Math@Mac Online Mathematics Competition

Wednesday, November 28, 2012

SOLUTIONS

---

1. A cubical box contains 64 identical small cubes that exactly fill the box. How many of these small cubes touch a side, the bottom, or the top of the box?

Answer: 56 (50% correct)

Hint:
It is easier to count the cubes that are in the interior of the box.

---

2. A bag contains 11 candy bars: three cost 50 cents each, four cost 1 dollar each and four cost 2 dollars each. Three candy bars are randomly chosen from the bag, without replacement. What is the probability that the total cost of the three candy bars is 4 dollars or more?

Answer: 14/33 (33% correct)

Solution:
First consider the combinations of the three types of candy bars whose total cost is at least \$4.00:

1. Selecting three bars that cost \$2.00 each: total cost \$6.00
2. Selecting two bars that cost \$2.00 each, and one bar that costs \$1.00 or \$0.50: total cost \$5.00 or \$4.50
3. Selecting one bar that costs \$2.00 and two bars that cost \$1.00 each: total cost \$4.00

The total cost of no other combination of three bars will be at least \$4.00.

Now consider the number of ways to select three bars for each of the above cases:

1. There are 4 ways to select 3 bars, each costing \$2.00,
2. There are "4 choose 2" ways to select 2 bars that each cost \$2.00 and 7 ways to select a bar that costs either \$1.00 or \$0.50, so in total there are $(6)(7) = 42$ ways,
3. There are 4 ways to choose one bar that costs \$2.00, and "4 choose 2" ways to select two bars that each cost \$1.00, so in total there are $(4)(6) = 24$ ways.

Altogether, there are $4 + 42 + 24 = 70$ ways to select 3 bars from the 11 whose total cost is at least \$4.00. Since the total number of ways to select 3 bars from the 11, without replacement, is "11 choose 3", or $(11)(10)(9)/(3)(2) = 165$, then the probability of selecting three bars whose total cost is at least \$4.00 is $70/165$, or $14/33$.

3. Jill rides her bike around a course in the shape of an equilateral triangle. Her speed is 10 kilometers per hour on the first side of the course, 15 kilometers per hour on the second side of the course, and 20 kilometers per hour on the third and final side of the course. Then Jill's average speed during her ride is

Answer: at least 13 but less than 14 kilometers per hour (25% correct)

Solution:
Let x be the length of each side in the triangle.
Then average speed $= \dfrac{3x}{\frac{x}{10}+\frac{x}{15}+\frac{x}{20}} \approx 13.8$ kilometers per hour.

---

4. The average of 9 numbers $a_1, a_2, \ldots, a_9$ is equal to 200. If the number a_1 is replaced by $4a_1$, then the new average is 400. What is the value of a_1?

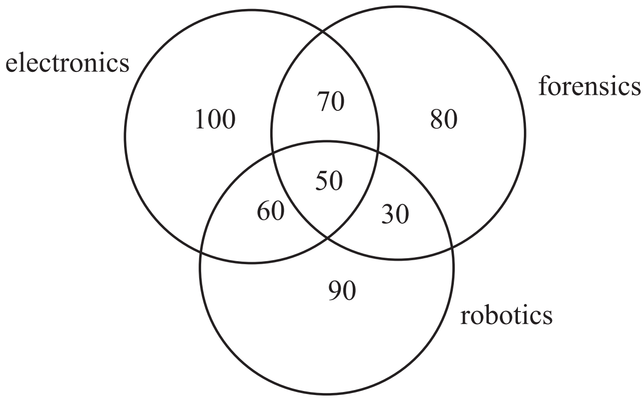
Answer: 600 (57% correct)

Solution:
The average of $4a_1, a_2, \ldots, a_9$ is $(4a_1 + a_2 + \cdots + a_9)/9 = (3a_1/9) + (a_1 + a_2 + \cdots + a_9)/9 = a_1/3 + 200$ and so $a_1/3 + 200 = 400$, or $a_1 = 600$.

5. In Hamilton Spy high school each student is required to take a specialized science course. 280 students took electronics, 230 forensics and 230 robotics classes. 120 students took both electronics and forensics, 110 took electronics and robotics, 80 took forensics and robotics, and 50 took all three courses. How many students are there in the school?

Answer: 480 (63% correct)

Solution:
Fill out the Venn diagram, starting with the intersection of all three courses, and then by using the information about the intersections of two courses.



Adding up the sizes of all subsets (each counted only once!), we get $100 + 70 + 80 + 60 + 50 + 30 + 90 = 480$.

---

6. What is the final digit of the product of all odd numbers *not divisible* by 5 from 1 to 999?

Answer: 1 (38% correct)

Solution:
We multiply 1 by 3, then by 7, then by 9, then by 11, etc. and record the last digits of the product: 1,3,1,9,9,7,9,1,1,3,1, etc. We notice the repeated pattern of length eight: 1,3,1,9,9,7,9,1. This pattern is produced by all odd numbers (not divisible by 5) in the interval of twenty numbers (from 1 to 20, 21 to 40, etc.). Since 999 is the last odd number in the group of numbers from 981 to 1000, the last digit of the product is the last digit of the repeated pattern, i.e., it is 1.

7. If the sides of a triangle satisfy $\dfrac{3}{a+b+c} = \dfrac{1}{a+b} + \dfrac{1}{a+c}$, what is the angle between the sides b and c?

Answer: 60 degrees (32% **correct)**

Solution:
We compute the common denominator and then cross-multiply:

$$3(a+b)(a+c) = (a+b+c)(2a+b+c)$$

Simplifying, we obtain $a^2 + bc - b^2 - c^2 = 0$. Combining with the law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$

we obtain $bc - 2bc\cos\alpha = 0$, i.e., $\cos\alpha = 1/2$ and $\alpha = \pi/3$ (60 degrees).

---

8. The sum of the solutions of the equation

$$(x+1)^2 + (2x+1)^2 + (3x+1)^2 + \cdots + (99x+1)^2 = 99$$

is

Answer: -6/199 (32% **correct)**

Solution:
It is a quadratic equation, so it has at most two real solutions. Squaring all terms on the left side, we get

$$(x^2+2x+1) + (2^2x^2 + 2\cdot 2x + 1) + (3^2x^2 + 2\cdot 3x + 1) + \cdots + (99^2x^2 + 2\cdot 99x + 1) = 99$$

and

$$(1^2 + 2^2 + 3^2 + \cdots + 99^2)x^2 + 2\cdot(1+2+3+\cdots+99)x = 0.$$

So, $x = 0$ or

$$x = -\frac{2\cdot(1+2+3+\cdots+99)}{1^2+2^2+3^2+\cdots+99^2}$$

Using the formulas for the sum of integers and the sum of their squares, we get

$$1^2+2^2+3^2+\cdots+99^2 = \frac{99\cdot 100\cdot 199}{6} \quad \text{and} \quad 1+2+3+\cdots+99 = \frac{99\cdot 100}{2}$$

and therefore

$$x = -2\,\frac{99\cdot 100}{2}\Big/\frac{99\cdot 100\cdot 199}{6} = -\frac{6}{199}$$

So the sum of the solutions is $-6/199$.

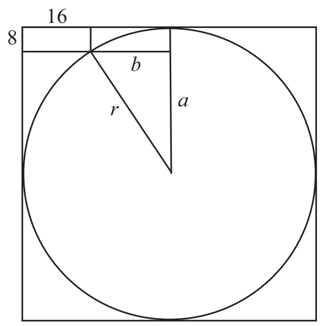
9. The rectangle in the corner measures 8 cm by 16 cm. What is the radius of the circle in cm?

Answer: 40 (73% correct)

Solution:
By Pythagora's Theorem, $a^2 + b^2 = r^2$. Since $a = r - 8$ and $b = r - 16$, it follows that

$$(r-8)^2 + (r-16)^2 = r^2$$

Simplifying, we ontain $r^2 - 48r + 320 = 0$. Of the two solutions, $r = 8$ and $r = 40$, only the latter makes sense.



---

10. Two numbers a and b each have three digits. You are asked to calculate the product ab, but instead you put a to the left of b to form a six-digit number d. It turns out that d is three times the correct answer. Find a.

Answer: 167 (40% correct)

Solution:
We are given that $1000a + b = 3ab$. Solve for a:

$$a = \frac{b}{3b - 1000}$$

Since b is a three-digit number, the only way a can have three digits is that b is divided by a number between 1 and 9. The only values of b for which $1 \leq 3b - 1000 \leq 9$ are 334, 335 and 336. However, only the division

$$\frac{334}{3(334) - 1000}$$

gives a three digit answer, $a = 167$.

The End