Math@Mac Online Mathematics Competition

Wednesday, November 18, 2015

SOLUTIONS

1. Write $2015! = 6^k m$, where m is not divisible by 6. What is k?
(Recall that $2015! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2014 \cdot 2015$.)

(A) 1002

(B) 996

(C) 994

(D) 988

Answer: 1002

Solution: Because $6 = 3 \cdot 2$, and in the expression for 2015! there are more factors of 2 than factors of 3, we have to count the number of factors of 3.

Since $2015/3 = 671.67$ we conclude that there are 671 numbers between 1 and 2015 which are divisible by 3. Now $2015/3^2 = 223.89$ so there are 223 numbers which are divisible by 9, and they contribute 223 additional factors of 3.

Keep going: $2015/3^3 = 74.63$ i.e., there are 74 numbers which are divisible by $3^3 = 27$, and they contribute 74 additional factors of 3.

$2015/3^4 = 24.88$, so there are 24 more factors of 3;

$2015/3^5 = 8.29$ so there are 8 more factors of 3

$2015/3^6 = 2.76$ so there are 2 more factors of 3

$2015/3^7 = 0.92$ no more!

Thus, the total number of factors of 6 in 2015! is $k = 671 + 223 + 74 + 24 + 8 + 2 = 1002$.

2. A list of consecutive positive numbers

$$1,\ 2,\ 3,\ 4,\ 5, \ldots, n-1,\ n$$

is written on a blackboard. One number is erased. The average (arithmetic mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased?

(A) 14

(B) 13

(C) 7

(D) 11

Answer: 7

Solution: If n is erased, then the average is

$$\frac{1+2+3+\cdots+(n-1)}{n-1} = \frac{(n-1)n}{2} \cdot \frac{1}{n-1} = \frac{n}{2}$$

If 1 is erased, then the average is

$$\frac{2+3+\cdots+n}{n-1} = \left(\frac{n(n+1)}{2} - 1\right) \cdot \frac{1}{n-1} = \frac{n^2+n-2}{2} \cdot \frac{1}{n-1}$$
$$= \frac{(n+2)(n-1)}{2} \cdot \frac{1}{n-1} = \frac{n+2}{2}$$

Thus the average of the sequence with one number erased must satisfy

$$\frac{n}{2} \leq 35\frac{7}{17} \leq \frac{n+2}{2} \quad \text{i.e.,} \quad n \leq 70\frac{14}{17} \leq n+2$$

and

$$68\frac{14}{17} \leq n \leq 70\frac{14}{17}$$

Since n is an integer, we conclude that $n = 69$ or $n = 70$. The average of the sequence with one number erased must satisfy

$$\frac{\text{some integer}}{n-1} = 35\frac{7}{17}$$

and so $n-1$ must be a multiple of 17. Thus, $n = 69$. Let x be the erased number. Then the average of the numbers

$$1,\ 2,\ 3,\ 4,\ 5, \ldots, x-1,\ x+1, \ldots, 69$$

is

$$\frac{\frac{69\cdot 70}{2} - x}{68} = 35\frac{7}{17}$$

Simplifying, we obtain $2415 - x = 2408$, and $x = 7$.

3. A box contains 26 black and 13 white balls. There is a reserve supply of black balls. You pick randomly two balls at the same time from the box. If they are both white, you throw both balls away and put one black ball into the box. If they are both black, you throw away one ball and put the other ball back into the box. If one ball is white and the other black, then you throw away the black ball and put the white ball back into the box.

Every time you do this, there will be one ball less in the box. What is the probability that the last ball left in the box is black?

(A) 0

(B) 2/3

(C) 1/3

(D) 1/2

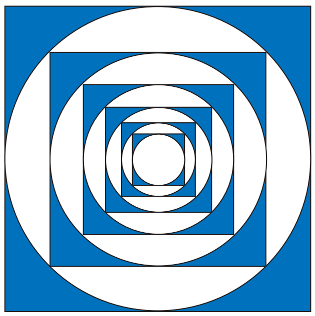
Answer: 0

Solution: After each step, the number of black balls increases by 1 (if two white balls are picked) or decreases by 1 (otherwise). This is not very useful. However - after each step, the number of white balls either remains the same (in case we picked a white ball and a black ball) or decreases by 2 (if we pick two white balls). Thus, at every moment the number of white balls in the box must be odd. Since the number of balls decreases, at some point there will be exactly one white ball left. If that's the only ball left, then the last ball is white.

Otherwise, there will be several black balls left, together with that one white ball. If two black balls are picked, they are replaced by one black ball. Otherwise a black and a white ball are picked and replaced by a white ball. Thus, there is one less black ball left in the box, together with the white ball. Continuing in the same way, we see that white ball will remain in the box.

Thus the chance that the last ball is black is zero.

4. The side of the outer square has length 1. Into this square a circle is inscribed, and into this circle a square is inscribed, and into this square a circle is inscribed, and so on. Find the exact area of the shaded region.



(A) $2 - \pi/3$

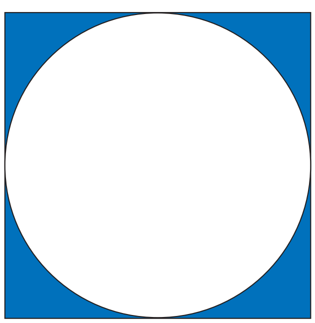
(B) $4 - 2\pi/3$

(C) $3 - \pi$

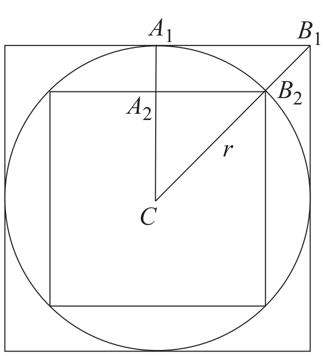
(D) $2 - \pi/2$

Answer: $2 - \pi/2$

Solution: The area of the shaded region below is $1 - \pi(1/2)^2 = 1 - \pi/4$.



Consider inscribing a square into a circle of radius r, which is inscribed into a square, as shown below.



Then $A_1B_1 = r$ and A_2B_2 is the side of a square whose diagonal is r; Thus, $A_2B_2 = r/\sqrt{2}$, i.e.,

$$A_2B_2 = \frac{1}{\sqrt{2}} A_1B_1$$

Think of the given figure as obtained through a sequence of iterations, where in each iteration we inscribe a square and then a cricle into it.

We have just shown that the dimensions of each iteration (consisting of a square and a circle inscribed into it) are scaled from the previous iteration by a factor of $1/\sqrt{2}$. Thus, the area of the shaded region in each iteration is $1/2$ of the area of the shaded region in the previous iteration, and so the total area is

$$\begin{aligned}
&\left(1 - \frac{\pi}{4}\right) + \frac{1}{2}\left(1 - \frac{\pi}{4}\right) + \frac{1}{2^2}\left(1 - \frac{\pi}{4}\right) + \frac{1}{2^3}\left(1 - \frac{\pi}{4}\right) + \cdots \\
&\qquad = \left(1 - \frac{\pi}{4}\right)\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) \\
&\qquad = \left(1 - \frac{\pi}{4}\right)\frac{1}{1 - 1/2} \\
&\qquad = 2\left(1 - \frac{\pi}{4}\right)
\end{aligned}$$

5. Scientists observing how children socialize in the playground have found that girls congregate in groups of two or three. Each time a new girl arrives, she randomly choses a group. If a chosen group has two girls in it, the new girl joins the group, forming a group of three. If the chosen group has three girls in it, the new girl takes one of the girls away from the group of three girls and forms a separate group of two girls (so, in this case, a group of three girls is replaced by two groups of two girls).

Initially there are five girls in the playground. Assuming that new girls arrive one at a time, what is the probability that the fourth new girl joins a group of two girls?

(A) 4/9

(B) 3/7

(C) 4/7

(D) 5/9

Answer: 5/9

Solution: After two new girls arrive, there will be 7 girls. There is only one way to divide seven girls into groups of two and three: two groups of two girls and one group of three girls. Thus, it is irrelevant what the first two new girls do!

With 1/3 chance, the third new girl joins a group of three. In that case, that group splits into two groups of two girls, thus there are four groups of two girls in the playground. In this case, the fourth new girl joins a group of two girls with 100% chance.

With 2/3 chance, the third new girl joins a group of two, and thus forms a group of three. In that case, there are two groups of three girls and one group of two girls in the playground. The fourth new girl then joins the group of two girls with 1/3 chance.

Thus, the chance that the fourth new girl joins the group of two girls is

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{3} = \frac{5}{9}.$$

6. A group of 500 high school students, 260 girls and 240 boys, is randomly divided into two rows of 250 students each. Each student in one row stands directly opposite from a student in the other row, and they shake hands. Which statement is true?

(A) Number of girl-girl handshakes is the same as the number of boy-boy handshakes

(B) Number of girl-girl handshakes is 5 more than the number of boy-boy handshakes

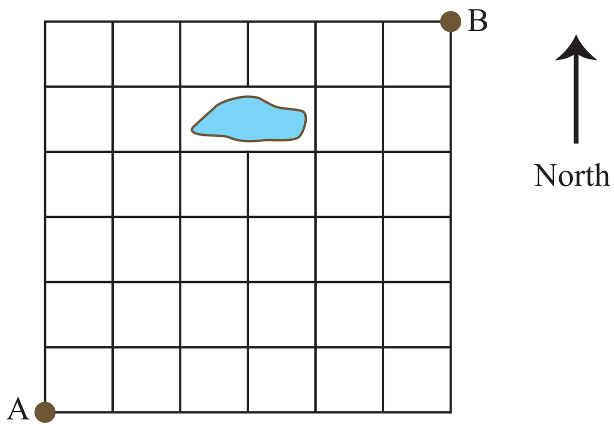
(C) Number of girl-girl handshakes is 10 more than the number of boy-boy handshakes

(D) Number of girl-girl handshakes is 20 more than the number of boy-boy handshakes

Answer: Number of girl-girl handshakes is 10 more than the number of boy-boy handshakes

Solution: Denote the number of girl-girl handshakes by G, the number of boy-boy handshakes by B, and the number of girl-boy handshakes by M. Then (count girls) $2G+M = 260$ and (count boys) $2B + M = 240$. Subtracting the two equations, we obtain $2G - 2B = 20$, and $G - B = 10$.

7. The streets in a town are laid out in a grid, as shown below. A small pond on the north side of town eliminates one segment of the grid. If you are allowed to move along the segments in the north and east directions only, in how many different ways can you walk from A to B?



(A) 924

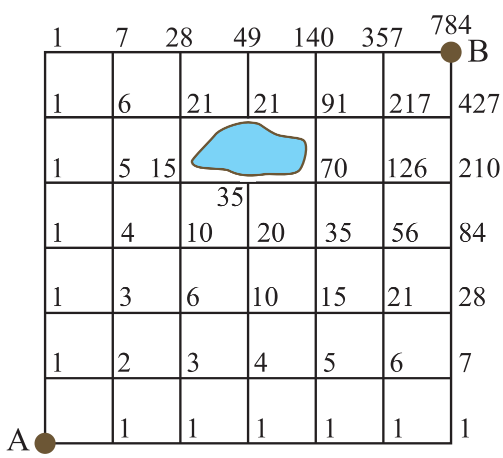
(B) 826

(C) 844

(D) 784

Answer: 784

Solution:

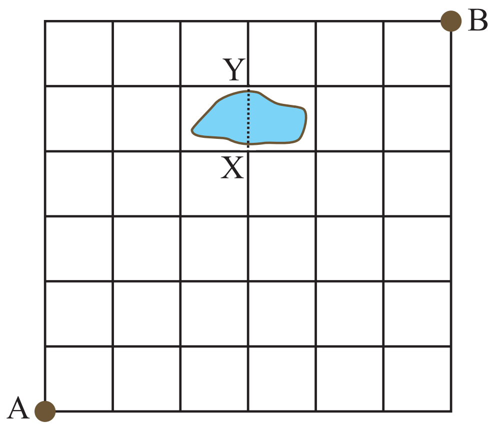


Start at A and label the number of ways in which one can arrive at each intersection. Clearly, all intersections in the bottom horizontal street and the leftmost vertical street can be arrived at in only one way. From there on, for each intersection we add the number below it and to the left of it (because these are the only two directions (south and east) from where it is possible to arrive at that intersection).

Here is another solution. Ignore the lake for now. In order to arrive to B from A we need to walk along the total of 12 segments. Think of a jar with numbers $1, 2, 3, \ldots 12$, from which we select six numbers - these numbers tell us along which segments we walk north until the next intersection.

Thus, the number of ways to arrive to B from A is $\binom{12}{6}$.

Now count the number of routes that use the missing segment XY.



To use the segment XY, we have to arrtive at X first; as above, we conclude that there are $\binom{7}{4}$ ways of doing it (there is a total of 7 segments, and we need to walk north along 4 segments). To walk from Y to B, we have $\binom{4}{1}$ choices.

Thus, the answer is $\binom{12}{6} - \binom{7}{4}\binom{4}{1} = 924 - (35)(4) = 784$.

8. Find the $12,345$ th term in the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$$

(A) 155

(B) 156

(C) 157

(D) 158

Answer: 157

Solution: Note the pattern: the last appearance of number 3 is in the 6th term, $6 = 1+2+3$; the last appearance of number 4 is in the 10th term, $10 = 1+2+3+4$; the last appearance of number 5 is in the 15th term, $15 = 1+2+3+4+5$.

Thus, the last appearance of number n is in term $1 + 2 + 3 + \cdots + n = n(n + 1)/2$.

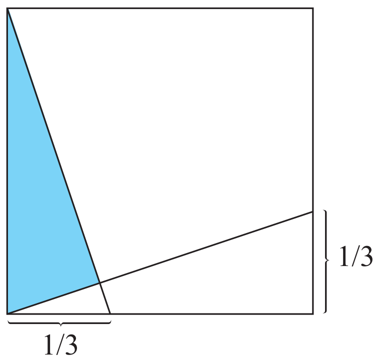
Solving $n(n + 1)/2 = 12345$ for a positive n we obtain

$$n^2 + n - 24690 = 0$$
$$n = \frac{-1 + \sqrt{1 + (4)(24690)}}{2} = \frac{-1 + \sqrt{98761}}{2} \approx 156.63$$

The last appearance of number 156 is in the term $156 \cdot 157/2 = 12256$, and the last appearance of number 157 is in the term $157 \cdot 158/2 = 12403$.

Thus, the 12345th term is 157.

9. The length of the side of the square is 1. Determine the area of the shaded triangle.



(A) 1/10

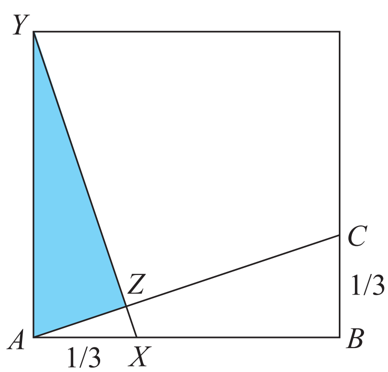
(B) 5/40

(C) 3/20

(D) 7/40

Answer: 3/20

Solution:



The area of the triangle ABC is 1/6. By Pythagoras' Theorem,

$$AC = \sqrt{1^2 + (1/3)^2} = \sqrt{10}/3.$$

Note that $\angle ACB = \angle AXZ$.

Since $\angle ZAX + \angle AXZ = \angle CAB + \angle ACB = 90$, it follows that $\angle AZX = 90$, and so the triangles ABC and AXZ are similar. Thus

$$\frac{AZ}{AX} = \frac{AB}{AC} \text{ and } \frac{AZ}{1/3} = \frac{1}{\sqrt{10}/3}$$

and $AZ = 1/\sqrt{10}$. Using Pythagoras' Theorem, we obtain

$$YZ = \sqrt{1^2 - (1/\sqrt{10})^2} = 3/\sqrt{10}$$

The area of the shaded triangle is

$$\frac{1}{2} \cdot \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} = \frac{3}{20}$$

10. Each card is covering up a positive real number. The number written on each card is the product of all numbers covered by all of the other cards. Find the product of the three smallest numbers covered up by the cards.

(A) 3

(B) 4

(C) 6

(D) 12

Answer: 3

Solution: Denote the numbers under five cards by a, b, c, d, and e. Then

$$\begin{aligned} bcde &= 36 \\ acde &= 9 \\ abde &= 24 \\ abce &= 12 \\ abcd &= 18 \end{aligned}$$

We multiply the five equations to get

$$\begin{aligned} a^4b^4c^4d^4e^4 &= 36 \cdot 9 \cdot 24 \cdot 12 \cdot 18 \\ (abcde)^4 &= 2^2 \cdot 3^2 \cdot 3^2 \cdot 2^3 \cdot 3 \cdot 2^2 \cdot 3 \cdot 2 \cdot 3^2 = 2^8 \cdot 3^8 \\ abcde &= 2^2 \cdot 3^2 = 36 \end{aligned}$$

Thus,

$$a = \frac{36}{bcde} = 1, \ b = \frac{36}{acde} = 4, \ c = \frac{36}{abde} = \frac{3}{2}, \ d = \frac{36}{abce} = 3, \ e = \frac{36}{abcd} = 2$$

and the product of the three smallest numbers is 3.

Here is an alternative way to solve the above system of equations: divide the frist equation by the second equation, to obtain

$$\frac{bcde}{acde} = \frac{36}{9} \quad \text{i.e.,} \quad \frac{b}{a} = 4$$

and $b = 4a$. Divide the first equation by the third equation, to obtain $c = 1.5a$. In the same way we obtain $d = 3a$ and $e = 2a$.
Using the second card, we obtain

$$acde = a(1.5a)(3a)(2a) = 9$$

i.e., $a^4 = 1$ and so $a = 1$. (We could have used any other card, did not have to be the second card.) The product of three smallest numbers is $a(1.5a)(2a) = 3$.

The End