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On the classification of topological 4-manifolds with finite fundamental group: corrigendum

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Abstract We correct a mistake in Lemma 2.3 of our paper *On the classification of topological* 4*-manifolds with finite fundamental group*. Math. Ann. 280:85–104 (1988). The main results of the paper are not affected.

1 Introduction

A key step in the proof of Theorem A in our 1988 paper [1] was the following result about Whitehead's Γ-functor.

Theorem 2.1 (Hambleton and Kreck [1, p. 91]) *Let π be a finite group. If L is either a finitely generated projective Λ-module, I or I^*, then $\Gamma(L) \otimes_\Lambda \mathbb{Z}$ is torsion free as an abelian group.*

In this statement $\Lambda = \mathbb{Z}[\pi]$ is the integral group ring, and I denotes the augmentation ideal, defined as the kernel $I = \ker(\Lambda \to \mathbb{Z})$ of the augmentation map.

The authors are grateful to Daniel Kasprowski and Peter Teichner, who recently pointed out a mistake in one part of the proof of this result (and outlined a correction).

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We incorrectly asserted in [1, Lemma 2.3] that

$$\Gamma(\Lambda) \cong \Gamma(I^*) \oplus \Lambda$$

and used this claim as input into the proof of [1, Theorem 2.1]. The mistake arose from our claim that the kernel $\ker(\Gamma(\Lambda^*) \to \Gamma(I^*))$ of the induced surjective map of duals was Λ-free. However, we do still have the following property:

Lemma 1.1 *Let I be the augmentation ideal of the integral group ring for a finite group. Then $\Gamma(I^*) \otimes_\Lambda \mathbb{Z}$ is torsion free as an abelian group.*

This is the result actually needed for the proof of [1, Theorem A].

2 The correction

Let π be a finite group and I^* be the dual of the augmentation ideal. We have an exact sequence

$$0 \to L \to \Gamma(\Lambda) \to \Gamma(I^*) \to 0 \tag{2.1}$$

and the kernel L has free $\mathbb{Z}$-basis given in [1, p. 91] by

$$N \otimes N \quad \text{and} \quad N \otimes g + g \otimes N, \qquad \text{for all } g \in \pi,\ g \neq 1,$$

where $N \in \mathbb{Z}[\pi]$ is the sum of the group elements. There is an exact sequence in Tate cohomology

$$0 \to \widehat{H}^{-1}(\pi; \Gamma(I^*)) \to H_0(\pi; \Gamma(I^*)) \to H^0(\pi; \Gamma(I^*)) \to \widehat{H}^0(\pi; \Gamma(I^*)) \to 0$$

and $H_0(\pi; \Gamma(I^*))$ is $\mathbb{Z}$-torsion free if and only if $\widehat{H}^{-1}(\pi; \Gamma(I^*)) = 0$. We compute this term via the Tate cohomology sequence

$$\widehat{H}^{-1}(\pi; \Gamma(\Lambda)) \to \widehat{H}^{-1}(\pi; \Gamma(I^*)) \to \widehat{H}^0(\pi; L) \to \widehat{H}^0(\pi; \Gamma(\Lambda)),$$

and note first that [1, Lemma 2.2] gives the structure of $\Gamma(\Lambda)$ as a Λ-module, namely the direct sum of a free Λ-module and summands of the form $\Lambda/\Lambda(1-g)$, where $1 \neq g \in \pi$, $g^2 = 1$. We have $\widehat{H}^{-1}(\pi; \Lambda) = 0$, and

$$\widehat{H}^{-1}(\pi; \Lambda/\Lambda(1-g)) = \widehat{H}^{-1}(\pi; \mathbb{Z}[\pi/H]) = \widehat{H}^{-1}(H; \mathbb{Z}) = 0,$$

by Shapiro's Lemma, since $\Lambda/\Lambda(1-g) = \mathbb{Z}[\pi/H]$, where $H = \langle g \rangle \cong \mathbb{Z}/2$. Therefore $\widehat{H}^{-1}(\pi; \Gamma(\Lambda)) = 0$.

Daniel Kasprowski and Peter Teichner pointed out that we actually have a short exact sequence

$$0 \to \Lambda \to L \to \mathbb{Z}/2 \to 0$$

in which the free submodule is generated by any element $g \otimes N + N \otimes g \in L$, and the cokernel is generated by the image of $N \otimes N$. We apply Tate cohomology to this sequence to obtain

$$\widehat{H}^0(\pi; L) \cong \widehat{H}^0(\pi; \mathbb{Z}/2),$$

which is zero when π has odd order and isomorphic to $\mathbb{Z}/2$ if π has even order.

If π has odd order we are done, so we assume that π has even order. It follows that the π-invariant element $N \otimes N$ represents the non-zero element of order two in $\widehat{H}^0(\pi; L) = \mathbb{Z}/2$, since the composite $\mathbb{Z} \to L \to \mathbb{Z}/2$ (with $1 \mapsto N \otimes N$) induces as surjection on $\widehat{H}^0$, and

$$2(N \otimes N) = \sum_{g \in \pi} N \otimes g + g \otimes N \in \Gamma(\Lambda)$$

is a norm in L. Moreover, we will show that $N \otimes N \in \Gamma(\Lambda)$ is not a norm by using the direct sum decomposition

$$\widehat{H}^0(\pi; \Gamma(\Lambda)) = \bigoplus_{1 \neq g, g^2=1} \widehat{H}^0(\pi; \Lambda/\Lambda(1-g)) = \bigoplus \{\mathbb{Z}/2 \mid 1 \neq g \in \pi, g^2 = 1\}.$$

The formula

$$N \otimes N = N(1 \otimes 1) + \sum_{1 \neq g \in \pi} N(1 \otimes g)$$

and the expression $N(1 \otimes g) = N(g^{-1} \otimes 1)$ show that (modulo norms) the right-hand side is the sum of terms of the form

$$\sum_{i=1}^{k} x_i(g \otimes 1 + 1 \otimes g), \qquad 1 \neq g \in \pi, \quad g^2 = 1,$$

where $|\pi/H| = k$ and $\{x_1, \ldots, x_k\}$ denotes a set of coset representatives for H in π. Each of these terms maps to $\sum \bar{x}_k \in \Lambda/\Lambda(1-g)$, and represents the generator of

$$\widehat{H}^0(\pi; \Lambda/\Lambda(1-g)) = \widehat{H}^0(\pi; \mathbb{Z}[\pi/H]) = \widehat{H}^0(H; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Hence the image of $N \otimes N$ is non-zero under the map $\widehat{H}^0(\pi; L) \to \widehat{H}^0(\pi; \Gamma(\Lambda))$. This completes the proof of Lemma 1.1. □

We can also determine the correct Λ-module structure of L.

Lemma 2.2 *Let* $L = \ker(\Gamma(\Lambda^*) \to \Gamma(I^*))$, *where* $\Lambda = \mathbb{Z}[\pi]$ *is the integral group ring of a finite group*, *and* $I \subset \mathbb{Z}[\pi]$ *is the augmentation ideal. Then* $L \cong \langle I, 2\rangle^*$.

Proof As noted above, we have an exact sequence

$$0 \to \Lambda \to L \to \mathbb{Z}/2 \to 0,$$

and the given $\mathbb{Z}$-basis for L shows that L is a free abelian group. Therefore the extension describing L is non-split. On the other hand, by dualizing the exact sequence

$$0 \to \langle I, 2\rangle \to \Lambda \to \mathbb{Z}/2 \to 0$$

we obtain the exact sequence

$$0 \to \Lambda \to \langle I, 2\rangle^* \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}/2, \Lambda) \to 0.$$

However, we can use the sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$ to determine that

$$\mathrm{Ext}^1_\Lambda(\mathbb{Z}/2, \Lambda) \cong \mathbb{Z}/2.$$

Therefore, the two exact sequences are congruent, and $L \cong \langle I, 2\rangle^*$. □

Remark 2.3 The exact sequence (2.1) is split if and only if π has odd order. Indeed, if π has odd order, then the module $\langle I, 2\rangle^*$ is cohomologically trivial, and hence projective. Therefore, if π has odd order we have

$$\Gamma(\Lambda) \cong \Gamma(I^*) \oplus \langle I, 2\rangle^*.$$

If π has even order, then one can check that the restriction of the sequence (2.1) to any subgroup of order two is non-split.

Reference

1. Hambleton, I., Kreck, M.: On the classification of topological 4-manifolds with finite fundamental group. Math. Ann. **280**, 85–104 (1988)