Math@Mac Online Mathematics Competition

Wednesday, November 29, 2017

Instructions:

There are **ten** questions. For multiple choice, select **one** of the selected options for each question. Otherwise, fill in the blank space(s) as required.

Check your answers carefully before submitting them. You will only be able to submit your answers once. Non-programmable, non-graphing calculators are permitted. You may not use any other resources including web-based ones.

Good luck!

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1. Assume that x_1 and x_2 are distinct solutions of the equation $x^2 + 2017x + 1 = 0$ and y_1 and y_2 are distinct solutions of the equation $y^2 + 2018y + 1 = 0$. Find the value of the expression

$$(x_1 - y_1)(x_2 - y_2)(x_1 - y_2)(x_2 - y_1)$$

2. Given are two distinct points in the plane, P and Q. A bug crawls in such a way that its distance from P is twice its distance from Q. What curve is the bug tracing?

(A) line

(B) parabola

(C) hyperbola

(D) circle

(E) none of the above

3. Emily and Johnny are playing a really exciting game, called "throw a coin ten times." They flip a coin, and if it's heads, Emily gets a point. If it's tails, Johnny gets a point. After ten throws, they compare their scores. What is the probability that the game is a tie? Express your answer as a fraction.

4. Denote by x_n the solution of the equation

$$\sqrt{x+1} - \sqrt{x} = \left(\sqrt{2} - 1\right)^n$$

where $n \geq 1$. Assume that m represents a certain integer. Then $x_{2018} - x_{2017}$ is equal to

(A) $\sqrt{2} + m$

(B) $2\sqrt{2} + m$

(C) m

(D) $m\sqrt{2} + 1$

(E) $m\sqrt{2} - 1$

(F) $m\left(\sqrt{2} + 1\right)$

5. Find all real numbers x which satisfy the equation

$$\left[(x-1)^2\right] = [x]$$

where [ ] denotes the greatest integer function. Recall the definition of $[x]$: If a real number x is written as $x = n + \alpha$, where n is an integer and $0 \leq \alpha < 1$, then $[x] = n$.
The solution is of the form $(a, b) \cup [c, d)$. Identify a, b, c, and d.

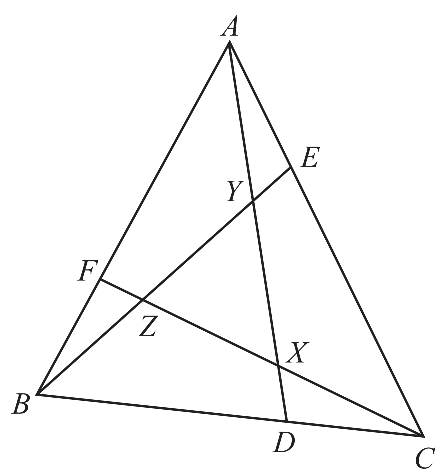
6. In the figure, CD, AE and BF are one-third of their respective sides. It can be shown that

$$AY : YX : XD = 3 : 3 : 1$$

with similar ratios for the segments BE and CF. If

$$\text{area } \Delta XYZ = x \cdot \text{area } \Delta ABC$$

what is x? Answer in the form of a fraction.



7. A jar contains at least two red balls and at least two white balls. If two balls are randomly drawn from the jar, the probability of drawing two red balls is five times the probability of drawing two white balls. Also, the probability of drawing one red and one white ball is six times the probability of drawing two white balls. How many red balls and how many white balls are in the jar?

8. Let $P(x)$ be a polynomial of degree four such that $P(2) = P(-2) = P(-3) = -1$ and $P(1) = P(-1) = 1$. What is $P(0)$?

9. Write

$$\frac{1}{196} = 0.d_1d_2d_3d_4\ldots$$

i.e., d_j is the j-th digit after the decimal point. What is d_{21}?

10. An equilateral triangle and a square are inscribed into a circle of radius 1, so that they have a common vertex. Find the area of the shaded region (i.e., the region common to both the triangle and the square).