ADDENDUM: HOMOTOPY SELF-EQUIVALENCES OF 4-MANIFOLDS

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Abstract. We add some details for a claim in the calculation of the braid diagram of [1] for 4-manifolds with finite odd order fundamental group.

1. A map in the braid diagram

In [1, §4] we claimed that there is an exact sequence

$$1 \to \mathcal{S}^h(M \times I, \partial) \to \mathcal{H}(M) \to \mathrm{Aut}_\bullet(M)$$

where $\mathcal{S}^h(M \times I, \partial)$ denotes the structure group of smooth or topological manifold structures on $M \times I$, relative to the given structure on $\partial(M \times I)$, and $\mathcal{H}(M)$ denotes the group of h-cobordisms from M to M. In particular, the first map was claimed to be an injection without any assumption on $\pi_1(M, x_0)$.

Later we realized that this map is not injective in general, due to the existence of elements in the structure set given by homotopy self-equivalences of $(M \times I, \partial)$ with non-trivial normal invariant. The purpose of this Addendum is to supply the details for the injectivity, under the assumption that $\pi_1(M)$ is a finite group of odd order.

Example 1.1. For $M = S^1 \times S^3$, consider the smoothing obtained by the pinch map to $M \times I \vee S^5 \to M \times I$ given by the identity and $\eta^2 \in \pi_5(S^3)$. This is a non-trivial element in the set of smoothings, as one can see by taking a codimension three Arf invariant. However, this element goes to zero in $\mathcal{H}(M)$.

Our main use of this claim was to compute a certain map in our braid diagram.

Lemma 1.2 ([1, Lemma 4.1])**.** *Suppose that $\pi_1(M)$ is finite of odd order. There is an injection $H_1(M; \mathbf{Z}) \to \widetilde{\mathcal{H}}(M, w_2)$, factoring through the map*

$$\Omega_5(M\langle w_2\rangle) \to \widetilde{\mathcal{H}}(M, w_2)$$

from the braid diagram.

With this result established, the diagram in the proof of [1, Lemma 4.1] now shows that the structure set does indeed inject into $\mathcal{H}(M)$, for odd order fundamental groups. This was used in the statement of one of our main results:

Theorem 1.3 ([1, Theorem B])**.** *Let M^4 be a connected, closed, oriented smooth (or topological) manifold of dimension 4. If $\pi_1(M, x_0)$ has odd order, then there is a short exact sequence of groups:*

$$1 \to \mathcal{S}^h(M \times I, \partial) \to \mathcal{H}(M) \to \mathrm{Isom}([\pi_1, \pi_2, k_M, s_M]) \to 1$$

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where the normal subgroup $\mathcal{S}^h(M \times I, \partial)$ is abelian and is determined up to extension by the short exact sequence

$$0 \to \tilde{L}_6(\mathbf{Z}[\pi_1(M, x_0)]) \to \mathcal{S}^h(M \times I, \partial) \to H_1(M; \mathbf{Z}) \to 0$$

of groups and homomorphisms.

In the statement of Lemma 1.2 for the spin case, the bordism groups of the normal 2-type are just the spin bordism groups $\Omega_*^{Spin}(M)$ or $\Omega_*^{Spin}(B)$, where $c\colon M \to B$ is the 2-type of M. We first discuss this case.

Lemma 1.4. *Suppose that $\pi := \pi_1(M)$ is a finite odd order group, $w_2(M) = 0$, and $u\colon M \to K(\pi, 1)$ is the classifying map for the universal covering.*

(i) *The induced map $u_*\colon \Omega_5^{Spin}(B) \to \Omega_5^{Spin}(K(\pi, 1))$ is an injection.*
(ii) $\Omega_5^{Spin}(K(\pi, 1)) = H_1(M) \oplus H_5(\pi)$.
(iii) $\Omega_5^{Spin}(M) = \mathbf{Z}/2 \oplus H_1(M)$.
(iv) *the map $\Omega_5^{Spin}(M) \to \Omega_5^{Spin}(B) \to \Omega_5^{Spin}(K(\pi, 1))$ is the projection onto the subgroup $H_1(M) \subset \Omega_5^{Spin}(K(\pi, 1))$.*

Proof. This follows from the calculation of the bordism group via the Atiyah-Hirzebruch spectral sequence as in [1, Proposition 4.5]. For the differential $d_5\colon E_{6,0} \to E_{1,4}$ note that the map $H_1(\pi) \to \bigoplus\{H_1(C) \,|\, \pi \twoheadrightarrow C \text{ cyclic quotient}\}$ is injective, but $H_6(C) = 0$. □

The proof of Lemma 1.2. According to the braid diagram, we must compute the image of the map $\pi_1(\mathcal{E}_\bullet(B)) \to \Omega_5^{Spin}(B)$ defined by sending the adjoint map $h\colon B \times S^1 \to B$, for a representative of an element in $\pi_1(\mathcal{E}_\bullet(B))$, to the bordism element $[M \times S^1, h \circ (c \times id)]$. We use the null-bordant spin structure on the S^1 factor (see [1, p. 153]). By the Lemma above, it is enough to consider the image of such an element in $\Omega_5^{Spin}(K(\pi, 1))$. But the reference map composed with $u\colon B \to K(\pi, 1)$ is determined just by the map on fundamental groups $M \times S^1 \to B \to K(\pi, 1)$. Since the adjoint map restricted to $S^1 \vee M$ is just projection onto M, by the base-point preserving conditions on $\pi_1(\mathcal{E}_\bullet(B))$, we see that the image of our element in $\Omega_5^{Spin}(K(\pi, 1))$ is just $[M \times S^1, u \circ p_1]$, where $p_1\colon M \times S^1 \to M$ is the first fator projection. Since we have used the null-bordant spin structure on the S^1-factor, such an element is zero in $\Omega_5^{Spin}(K(\pi, 1))$ and hence also in $\Omega_5^{Spin}(B)$. □

In the non-spin case, note that the normal 1-type is $K(\pi, 1) \times BSO$, so we have a natural map $u_*\colon \Omega_5(B\langle w_2\rangle) \to \Omega_5^{SO}(K(\pi, 1))$. In comparing the spectral sequences, the map on the $E_{1,4}$ terms is multiplication by 16, and hence an isomorphism on $H_1(M)$.

References

[1] I. Hambleton and M. Kreck, *Homotopy self-equivalences of 4-manifolds*, Math. Z. **248** (2004), 147–172; Erratum: Math. Z. **262** (2009), 473–474.

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