ASSEMBLY MAPS FOR GROUP EXTENSIONS IN K-THEORY AND L-THEORY WITH TWISTED COEFFICIENTS

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Abstract. In this paper we show that the Farrell-Jones isomorphism conjectures are inherited in group extensions for assembly maps in K-theory and L-theory with twisted coefficients.

Introduction

Under what assumptions are the Farrell-Jones isomorphism conjectures inherited by group extensions or subgroups? We will formulate a version of the standard conjectures (see Farrell-Jones [10]) with twisted coefficients in an additive category, and then study these questions via the continuously controlled assembly maps of [11, §7]. A formulation using the Davis-Lück assembly maps [9] has already been given by Bartels and Reich [4], and applied there to show inheritance by subgroups. Recall that the Farrell-Jones conjecture in algebraic K-theory asserts that certain "assembly" maps

$$H_n^G(E_{\mathcal{VC}}G; \mathbb{K}_R) \to K_n(RG)$$

are isomorphisms, for a given ring R, and all $n \in \mathbf{Z}$. Here the space $E_{\mathcal{VC}}G$ is the universal G-CW-complex for G-actions with virtually cyclic isotropy, and the left-hand side denotes equivariant homology with coefficients in the non-connective K-theory spectrum for the ring R.

Theorem A. *Let $N \to G \xrightarrow{\pi} K$ be a group extension, where $N \triangleleft G$ is a normal subgroup, and K is the quotient group. Let $\mathcal{A}$ be an additive category with G-action. Suppose that*

(i) *The group K satisfies the Farrell-Jones conjecture in algebraic K-theory, with twisted coefficients in any additive category with K-action.*
(ii) *Every subgroup of G containing N as a subgroup, with virtually cyclic quotient, satisfies the Farrell-Jones conjecture in algebraic K-theory, with twisted coefficients in $\mathcal{A}$.*

Then the group G satisfies the Farrell-Jones conjecture in algebraic K-theory, with twisted coefficients in $\mathcal{A}$.

This is a special case of a more general result (see Theorem 4.7). The same statement holds for algebraic L-theory as well, where the coefficient categories are additive categories with involution. The corrresponding result for the Baum-Connes conjecture was

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obtained by Oyono-Oyono [12], and our proof follows the outline given there. One of the main points is that the most effective methods known for proving the standard Farrell-Jones conjectures (for particular groups G) also work for the twisted coefficient versions (compare [1], [3], [6], [7], [15], [16], and [17]). An immediate corollary to Theorem A is the following.

Corollary (Corollary 4.10)**.** *The Farrell-Jones conjecture with twisted coefficients is true for $G_1 \times G_2$ if and only if it is true for G_1, G_2, and every product $V_1 \times V_2$, where $V_1 \leq G_1$ and $V_2 \leq G_2$ are virtually cyclic subgroups.*

The fibered isomorphism conjecture of Farrell and Jones [10] for a group G and a ring R asserts that for every group homomorphism, $\phi\colon H \to G$, the assembly map for H relative to the family generated by the subgroups $\phi^{-1}(V)$, $V \subset G$ virtually cyclic, is an isomorphism. This conjecture implies the Farrell-Jones conjecture and has better inheritance properties. For example, the fibered version of our Theorem A is also true (see, for example, [2, Section 2.3]). The following result shows that the Farrell-Jones conjecture with twisted coefficients implies the Fibered Farrell-Jones conjecture.

Theorem B. *Suppose that $\phi\colon H \to G$ is a group homomorphism. Then the Farrell-Jones conjecture for G, with twisted coefficients in any G-category, implies that the assembly map for H relative to the family generated by the subgroups $\phi^{-1}(V)$, $V \subset G$ virtually cyclic, is an isomorphism with twisted coefficients in any H-category.*

The corresponding result for the Davis-Lück assembly maps was obtained by Bartels-Reich [4], who also pointed out a number of applications of the assembly map with twisted coefficients, including the study the K- and L-theory of twisted group rings (see also Example 4.8 and Example 4.9 below). One can check as in [11] that those assembly maps are equivalent to the continuously controlled assembly maps used in this paper.

1. Assembly via Controlled Categories

The controlled categories of Pedersen [13], Carlsson-Pedersen [6], [8] are our main tool for identifying various different assembly maps. We will recall the definition of these categories, and then the usual assembly maps are obtained by applying functors

$$H\colon G\text{-}CW\text{-Complexes} \to \mathit{Spectra}$$

as described in [11]. We will extend the earlier definitions in order to allow an additive category as coefficients, instead of just working with modules over a ring R. A formulation for assembly maps with coefficients in the setting of [9] has already been given in [4]. Following the method of [11], one can check that the two different descriptions give the same assembly maps.

Let G be any discrete group, and let X be a G-CW complex (we will use a left G-action). Subspaces of the form $G\cdot D \subset X$, with D compact in X, are called *G-compact* subspaces of X. More generally, a subspace whose closure has this form is called relatively G-compact. A *resolution* of X is a pair $(\overline{X}, p)$, where $\overline{X}$ is a free G-CW complex and $p\colon \overline{X} \to X$ is a continuous G-equivariant map, such that for every G-compact set $G \cdot D \subset X$ there exists a G-compact set $G \cdot \overline{D} \subset \overline{X}$ such that $p(G \cdot \overline{D}) = G \cdot D$. The notion of resolution comes from [13], and was developed further in [1, §3]. The original example was $\overline{X} = G \times X$, with the diagonal G-action and first factor projection.

Let $\mathcal{A}$ be an additive category with involution, and suppose that $\mathcal{A}$ has a right G-action compatible with the involution. This is a collection of covariant functors $\{g^*\colon \mathcal{A} \to \mathcal{A}, \forall g \in G\}$, such that $(g \circ h)^* = h^* \circ g^*$ and $e^* = id$. We require that the functors g^* commute with the involution $*\colon \mathcal{A} \to \mathcal{A}$ (an involution is a contravariant functor with square the identity).

Definition 1.1. Let (Z, X) be a G-CW pair, where X is a closed G-invariant subspace. Let $Y = Z - X$, and fix a resolution $p\colon \overline{Z} \to Z$, whose restriction to Y is denoted $\overline{Y}$. The category $\mathcal{D}(Z, X; \mathcal{A})$ has objects $A = (A_y)$ consisting of a collection of objects of $\mathcal{A}$, indexed by $y \in \overline{Y}$, and morphisms $\phi\colon A \to B$ consisting of collections $\phi = (\phi_y^z)$ of morphisms $\phi_y^z\colon A_y \to B_z$ in $\mathcal{A}$, indexed by $y, z \in \overline{Y}$, satisfying:

(i) the support $\{y \in \overline{Y} \mid A_y \neq 0\}$ is *locally finite* in $\overline{Y}$, and relatively G-compact in $\overline{Z}$.

(ii) for each morphism $\phi\colon A \to B$, and for each $y \in \overline{Y}$, the set $\{z \mid \phi_y^z \neq 0 \text{ or } \phi_z^y \neq 0\}$ is finite.

(iii) the morphisms $\phi\colon A \to B$ are *continuously controlled* at $X \subset Z$. For every $x \in X$, and for every G_x-invariant neighbourhood U of x in Z, there is a G_x-invariant neighbourhood V of x in Z so that $\phi_y^z = 0$ and $\phi_z^y = 0$ whenever $p(y) \in (Y - U)$ and $p(z) \in (V \cap U \cap Y)$.

If $X = \emptyset$, we use the shorter notation $\mathcal{D}(Z; \mathcal{A}) := \mathcal{D}(Z, \emptyset; \mathcal{A})$, and in this case the continous control condition (iii) on morphisms is vacuous. If S is a discrete left G-set, we denote by $\mathcal{D}_l(S \times Z, S \times X; \mathcal{A})$ the subcategory where the morphisms are S-level-preserving: $\phi_{(s,y)}^{(s',z)} = 0$ if $s \neq s' \in S$, for any $y, z \in Y$.

The category $\mathcal{D}(Z, X; \mathcal{A})$ is an additive category with involution, where the dual of A is given by $(A^*)_y = A_y^*$ for all $y \in \overline{Y}$. It depends functorially on the pair (Z, X) of G-CW complexes. The actions of G on $\mathcal{A}$ and Z induce a right G-action on $\mathcal{D}(Z, X; \mathcal{A})$. For $g \in G$, we set $(gA)_y = g^* A_{gy}$ and $(g\phi)_y^z = g^*(\phi_{gy}^{gz})$. The fixed subcategory will be denoted $\mathcal{D}^G(Z, X; \mathcal{A})$. If $G = \{e\}$ is the trivial group, we use the notation $\mathcal{D}^0(Z, X; \mathcal{A})$. We have not included the resolution $(\overline{Z}, p)$ in the notation, because two different resolutions give G-equivalent categories (see [1, Prop. 3.5]). We can compare these fixed subcategories to the equivariant category $\mathcal{B}_G(Z, X; R)$ defined in [11, §7].

Lemma 1.2. *There is an equivalence of categories $\mathcal{B}_G(Z, X; R) \simeq \mathcal{D}^G(Z, X; \mathcal{A})$, when $\mathcal{A}$ is the category of finitely-generated free R-modules.*

Proof. We define a functor $F\colon \mathcal{D}^G(Z, X; \mathcal{A}) \to \mathcal{B}_G(Z, X; R)$ by sending an object A to the free R-module $F(A)_y = \oplus_{g \in G_y} A_{(g,y)}$, for all $y \in Y$, with the obvious reference map to Y. Similarly, for a morphism $\phi\colon A \to B$, we define $F(\phi)_y^z = (\phi_{g,y}^{g',z})_{g,g' \in G}$, for all $y, z \in Y$. The verification that this definition makes sense will be left to the reader.

Conversely, we can define a functor $F'\colon \mathcal{B}_G(Z, X; R) \to \mathcal{D}^G(Z, X; \mathcal{A})$ on objects by decomposing an object $A = (A_y)$ of $\mathcal{B}_G(Z, X; R)$ as $A_y = \oplus_{g \in G_y} (A_y)_g$, since A_y is a finitely-generated free RG_y-module. Now we let $F'(A)_{(g,y)} = (A_y)_g$, for all $y \in Y$, $g \in G$, and on morphisms by letting $F'(\phi)^{g',z}_{g,y} = \phi^{g'z}_{gy}$. Again the verifications will be left to the reader (technically we should work with a category equivalent to $\mathcal{B}_G(Z, X; R)$, in which the objects are based: each $A = R[T]$, where T is a free G-set, and T is equipped with a reference map to $X \times [0, 1]$). □

For applications to assembly maps, we will let X be a G-CW complex and $Z = X \times [0, 1]$ so that $Y = X \times [0, 1)$. The category just defined will be denoted

$$\mathcal{D}^G(X \times [0, 1); \mathcal{A}) := \mathcal{D}^G(X \times [0, 1], X \times 1; \mathcal{A}) \ .$$

Let $\mathcal{D}^G(X \times [0, 1); \mathcal{A})_\emptyset$ denote the full subcategory of $\mathcal{D}^G(X \times [0, 1); \mathcal{A})$ with objects A such that the intersection with the closure

$$\operatorname{supp}(A) = \overline{\{(x, t) \in \overline{X} \times [0, 1) \mid A_{(x,t)} \neq 0\}} \cap (X \times 1)$$

is the empty set.

Example 1.3. If $\mathcal{A}$ is the additive category of finitely generated free R-modules, then $\mathcal{D}^G(X \times [0, 1); \mathcal{A})_\emptyset$ is equivalent to the category of finitely generated free RG-modules, for any G-CW complex X.

The quotient category will be denoted $\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}$, and we remark that this is a germ category (see [11, §7], [14], [6]). The objects are the same as in $\mathcal{D}^G(X \times [0, 1); \mathcal{A})$ but morphisms are identified if they agree close to $\overline{X} = \overline{X} \times 1$ (i.e. on the complement of a neighbourhood of $\overline{X} \times 0$). Here is a useful remark.

Lemma 1.4 ([11])**.** *Let S be a discrete left G-set. The forgetful functor*

$$\mathcal{D}^G_l(S \times X \times [0, 1); \mathcal{A})^{>0} \to \mathcal{D}^G(S \times X \times [0, 1); \mathcal{A})^{>0}$$

is an equivalence of categories.

Proof. In the germ category, every morphism has a representative which is level-preserving with respect to projection on S. □

The category $\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}$ is an additive category with involution, and we obtain a functor G-CW-Complexes $\to$ AddCat$^-$. The results of [5, 1.28, 4.2] now show that the functors F^λ: G-CW-Complexes $\to$ *Spectra* defined by

$$F^\lambda_G(X; \mathcal{A}) := \begin{cases} \mathbb{K}^{-\infty}(\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}) \\ \mathbb{L}^{-\infty}(\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}) \end{cases} , \tag{1.5}$$

where $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$ respectively, are G-homotopy invariant and G-excisive.

We can now extend the definition of the assembly maps to allow coefficients in any additive category with G-action.

Definition 1.6. We define the *continuously controlled assembly map with coefficients in* $\mathcal{A}$ to be the map $F^\lambda_G(X; \mathcal{A}) \to F^\lambda_G(\bullet; \mathcal{A})$.

From the methods of [11], the continuously controlled assembly map with coefficients is homotopy equivalent to the assembly map with coefficients constructed in [4]. The most important example to consider is when $X = E_{\mathcal{VC}}G$, in which case the *Farrell-Jones conjecture with coefficients* asserts that this assembly map is an equivalence. Given a discrete group G, a family of subgroups $\mathcal{F}$ of G, and coefficients $\mathcal{A}$, we will refer to

$$F^\lambda_G(E_{\mathcal{F}}G;\mathcal{A}) \to F^\lambda_G(\bullet;\mathcal{A})$$

as the $(G,\mathcal{F},\mathcal{A})$-*assembly map.*

By applying $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$ to the sequence of additive categories (with involution):

$$\mathcal{D}^G(X\times[0,1);\mathcal{A})_\emptyset \to \mathcal{D}^G(X\times[0,1);\mathcal{A}) \to \mathcal{D}^G(X\times[0,1);\mathcal{A})^{>0}$$

we obtain a fibration of spectra [6]. As in [11], we have the following description for the assembly map.

Theorem 1.7 ([11, §7])**.** *The continuously controlled assembly map*

$$F^\lambda_G(X;\mathcal{A}) \to F^\lambda_G(\bullet;\mathcal{A})$$

is homotopy equivalent to the connecting map

$$\lambda(\mathcal{D}^G(X\times[0,1);\mathcal{A})^{>0}) \to \Omega^{-1}\lambda(\mathcal{D}^G(X\times[0,1);\mathcal{A})_\emptyset$$

for $\lambda = \mathbb{K}^{-\infty}$ *or* $\lambda = \mathbb{L}^{-\infty}$*.*

See [11, §2] for the definition of homotopy equivalent functors from

$$G\text{-}CW\text{-Complexes} \to \mathit{Spectra},$$

and [9, 5.1] for the result that any functor $E\colon \mathbf{Or}(G) \to \mathit{Spectra}$ out of the orbit category of G may be extended uniquely (up to homotopy) to a functor $E_\%\colon G\text{-}CW\text{-Complexes} \to \mathit{Spectra}$ which is G-homotopy invariant and G-excisive. This will be our method for comparing functors. The *orbit category* $\mathbf{Or}(G)$ is the category with objects G/K, for K any subgroup of G, and the morphisms are G-maps.

2. Change of Coefficients

We will need some 'change of coefficient' properties for the categories defined in the last section. The first three properties are essentially just translations of [4, Proposition 2.8] into our language. The corresponding versions for additive categories with involution are needed to apply these change of coefficient functors to L-theory.

Definition 2.1. Let K and G be groups, $\mathcal{A}$ an additive category with commuting right K and G-actions, and S a K-G biset. Then, the category $\mathcal{D}^K(S;\mathcal{A})$ has a right G-action via $(g\cdot A)_y = g^*A_{yg^{-1}}$ and $(g\cdot\phi)^z_y = g^*\phi^{zg^{-1}}_{yg^{-1}}$, for all $y,z\in\overline{S}$. We will mostly use the level-preserving subcatetory $\mathcal{D}^K_l(S;\mathcal{A})$.

If T is a left G-set, and S is a transitive K-G biset (meaning that $K\backslash S/G$ is a point), we define a $K\times G$-action on $S\times T$ by the formula $(k,g)\cdot(s,t) := (ksg^{-1},gt)$ for all $(k,g)\in K\times G$ and all $(s,t)\in S\times T$. This action is used in the statements below.

Lemma 2.2. *Let T be a left G-set, and S be a transitive K-G biset. Then there is an additive functor*

$$F\colon \mathcal{D}_l^{K\times G}(S\times T\times[0,1);\mathcal{A})\to \mathcal{D}_l^{G}(T\times[0,1);\mathcal{D}_l^K(S;\mathcal{A}))$$

which induces an equivalence of categories

$$\mathcal{D}_l^{K\times G}(S\times T;\mathcal{A})\simeq \mathcal{D}_l^{G}(T;\mathcal{D}_l^K(S;\mathcal{A}))\ .$$

Proof. We will take the standard resolutions $\overline{S}=K\times S$, with elements denoted (k,s), for $k\in K$ and $s\in S$, and $\overline{T}=G\times T\times[0,1]$, with elements denoted (g,t), for $g\in G$ and $t\in T\times[0,1]$. Therefore

$$\overline{S}\times\overline{T}=K\times G\times S\times T\times[0,1]$$

is a resolution for $S\times T\times[0,1]$. We define the functor

$$F\colon \mathcal{D}_l^{K\times G}(S\times T\times[0,1);\mathcal{A})\to \mathcal{D}_l^{G}(T\times[0,1);\mathcal{D}_l^K(S;\mathcal{A}))$$

on objects by setting $B=F(A)_{(g,t)}$ in $\mathcal{D}_l^K(S;\mathcal{A})$ as the object $B=(B_{(k,s)})$ with $B_{(k,s)}=A_{(k,g,s,t)}$ in $\mathcal{A}$. We use a similar formula for morphisms:

$$\left(F(\phi)_{(g,t)}^{(g',t')}\right)_{(k,s)}^{(k',s')}=\phi_{(k,g,s,t)}^{(k',g',s',t')}$$

The proof that this is a well-defined functor is given in Section 5, where step $(5''')$ of the argument depends on the assumption that S is a transitive K-G biset.

Since $\mathcal{D}_l^{K\times G}(S\times T;\mathcal{A})\simeq\mathcal{D}_l^{K\times G}(S\times T\times[0,1);\mathcal{A})_\emptyset$ and $\mathcal{D}_l^{G}(T;\mathcal{D}_l^K(S;\mathcal{A}))\simeq\mathcal{D}_l^{G}(T\times[0,1);\mathcal{D}_l^K(S;\mathcal{A}))_\emptyset$, the functor F induces an additive functor

$$F\colon \mathcal{D}_l^{K\times G}(S\times T;\mathcal{A})\to \mathcal{D}_l^{G}(T;\mathcal{D}_l^K(S;\mathcal{A})).$$

On this subcategory, we define an inverse additive functor

$$F'\colon \mathcal{D}_l^{G}(T;\mathcal{D}_l^K(S;\mathcal{A}))\to\mathcal{D}_l^{K\times G}(S\times T;\mathcal{A})$$

on objects by setting $F'(B)_{(k,g,s,t)}=\left(B_{(g,t)}\right)_{(k,s)}$, and a similar formula for morphisms:

$$F'(\phi)_{(k,g,s,t)}^{(k',g',s',t')}=\left(\phi_{(g,t)}^{(g',t')}\right)_{(k,s)}^{(k',s')}$$

It is easy to check that F' is a well-defined functor. The functors F and F' are inverses, so give an equivalence of categories. $\square$

Corollary 2.3. *Let G and K be groups, and $\mathcal{A}$ be an additive category with commuting right K and G-actions,. Then*

$$\mathcal{D}^{K\times G}(\bullet;\mathcal{A})\simeq\mathcal{D}^{G}(\bullet;\mathcal{D}^K(\bullet;\mathcal{A}))\ .$$

Proof. We substitute $S=\bullet$ and $T=\bullet$ in the statement above. Note that morphisms are automatically level-preserving in this case. $\square$

Lemma 2.4. *Let K and G be groups, $\mathcal{A}$ an additive category with commuting right K and G-actions, and S a transitive K-G biset. Then, for any G-CW complex X, the functors*

$$F^{\lambda}_{K\times G}(S\times X;\mathcal{A})$$

and

$$F^{\lambda}_{G}(X;\mathcal{D}^K_l(S;\mathcal{A}))$$

are homotopy equivalent, where $\lambda = \mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$. Here $K\times G$ acts on $S\times X$ by the formula $(k,g)\cdot(x,s) := (ksg^{-1}, gx)$.

Proof. By [9, 5.1] it is enough to show that the two functors are G-homotopy invariant, G-excisive, and homotopy equivalent when restricted to the orbit category $\mathbf{Or}(G)$. For the first two properties, we apply [5, 1.28, 4.2]. For the last property, we follow the method of [11, §8]. Let $T = G/H$ and consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{K\times G}_l(S\times T\times[0,1);\mathcal{A})_{\emptyset} \arrow[r] \arrow[d, "F", "\simeq"'] & \mathcal{D}^{K\times G}_l(S\times T\times[0,1);\mathcal{A}) \arrow[r] \arrow[d, "F"] & \mathcal{D}^{K\times G}_l(S\times T\times[0,1);\mathcal{A})^{>0} \arrow[d, "F"] \\
\mathcal{D}^{G}_l(T\times[0,1);\mathcal{D}^K_l(S;\mathcal{A}))_{\emptyset} \arrow[r] & \mathcal{D}^{G}_l(T\times[0,1);\mathcal{D}^K_l(S;\mathcal{A})) \arrow[r] & \mathcal{D}^{G}_l(T\times[0,1);\mathcal{D}^K_l(S;\mathcal{A}))^{>0}
\end{tikzcd}$$

where the vertical maps are induced by the additive functors of Lemma 2.2. We apply $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$ to obtain fibrations of spectra. Note that λ applied to either of the middle two categories gives a spectrum with trivial homotopy groups (by an Eilenberg swindle). Therefore the first and third vertical maps induce a homotopy equivalence of spectra. Since the level-preserving condition is automatic on the germ categories, we are done. □

The next property allows us to divide out a normal subgroup in suitable circumstances.

Lemma 2.5. *Let N be a normal subgroup of G, and $\mathcal{A}$ be an additive category with right G-action such that N acts trivially. Let X be a G-CW complex such that N acts freely on X. Then there is an additive functor*

$$\mathcal{D}^G(X\times[0,1);\mathcal{A}) \;\to\; \mathcal{D}^{G/N}(N\backslash X\times[0,1);\mathcal{A})$$

which induces an isomorphism on K-theory after taking germs away from the empty set.

Proof. We will construct a functor $F = F_2\circ F_1$ inducing this isomorphism in two steps. First, we have a functor $F_1\colon \mathcal{D}^G(X\times[0,1);\mathcal{A}) \to \mathcal{D}^G(N\backslash X\times[0,1);\mathcal{A})$, which is the identity on objects and morphisms. The continuous control condition measured in X is stronger than the continuous control condition measured in $N\backslash X$, so this is well-defined. This functor induces a homotopy invariant and G-excisive functor

$$F_1\colon \mathcal{D}^G(G/H\times[0,1);\mathcal{A})^{>0} \to \mathcal{D}^G(N\backslash G/H\times[0,1);\mathcal{A})^{>0}$$

for $X = G/H$, and an equivalence $\mathcal{D}^G(G/H;\mathcal{A}) \simeq \mathcal{D}^G(N\backslash G/H;\mathcal{A})$. Therefore F_1 induces isomorphisms on K-theory after taking germs away from the empty set (as in the proof of Lemma 2.4). Secondly, there is a functor

$$F_2\colon \mathcal{D}^G(N\backslash X\times[0,1);\mathcal{A}) \to \mathcal{D}^{G/N}(N\backslash X\times[0,1);\mathcal{A})$$

defined on objects by $F_2(A)_{(gN,\bar{y})} = A_{(g,\bar{y})}$, where $\bar{y} \in N\backslash X \times [0,1)$. We define the functor on morphisms by $F_2(\phi)_{(gN,\bar{y})}^{(g'N,\bar{y}')} = \phi_{(g,\bar{y})}^{(g',\bar{y}')}$. This is well-defined by G-invariance of the objects and morphisms in the domain, and the continuous control conditions on morphisms agree since both are measured in $N\backslash X$. We also have an inverse functor F_2' defined by $F_2'(A)_{(e,\bar{y})} = A_{(eN,\bar{y})}$ on objects, extended by G-equivariance, and similarly for morphisms. It follows that F_2 is an equivalence of categories. □

In the next statement, if $\mathcal{A}$ is an additive G-catgeory, we denote by $\operatorname{Res}_H \mathcal{A}$ the same category considered as an H-category under restriction to a subgroup H of G. The following is "Shapiro's Lemma" in our setting.

Proposition 2.6. *Let H be a subgroup of G, $\mathcal{A}$ be an additive category with G-action, and X be an H-CW complex. There is an additive functor*

$$\mathcal{D}^H(X \times [0,1); \operatorname{Res}_H \mathcal{A}) \to \mathcal{D}^G(G \times_H X \times [0,1); \mathcal{A})$$

which induces an equivalence of categories after taking germs.

Proof. This proposition is proven in [1, Proposition 8.3] in the case where $\mathcal{A}$ is the category of finitely generated free R-modules. The same proof works for any coefficient category once the functor $\operatorname{Ind}\colon \mathcal{D}^H(X \times [0,1); \operatorname{Res}_H \mathcal{A}) \to \mathcal{D}^G(G \times_H X \times [0,1); \mathcal{A})$ is defined for general $\mathcal{A}$. Let $\phi\colon A \to B$ be a morphism in $\mathcal{D}^H(X \times [0,1); \operatorname{Res}_H \mathcal{A})$. Then

$$\operatorname{Ind}\colon \mathcal{D}^H(X \times [0,1); \operatorname{Res}_H \mathcal{A}) \to \mathcal{D}^G(G \times_H X \times [0,1); \mathcal{A})$$

is defined by $\operatorname{Ind}(A)_{[g,y]} = (g^{-1})^* A_y$, and $\operatorname{Ind}(\phi)_{[g,y]}^{[g',y']} = (g^{-1})^* \phi_y^{g^{-1}g'y'}$ if $g^{-1}g' \in H$, and is zero otherwise. The inverse of this functor on the corresponding germ categories is induced by the inclusion $i\colon X \to G \times_H X$. That is, $\operatorname{Ind}^{-1}(M)_y = M_{i(y)}$ and $\operatorname{Ind}^{-1}(\psi)_y^{y'} = \psi_{i(y)}^{i(y')}$. □

Remark 2.7. The equivalences given in these three properties are natural with respect to equivariant maps $X \to X'$. If $\mathcal{A}$ is an additive category with involution, one can check that the above properties continue to hold in this context. This is needed for applications to the L-theory assembly maps.

3. Assembly and subgroups

The properties of the continuously controlled categories given so far lead to a formal statement about assembly and subgroups. This is just our version of [4, Proposition 4.2]. If H is a subgroup of G, and $\mathcal{A}$ is an additive H-category, we denote $\operatorname{Ind}_H^G \mathcal{A} := \mathcal{D}_l^H(G; \mathcal{A})$ considered as a G-category by using the H-G biset structure of G.

Proposition 3.1. *Let $f\colon X \to X'$ be a G-equivariant map between G-CW complexes. Let H be a subgroup of G, and let $\mathcal{A}$ be an additive category with H-action. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{D}^H(\operatorname{Res}_H X \times [0,1); \mathcal{A})^{>\emptyset} \arrow[r, "f_*"] \arrow[d, leftrightarrow, "\simeq"] & \mathcal{D}^H(\operatorname{Res}_H X' \times [0,1); \mathcal{A})^{>\emptyset} \arrow[d, leftrightarrow, "\simeq"] \\
\mathcal{D}^G(X \times [0,1); \operatorname{Ind}_H^G \mathcal{A})^{>\emptyset} \arrow[r, "f_*"] & \mathcal{D}^G(X' \times [0,1); \operatorname{Ind}_H^G \mathcal{A})^{>\emptyset}
\end{tikzcd}$$

Proof. By Lemma 2.4 with $K = H$ and $S = G$, we have

$$\mathcal{D}^G(X \times [0,1); \operatorname{Ind}_H^G \mathcal{A})^{>\emptyset} \simeq \mathcal{D}^{H\times G}(G \times X \times [0,1); \mathcal{A})^{>\emptyset}$$

where $1 \times G$ acts trivially on $\mathcal{A}$ in the right-hand side. Finally,

$$\mathcal{D}^{H\times G}(G \times X \times [0,1); \mathcal{A})^{>\emptyset} \simeq \mathcal{D}^H(\operatorname{Res}_H X \times [0,1); \mathcal{A})^{>\emptyset}$$

by applying Lemma 2.5 to $H \times G$ with $N = G$. Note that G acts freely on $G \times X$, with quotient isomorphic to $\operatorname{Res}_H X$. □

Corollary 3.2. *Let H be a subgroup of G and $\mathcal{F}$ be a family of subgroups of G. Suppose that the K-theory or L-theory $(G, \mathcal{F}, \mathcal{B})$-assembly map is an isomorphism (respectively injection or surjection) for every additive coefficient category $\mathcal{B}$ with G-action. Then the $(H, \mathcal{F}|_H, \mathcal{A})$-assembly map is an isomorphism (respectively injection or surjection) for any additive coefficient category $\mathcal{A}$ with H-action.*

Proof. Just substitute $X = E_{\mathcal{F}}G$ and $X' = \bullet$ in the diagram above. □

In particular, this says that the Farrell-Jones conjecture with coefficients is stable under taking subgroups. These ideas can be extended further to obtain a version of the fibered isomorphism conjecture.

Proposition 3.3. *Let $\phi\colon H \to G$ be a group homomorphism, and let $\mathcal{F}$ be a family of subgroups of G. If the K-theory or L-theory assembly map for G relative to the family $\mathcal{F}$ is an isomorphism (respectively injective or surjective), with twisted coefficients in any additive G-category, then the assembly map for H relative to the pull-back family $\phi^*\mathcal{F} = \{K \leq H \mid \phi(K) \in \mathcal{F}\}$ is an isomorphism (respectively injection or surjection), with twisted coefficients in any additive H-category.*

Proof. The proof is the same as for Proposition 3.1 using $X = E_{\mathcal{F}}G$ and $X' = \bullet$, with the action of H on $S = G$ and on X defined via ϕ, and $\operatorname{Res}_\phi X = E_{\phi^*\mathcal{F}}G$. □

4. Assembly for Extensions

In [12] the Baum-Connes conjecture for topological K-theory is shown to pass to extensions. We show that there is a similar statement for algebraic K- and L-theory.

The proof outline used in [12] has two main steps, which we now translate into our setting. In the first step we use a discrete transitive right G-set S, which can be expressed as a single orbit $S = \{s\} \cdot G$.

Proposition 4.1. *Let X be a G-CW complex, $S = \{s\} \cdot G$, and $\mathcal{A}$ be an additive G-category with involution. Then there is an additive functor*

$$\mathcal{D}^{G_s}(\operatorname{Res}_{G_s} X \times [0,1); \operatorname{Res}_{G_s} \mathcal{A}) \to \mathcal{D}^G(X \times [0,1); \mathcal{D}_l^0(S; \mathcal{A}))^{>0}$$

which induces a homotopy equivalence of spectra after applying $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$. This equivalence is natural with respect to maps $X \to X'$ of G-CW complexes.

Proof. By Proposition 2.6,

$$\mathbb{K}^{-\infty}(\mathcal{D}^{G_s}(\mathrm{Res}_{G_s} X \times [0,1); \mathrm{Res}_{G_s} \mathcal{A})^{>0}) \simeq \mathbb{K}^{-\infty}(\mathcal{D}^G(G \times_{G_s} X \times [0,1); \mathcal{A})^{>0}).$$

Since $G \times_{G_s} X$ is G-equivariantly homeomorphic to $(G_s \backslash G) \times X = S \times X$, via the map $[g,x] \mapsto (Hg^{-1}, gx)$, and so

$$\mathcal{D}^G(G \times_{G_s} X \times [0,1); \mathcal{A})^{>0} \cong \mathcal{D}^G(S \times X \times [0,1); \mathcal{A})^{>0},$$

where $S \times X$ has the usual left G-action $g \cdot (s,x) = (sg^{-1}, gx)$. Finally, by Lemma 2.4,

$$\mathbb{K}^{-\infty}(\mathcal{D}^G(S \times X \times [0,1); \mathcal{A})^{>0}) \simeq \mathbb{K}^{-\infty}(\mathcal{D}^G(X \times [0,1); \mathcal{D}_l^0(S; \mathcal{A}))^{>0}).$$

The same proof works if we replace $\mathbb{K}^{-\infty}$ by $\mathbb{L}^{-\infty}$. $\square$

Example 4.2. Let $\pi\colon G \to K$ be a surjection of groups, and $V \subset K$ be a subgroup. We consider $S = K$ as a right-$(G \times V)$-set via the transitive action $k \cdot (g,v) := \pi(g)^{-1}kv$, where $g \in G$, $v \in V$, and $k \in K$. Let X be a $(G \times K)$-CW complex, and let $V' \subset G \times V$ denote the stabilizer subgroup of $e \in K$. Notice that $V' \cong \pi^{-1}(V)$, since $\pi(g)^{-1}v = e$ implies $g \in \pi^{-1}(v)$. By Proposition 4.1, we have a commutative diagram

$$\begin{tikzcd}
F^\lambda_{V'}(X; \mathcal{A}) \arrow[r] \arrow[d, "\simeq"] & F^\lambda_{V'}(\bullet; \mathcal{A}) \arrow[d, "\simeq"] \\
F^\lambda_{G \times V}(X; \mathcal{D}_l^0(K; \mathcal{A})) \arrow[r] & F^\lambda_{G \times V}(\bullet; \mathcal{D}_l^0(K; \mathcal{A}))
\end{tikzcd}$$

for $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$, which shows that the lower assembly map is a homotopy equivalence of spectra whenever the upper map is an equivalence.

Remark 4.3. In the proof of Theorem A, we will be using Example 4.2 with $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$, where $\mathcal{F}_G$ is a family of subgroups of G and $\mathcal{F}_K$ is a family of subgroups of K such that $\pi(H) \in \mathcal{F}_K$ for every $H \in \mathcal{F}_G$. If $V \in \mathcal{F}_K$, then the map $E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K \to E_{\mathcal{F}_G} G \times \bullet$ is a $G \times V$-equivariant homotopy equivalence. Therefore, it is a V'-equivariant homotopy equivalence. Since $V' \cong \pi^{-1}(V)$, we have the homotopy commutative diagram:

$$\begin{tikzcd}
F^\lambda_{\pi^{-1}(V)}(E_{\mathcal{F}_G} G; \mathcal{A}) \arrow[r, "a"] \arrow[d, "\simeq"] & F^\lambda_{\pi^{-1}(V)}(\bullet; \mathcal{A}) \arrow[d, "\simeq"] \\
F^\lambda_{G \times V}(X; \mathcal{D}_l^0(K; \mathcal{A})) \arrow[r, "b"] & F^\lambda_{G \times V}(\bullet; \mathcal{D}_l^0(K; \mathcal{A}))
\end{tikzcd}$$

where $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$.

If $V = K$, then $G \cong V' \subset G \times K$ and G acts on $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$ via this isomorphism. Since we are assuming that $\pi(H) \in \mathcal{F}_K$ for every $H \in \mathcal{F}_G$, X is a model for $E_{\mathcal{F}_G} G$. Thus, we have the homotopy commutative diagram:

$$\begin{tikzcd}
F^\lambda_{G}(E_{\mathcal{F}_G} G; \mathcal{A}) \arrow[r, "c"] \arrow[d, "\simeq"] & F^\lambda_{G}(\bullet; \mathcal{A}) \arrow[d, "\simeq"] \\
F^\lambda_{G \times K}(X; \mathcal{D}_l^0(K; \mathcal{A})) \arrow[r, "d"] & F^\lambda_{G \times K}(\bullet; \mathcal{D}_l^0(K; \mathcal{A}))
\end{tikzcd}$$

Definition 4.4. Let G_1 and G_2 be discrete groups, and let X_1 and X_2 be G_1- and G_2-CW complexes, respectively. Let $\mathcal{A}$ be a $G_1 \times G_2$-additive category with involution. The *partial assembly map*,

$$\mu^{G_1,G_2} \colon F^{\lambda}_{G_1\times G_2}(X_1 \times X_2; \mathcal{A}) \to F^{\lambda}_{G_2}(X_2; \mathcal{D}^{G_1}(\bullet; \mathcal{A})),$$

is the map induced by the second factor projection $X_1 \times X_2 \to \bullet \times X_2$, composed with the homotopy equivalence from Lemma 2.4 with $S = \bullet$.

Lemma 4.5. *The partial assembly map is natural in the control spaces and involution invariant.* □

Now the second step of the proof outline gives a criterion for the partial assembly map to be an equivalence.

Proposition 4.6. *Let G and K be groups, and let $\mathcal{B}$ be an additive $G \times K$-category. Let $\mathcal{F}_K$ be a family of subgroups of K. Let X_1 be a G-CW complex and X_2 be a K-CW complex with isotropy in $\mathcal{F}_K$. Suppose that*

$$F^{\lambda}_{G\times V}(X_1 \times \bullet; \mathcal{B}) \to F^{\lambda}_{G\times V}(\bullet; \mathcal{B})$$

is a homotopy equivalence for all subgroups $V \in \mathcal{F}_K$. Then the partial assembly map

$$\mu^{G,K} \colon F^{\lambda}_{G\times K}(X_1 \times X_2; \mathcal{B}) \to F^{\lambda}_{K}(X_2; \mathcal{D}^{G}(\bullet; \mathcal{B}))$$

is also an equivalence for $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$.

Proof. Suppose that $X_2 = K/V$ for some $V \in \mathcal{F}_K$. Then, by Shapiro's Lemma (Proposition 2.6),

$$\begin{array}{ccc} F^{\lambda}_{G\times V}(X_1 \times \bullet; \mathcal{B}) & \xrightarrow{\mu^{G,V}} & F^{\lambda}_{V}(\bullet; \mathcal{D}^{G}(\bullet; \mathcal{B})) \\ \downarrow \simeq & & \downarrow \simeq \\ F^{\lambda}_{G\times K}(X_1 \times K/V; \mathcal{B}) & \xrightarrow{\mu^{G,K}} & F^{\lambda}_{G\times K}(K/V; \mathcal{D}^{G}(\bullet; \mathcal{B})) \end{array}$$

and the upper map is an equivalence by assumption, since $F^{\lambda}_{V}(\bullet; \mathcal{D}^{G}(\bullet; \mathcal{B})) \simeq F^{\lambda}_{G\times V}(\bullet; \mathcal{B})$. The functors $H(X_2) := F^{\lambda}_{G\times K}(X_1 \times X_2; \mathcal{B})$ and $H'(X_2) := F^{\lambda}_{K}(X_2; \mathcal{D}^{G}(\bullet; \mathcal{B}))$ are homotopy-invariant and K-excisive functors from K-CW complexes to spectra. Since $H(K/V) \simeq H'(K/V)$ for all $V \in \mathcal{F}_K$, we conclude that $H(X_2) \simeq H'(X_2)$ for all K-CW complexes with isotropy in $\mathcal{F}_K$. □

The following is our main result about extensions:

Theorem 4.7. *Let $N \to G \xrightarrow{\pi} K$ be a group extension, where $N \triangleleft G$ is a normal subgroup, and K is the quotient group. Let $\mathcal{F}_G$ be a family of subgroups of G and $\mathcal{A}$ an additive category with right G-action. Let $\mathcal{F}_K$ be a family of subgroups of K such that $\pi(H) \in \mathcal{F}_K$ for every $H \in \mathcal{F}_G$. Suppose that for every $V \in \mathcal{F}_K$ the $(\pi^{-1}(V), \mathcal{F}_G|_{\pi^{-1}(V)}, \mathcal{A})$-assembly map in algebraic K-theory is an isomorphism, and that for every additive category $\mathcal{B}$ with right K-action the $(K, \mathcal{F}_K, \mathcal{B})$-assembly map in algebraic K-theory is injective (resp. surjective). Then the $(G, \mathcal{F}_G, \mathcal{A})$-assembly map in algebraic K-theory is injective (resp. surjective).*

The same statement holds for algebraic L-theory as well.

Example 4.8. Suppose that N is *finite* normal subgroup of G. Then the Farrell-Jones conjecture with twisted coefficients holds for G if it holds for $K = G/N$.

Example 4.9. Suppose that $1 \to N \to G \to K \to 1$ is a group extension, and $\mathcal{F}_G$ and $\mathcal{F}_K$ both denote the family of finite subgroups of their respective groups. Then the conclusions of Theorem 4.7 hold provided that the assembly map is injective (resp. surjective) for K and for every subgroup of G containing N as a subgroup of finite index.

The Proof of Theorem 4.7. Let $X = E_{\mathcal{F}_G}G \times E_{\mathcal{F}_K}K$. Let $V \in \mathcal{F}_K$ be given. By Remark 4.3, we have a homotopy commutative diagram:

$$\begin{tikzcd}
F^{\lambda}_{\pi^{-1}(V)}(E_{\mathcal{F}_G}G;\mathcal{A}) \arrow[r, "a"] \arrow[d, "\simeq"] & F^{\lambda}_{\pi^{-1}(V)}(\bullet;\mathcal{A}) \arrow[d, "\simeq"] \\
F^{\lambda}_{G\times V}(X;\mathcal{D}^0_l(K;\mathcal{A})) \arrow[r, "b"] & F^{\lambda}_{G\times V}(\bullet;\mathcal{D}^0_l(K;\mathcal{A}))
\end{tikzcd}$$

Let $\mathcal{B} = \mathcal{D}^0_l(K;\mathcal{A})$, and note that the upper map a is an equivalence by assumption, since $\operatorname{Res}_{\pi^{-1}(V)} E_{\mathcal{F}_G}G$ is a universal space for the family $\mathcal{F}_G|_{\pi^{-1}(V)}$. Hence, the lower map b is also an equivalence. By Proposition 4.6, we have the homotopy commutative diagram:

$$\begin{tikzcd}
F^{\lambda}_{G\times K}(X;\mathcal{B}) \arrow[r, "d"] \arrow[d, "\mu^{G,K}"', "\simeq"] & F^{\lambda}_{G\times K}(\bullet;\mathcal{B}) \arrow[d, "\simeq"] \\
F^{\lambda}_{K}(E_{\mathcal{F}_K}K;\mathcal{D}^G(\bullet;\mathcal{B})) \arrow[r, "e"] & F^{\lambda}_{K}(\bullet;\mathcal{D}^G(\bullet;\mathcal{B}))
\end{tikzcd}$$

By assumption, the map e is injective (resp. surjective), which implies that d is injective (resp. surjective).

Using Remark 4.3 again, we have the homotopy commutative diagram:

$$\begin{tikzcd}
F^{\lambda}_{G}(E_{\mathcal{F}_G}G;\mathcal{A}) \arrow[r, "c"] \arrow[d, "\simeq"] & F^{\lambda}_{G}(\bullet;\mathcal{A}) \arrow[d, "\simeq"] \\
F^{\lambda}_{G\times K}(X;\mathcal{D}^0_l(K;\mathcal{A})) \arrow[r, "d"] & F^{\lambda}_{G\times K}(\bullet;\mathcal{D}^0_l(K;\mathcal{A}))
\end{tikzcd}$$

Therefore, the assembly map c is injective (resp. surjective). □

Corollary 4.10. *The Farrell-Jones conjecture with twisted coefficients is true for $G_1 \times G_2$ if and only if it is true for G_1, G_2, and every product $V_1 \times V_2$, where $V_1 \le G_1$ and $V_2 \le G_2$ are virtually cyclic subgroups.*

Proof. By our main result applied to the projection $G_1 \times G_2 \to G_2$, we may assume that G_2 is virtually cyclic. Similarly, we may assume that G_1 is virtually cyclic. Thus, we are reduced to knowing the conjecture for products $V_1 \times V_2$ of virtually cyclic subgroups of G_1 and G_2 respectively. □

Remark 4.11. A product $V_1 \times V_2$ of virtually cyclic subgroups can be further reduced to the basic cases $\mathbf{Z} \times \mathbf{Z}$, $\mathbf{Z} \times D_\infty$ and $D_\infty \times D_\infty$ after quotients by finite normal subgroups.

5. The proof of Lemma 2.2

We will check the details of Lemma 2.2, which asserts that there is an additive functor

$$F\colon \mathcal{D}_l^{K\times G}(S\times T\times[0,1);\mathcal{A})\to \mathcal{D}_l^{G}(T\times[0,1);\mathcal{D}_l^{K}(S;\mathcal{A}))$$

defined by

$$\begin{aligned}(F(A)_{(g,t)})_{(k,s)} &:= A_{(k,g,s,t)}\\ \Big(F(\phi)_{(g,t)}^{(g',t')}\Big)_{(k,s)}^{(k',s')} &:= \phi_{(k,g,s,t)}^{(k',g',s',t')}.\end{aligned}$$

Here $\mathcal{A}$ is an additive category with commuting right K and G-actions, T a left G-set and S a transitive K-G biset. The group $K\times G$ acts on $S\times T$ by the formula $(k,g)\cdot(s,t) := (ksg^{-1},gt)$. Recall the notation (k,s) for elements of $K\times S$, and (g,t) for elements of $G\times T\times[0,1]$. We will let $\epsilon\colon T\times[0,1]\to T$ denote the projection map. Notice that $\phi_{(k,g,s,t)}^{(k',g',s',t')}=0$ unless $s=s'$ and $\epsilon(t)=\epsilon(t')$, since the morphisms $\phi\colon A\to B$ in the domain category are assumed to be level-preserving. The free $(K\times G)$-space

$$\overline{S}\times\overline{T}=K\times G\times S\times T\times[0,1]$$

is a resolution for $S\times T\times[0,1]$. The proof that F is a functor is done in the following steps.

(1). $F(\phi\circ\psi)=F(\phi)\circ F(\psi)$. Since

$$\Big(F(\phi)\circ F(\psi)\Big)_{(g,t)}^{(g',t')} = \sum_{(g'',t'')} F(\phi)_{(g'',t'')}^{(g',t')}\circ F(\psi)_{(g,t)}^{(g'',t'')}$$

we have that:

$$\begin{aligned}\Big(\big(F(\phi)\circ F(\psi)\big)_{(g,t)}^{(g',t')}\Big)_{(k,s)}^{(k',s')} &= \Big(\sum_{(g'',t'')} F(\phi)_{(g'',t'')}^{(g',t')}\circ F(\psi)_{(g,t)}^{(g'',t'')}\Big)_{(k,s)}^{(k',s')}\\ &= \sum_{(g'',t'')}\Big(F(\phi)_{(g'',t'')}^{(g',t')}\circ F(\psi)_{(g,t)}^{(g'',t'')}\Big)_{(k,s)}^{(k',s')}\\ &= \sum_{(g'',t'')}\sum_{(k'',s'')}\phi_{(k'',g'',s'',t'')}^{(k',g',s',t')}\circ\psi_{(k,g,s,t)}^{(k'',g'',s'',t'')}\\ &= (\phi\circ\psi)_{(k,g,s,t)}^{(k',g',s',t')}\\ &= \Big(F(\phi\circ\psi)_{(g,t)}^{(g',t')}\Big)_{(k,s)}^{(k',s')}\end{aligned}$$

(2). *$F(A)_{(g,t)}$ is an object of $\mathcal{D}_l^K(S;\mathcal{A})$, for every $(g,t)\in G\times T\times[0,1)$.*

(2′). *$F(A)_{(g,t)}$ is K-invariant.* For each $h \in K$,

$$\begin{aligned}\big(h^*(F(A)_{(g,t)})\big)_{(k,s)} &= h^*\big((F(A)_{(g,t)})_{(hk,hs)}\big)\\ &= h^*\big(A_{(hk,g,hs,t)}\big)\\ &= (h^*A)_{(k,g,s,t)}\\ &= A_{(k,g,s,t)}\\ &= (F(A)_{(g,t)})_{(k,s)}\end{aligned}$$

(2″). *The support of $F(A)_{(g,t)}$ is K-compact in $K \times S$.*

Since a discrete K-set is K-compact if and only if its image under the quotient map is finite, we need to show that $K\backslash \operatorname{supp}(F(A)_{(g,t)})$ is finite. Let p be the projection map from $K \times G \times S \times T \times [0,1)$ to $K \times G \times S \times T$, $M = p\big(\operatorname{supp}(A)\big)$, and $N = p\big(\operatorname{supp}(A) \cap K \times \{g\} \times S \times \{t\}\big) \subset M$. Consider the following commutative diagram, in which $f(k', g', s', t') = (k', s'g')$, $m_g(k,s) = (k, sg^{-1})$, and the vertical arrows are quotient maps.

$$\begin{tikzcd} K \times G \times S \times T \arrow[r, "f"] \arrow[d, "q_{K\times G}"] & K \times S \arrow[r, "m_g"] \arrow[d, "q_K"] & K \times S \arrow[d, "q_K"] \\ (K \times G)\backslash(K \times G \times S \times T) \arrow[r, "\bar f"] & K\backslash(K \times S) \arrow[r, "\bar m_g"] & K\backslash(K \times S) \end{tikzcd}$$

Since M is discrete and $(K \times G)$-compact, $q_{K\times G}(M)$ is finite. Since $N \subset M$, $q_{K\times G}(N)$ is also finite. Therefore, $(\bar m_g \circ \bar f \circ q_{K\times G})(N) = (q_K \circ m_g \circ f)(N) = q_K(\operatorname{supp}(F(A)_{(g,t)}))$ is finite.

(3). *$F(\phi)_{(g,t)}^{(g',t')}$ is a morphism of $\mathcal{D}_l^K(S;\mathcal{A})$, for every $(g,t),(g',t') \in G \times T \times [0,1)$.*

(3′). *$F(\phi)_{(g,t)}^{(g',t')}$ is K-invariant.* The proof is similar to the proof of **(2′)**.

(3″). *Fix $(k,s) \in K \times S$. Then, the following set is finite:*

$$P = \left\{(k',s') \in K \times S \,\middle|\, \left(F(\phi)_{(g,t)}^{(g',t')}\right)_{(k,s)}^{(k',s')} \neq 0 \text{ or } \left(F(\phi)_{(g,t)}^{(g',t')}\right)_{(k',s')}^{(k,s)} \neq 0\right\}.$$

The sets $\left\{(k',g',s',t') \in K \times G \times S \times T \times [0,1) \,\middle|\, \phi_{(k,g,s,t)}^{(k',g',s',t')} \neq 0\right\}$ and $\left\{(k',g,s',t) \in K \times G \times S \times T \times [0,1) \,\middle|\, \phi_{(k',g,s',t)}^{(k,g',s,t')} \neq 0\right\}$ are finite and their union projects onto P.

(3‴). *$F(\phi)_{(g,t)}^{(g',t')}$ is level preserving in S.* This is because ϕ is level-preserving in $S \times T$.

(4). *$F(A)$ is an object of $\mathcal{D}_l^G(T \times [0,1);\mathcal{D}_l^K(S;\mathcal{A}))$.*

(4′). *$F(A)$ is G-invariant.* For each $\gamma \in G$,

$$\begin{aligned}
\big(\gamma^*(F(A))_{(g,t)}\big)_{(k,s)} &= \big(\gamma^*(F(A)_{(\gamma g,\gamma t)})\big)_{(k,s)} \\
&= \gamma^*\big((F(A)_{(\gamma g,\gamma t)})_{(k,s\gamma^{-1})}\big) \\
&= \gamma^*\big(A_{(k,\gamma g,s\gamma^{-1},\gamma t)}\big) \\
&= (\gamma^* A)_{(k,g,s,t)} \\
&= A_{(k,g,s,t)} \\
&= (F(A)_{(g,t)})_{(k,s)}
\end{aligned}$$

(4″). *The support of $F(A)$ is relatively G-compact in $G \times T \times [0,1)$.*

Let $p\colon K \times G \times S \times T \times [0,1) \to G \times T \times [0,1)$ be the projection map. Since $\mathrm{supp}(A)$ is relatively $(K \times G)$-compact and $p(\mathrm{supp}(A)) = \mathrm{supp}(F(A))$, $\mathrm{supp}(F(A))$ is relatively G-compact in $G \times T \times [0,1)$.

(4‴).*The support of $F(A)$ is locally finite in $G \times T \times [0,1)$.*

Let $(g,t) \in \mathrm{supp}(F(A))$ be given. We must find an open neighborhood $U \subset G \times T \times [0,1)$ of (g,t) such that $U \cap \mathrm{supp}(F(A)) = \{(g,t)\}$. Let

$$L = \{(k,s) \in K \times S \mid (k,g,s,t) \in \mathrm{supp}(A)\}.$$

From **(1″)**, we know that L is K-compact. That is, $L = K \cdot (K_0 \times S_0)$, where $K_0 \subset K$ and $S_0 \subset S$ are finite sets. Since $\mathrm{supp}(A)$ is locally finite in $K \times G \times S \times T \times [0,1)$, for each $(k_i, s_i) \in K_0 \times S_0$, there is a neighborhood $U_i \subset T \times [0,1)$ of t, such that

$$(\{k_i\} \times \{g\} \times \{s_i\} \times U_i) \cap \mathrm{supp}(A) = \{(k_i, g, s_i, t)\}.$$

Thus, for each $(k,s) \in L$, there is an i, such that

$$(\{k\} \times \{g\} \times \{s\} \times U_i) \cap \mathrm{supp}(A) = \{(k,g,s,t)\}.$$

Therefore, if we let $U = \{g\} \times (\cap_i U_i)$, then $U \cap \mathrm{supp}(F(A)) = \{(g,t)\}$.

(5). *$F(\phi)$ is a morphism in $\mathcal{D}^G_l(T \times [0,1); \mathcal{D}^K_l(S;\mathcal{A}))$.*

(5′). *$F(\phi)$ is G-invariant.* The proof is similar to the proof of **(3′)**.

(5″). *Fix $(g,t) \in G \times T \times [0,1)$. Then, the following set is finite*

$$\Big\{(g',t') \in G \times T \times [0,1) \,\Big|\, F(\phi)^{(g',t')}_{(g,t)} \neq 0 \text{ or } F(\phi)^{(g,t)}_{(g',t')} \neq 0\Big\}.$$

As we saw in **(2″)**, $\mathrm{supp}(A) \cap K \times \{g\} \times S \times \{t\}$ is K-compact. Therefore, it is contained in $K \cdot (K_0 \times \{g\} \times S_0 \times \{t\})$, for some finite subsets $K_0 \in K$ and $S_0 \in S$. Notice that by K-equivariance, $F(\phi)^{(g',t')}_{(g,t)} \neq 0$ if and only if there exists an $s_0 \in S_0$, $k_0 \in K_0$ and $k' \in K$ such that $\phi^{(k',g',s,t')}_{(k_0,g,s_0,t)} \neq 0$. But for each $k_0 \in K_0$ and each $s_0 \in S_0$, there are only finitely many $k' \in K$, $g' \in G$ and $t' \in T \times [0,1)$ such that $\phi^{(k',g',s_0,t')}_{(k_0,g,s_0,t)} \neq 0$. Therefore, there are only finitely many $g' \in G$ and $t' \in T \times [0,1)$ such that $F(\phi)^{(g',t')}_{(g,t)} \neq 0$. A similar argument shows that there are only finitely many $g' \in G$ and $t' \in T \times [0,1)$ such that $F(\phi)^{(g,t)}_{(g',t')} \neq 0$.

(5‴). *$F(\phi)$ is continuously controlled in $T\times[0,1)$.*

Let $\phi\colon A\to B$ be a morphism in $\mathcal{D}_l^{K\times G}(S\times T\times[0,1);\mathcal{A})$. Let $(x_0,1)\in T\times[0,1]$ and a G_{x_0}-invariant neighborhood $U\subset T\times[0,1]$ of $(x_0,1)$ be given. We must find a G_{x_0}-invariant neighborhood $V\subset T\times[0,1]$ of $(x_0,1)$, such that $F(\phi)_{(g,t)}^{(g',t')}=0=F(\phi)_{(g',t')}^{(g,t)}$ whenever $(g,t)\in G\times V$ and $(g',t')\notin G\times U$.

By definition, $\left(F(\phi)_{(g,t)}^{(g',t')}\right)_{(k,s)}^{(k',s)}=\phi_{(k,g,s,t)}^{(k',g',s,t')}$. Let $s_0\in S$ with $K\cdot s_0\cdot G=S$, and let $H\le K\times G$ be the stabilizer subgroup of s_0. We will identify $G\times T\times[0,1]$ with the level $\{s_0\}\times G\times T\times[0,1]$. Notice that the intersection of $\operatorname{supp}(A)$ with $K\times G\times\{s_0\}\times T\times[0,1)$ is contained in,

$$\bigcup_{(a,b)\in H} a\cdot K_0\times b\cdot G_0\times\{s_0\}\times b\cdot T_0\times[0,1),$$

where $K_0\subset K$, $G_0\subset G$ and $T_0\subset T$ are finite sets. This holds since $\operatorname{supp}(A)$ is relatively $(K\times G)$-compact and any element of $(K\times G)-H$ will move s_0 to another level in S.

Suppose that $\phi_{(k,g,s,t)}^{(k',g',s,t')}\neq 0$ for some $k\in K$, $g\in G$ and $t\in U$. Then we can write $\tau s\gamma^{-1}=s_0$, for some $\tau\in K$ and some $\gamma\in G$. By equivariance, $\phi_{(\tau k,\gamma g,s_0,\gamma t)}^{(\tau k',\gamma g',s_0,\gamma t')}\neq 0$. For this to happen, $(\tau k,\gamma g,s_0,\gamma t)\in\operatorname{supp}(A)$. This implies that there exists $(a,b)\in H$ such that

$$(\tau k,\gamma g,s_0,\gamma t)\in a\cdot K_0\times b\cdot G_0\times\{s_0\}\times b\cdot T_0\times[0,1),$$

which is equivalent to saying that

$$(a^{-1}\tau k,b^{-1}\gamma g,s_0,b^{-1}\gamma t)\in K_0\times G_0\times\{s_0\}\times T_0\times[0,1)$$

In particular, $b^{-1}\gamma t\in b^{-1}\gamma U\cap(T_0\times[0,1))$.

Since T_0 is finite, there are only finitely many elements of G, say $\{g_1,g_2,\dots,g_r\}$, such that $g_iU\cap(T_0\times[0,1))\neq\emptyset$. Therefore, $\gamma=bg_i$ for some $(a,b)\in H$ that fixes s_0 and some i with $1\le i\le r$.

Since ϕ is continuously controlled at $g_i\cdot(x_0,1)$ along $S\times T\times 1$, there is a neighborhood $V_i\subset T\times[0,1]$ of $(x_0,1)$ such that $\phi_{(k,g,s_0,g_it)}^{(k',g',s_0,g_it')}=0$ if $t\in V_i$ and $t'\notin U$, for $1\le i\le r$.

Let $V=\cap_i V_i$. Then, if $t\in V$ and $t'\notin U$, we have

$$\phi_{(a^{-1}\tau k,g_ig,s_0,g_it)}^{(a^{-1}\tau k',g_ig',s_0,g_it')}=0$$

and hence

$$0=\phi_{(\tau k,bg_ig,s_0,bg_it)}^{(\tau k',bg_ig',s_0,bg_it')}=\phi_{(\tau k,\gamma g,s_0,\gamma t)}^{(\tau k',\gamma g',s_0,\gamma t')}=\phi_{(k,g,s,t)}^{(k',g',s,t')}$$

by equivariance of the morphisms, and the relations $\gamma=bg_i$, $s_0=\tau s\gamma^{-1}$. A similar argument shows that $F(\phi)_{(g',t')}^{(g,t)}=0$ if $t\in V$ and $t'\notin U$. Therefore $F(\phi)$ is continuously controlled along $T\times 1$. □

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