IsAbelian(G) then continue; fi;
  Sb_G:=List(ConjugacyClassesSubgroups(G), Representative);
  Normal_Sb_G:=Filtered(Sb_G, x -> IsNormal(G,x));
  Cyclic_Quo:=Filtered(Normal_Sb_G, x -> IsCyclic(G/x));
  conj_G:=List(ConjugacyClasses(G),Representative);
  for N in Cyclic_Quo do
   if  Size(G)<4*Size(N) then continue; fi;
   W:=Filtered(Cyclic_Quo, x -> IsSubgroup(x,N) and Size(G)=2*Size(x))[1];
   T:=Filtered(Cyclic_Quo, x -> IsSubgroup(x,N) and Size(x)=2*Size(N))[1];
   p:=NaturalHomomorphismByNormalSubgroup(G,N);
   G_N:=Image(p);
   poi:=GroupHomomorphismByFunction(W,G_N,x->Image(p,x));;
   H_2_poi:=GroupHomology(poi,2,2);
   if Size(Image(H_2_poi))=1 then continue; fi;
   H_4_poi:=GroupHomology(poi,4,2);
   if Size(Image(H_4_poi))>1 then continue; fi;
   H_4_p:=GroupHomology(p,4,2);
   if Size(Image(H_4_p))=1 then continue; fi;
   conj_cl:=Filtered(conj_G,x -> (x in T) and not(x in N));
   Print("\n","A sequence N < T <= W < G = SG[",n,",",i,"]",
     " that satisfy hypothesis of conjecture is \n", [N,T,W,G],"\n",
     "For this sequence the number of conjugacy classes in T-N is ",
     Size(conj_cl),"\n", "They are represented by the elements: ",
     conj_cl,"\n");
  od;
 od;
od;
```

LISTING 8. Code for finding examples that satisfy the hypothesis of the conjecture.

The GAP code above shows that smallest groups that contains such a increasing sequence are $SG[32,8]$, $SG[64,13]$ and $SG[64,14]$. In these groups, each such increasing sequence of subgroups $T-N$ has an odd number of $\pi$-conjugacy classes. However, this is not new information about $\kappa_4'$ for these groups, since $H^1(\mathrm{Wh}'(\widehat{\mathbb{Z}}_2\pi)) = 0$.

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