Math@Mac Online Mathematics Competition

Wednesday, November 27, 2019

SOLUTIONS

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1. What is the sum of all positive integers from 1 to 500 which are NOT divisible by 4 or 5?

(A) $56,750$

(B) $68,500$

(C) $66,200$

(D) $75,000$

(E) $72,250$

Answer: (D) 75,000

Solution: The sum of all numbers from 1 to 500 is

$$a = \frac{500 \cdot 501}{2} = 125,250$$

From this sum we need to subtract all multiples of 4:

$$b = 4 + 8 + 12 + \cdots + 500 = 4(1 + 2 + 3 + \cdots + 125) = 4 \cdot \frac{125 \cdot 126}{2} = 31,500$$

and all multiples of 5:

$$c = 5 + 10 + 15 + \cdots + 500 = 5(1 + 2 + 3 + \cdots + 100) = 5 \cdot \frac{100 \cdot 101}{2} = 25,250$$

We subtracted too much! The multiples of $4 \cdot 5 = 20$ were subtracted twice. Compute

$$d = 20 + 40 + 60 + \cdots + 500 = 20(1 + 2 + 3 + \cdots + 25) = 20 \cdot \frac{25 \cdot 26}{2} = 6,500$$

Thus, the sum is

$$a - b - c + d = 125,250 - 31,500 - 25,250 + 6,500 = 75,000$$

2. Two polynomials $f(x)$ and $g(x)$ satisfy

$$f(x) \cdot g(x) = 4x^4 + 4x^3 + 13x^2 + 6x + 9 \qquad (P)$$
$$f(x) + g(x) = 4x^2 + 2x + 6 \qquad (S)$$

What is the product of all coefficients of $f(x)$? (Note: *all coefficients* includes free terms as well; for example, the product of all coefficients of the polynomial $x^3 - 3x^2 + 2x + 4$ is $(1)(-3)(2)(4) = -24$.)

Answer: 6

Solution: From (S), we deduce that neither polynomial is of degree larger than 2. The product formula (P) implies that both $f(x)$ and $g(x)$ are of degree 2.

Write $f(x) = ax^2 + bx + c$ and $g(x) = dx^2 + ex + f$.

Comparing the coefficients of x^4 in (P), we conclude that $4 = ad$, and comparing the coefficients of x^2 in (S), we get $4 = a + d$. Combining the two equations we obtain $ad = a(4 - a) = 4$, thus $a^2 - 4a + 4 = 0$, and the only solution is $a = 2$. Thus, $d = 2$.

Likewise, comparing free coefficients (i.e., no x terms) in (P) we obtain $cf = 9$, and by comparing free coefficients in (S) we obtain $c + f = 6$. Combining the two equations, $cf = c(6 - c) = 9$ and $c^2 - 6c + 9 = 0$; the only solution is $c = 3$, and so $f = 3$.
So we already know four (of six) coefficients.

From

$$f(x) + g(x) = (2x^2 + bx + 3) + (2x^2 + ex + 3) = 4x^2 + (b + e)x + 6$$

by comparing the coefficients of x, we get $b + e = 2$.
Likewise,

$$\begin{aligned} f(x) \cdot g(x) &= (2x^2 + bx + 3) \cdot (2x^2 + ex + 3) \\ &= 4x^4 + 2(b + e)x^3 + (6 + 6 + be)x^2 + 3(b + e)x + 9 \end{aligned}$$

Comparing the coefficients of x^2, we obtain $12 + be = 13$ and $be = 1$. Solving this system we obtain we obtain $b = 1$ and $e = 1$.
Thus, $abc = 6$.

3. Consider the following list of numbers:

$$y^2 + 9,\ \sqrt{x},\ 68,\ 111,\ 182,\ 296,\ 481$$

The numbers x and y also satisfy the equation

$$3y^2 - 9y + \frac{x}{4} = 484,$$

which makes the value of y unique. Find that value of y.

Answer: -4

Solution: Observing the last five integers of the list, we notice that

$$\begin{aligned} 68 + 111 + 3 &= 182 \\ 111 + 182 + 3 &= 296 \\ 182 + 296 + 3 &= 481 \end{aligned}$$

This pattern suggests that

$$\sqrt{x} + 68 + 3 = 111$$

and

$$y^2 + 9 + \sqrt{x} + 3 = 68$$

Solving the first equation for x, we get

$$\sqrt{x} = 111 - 68 - 3 = 40$$

and $x = 1600$. Substituting $\sqrt{x} = 40$ into the second equation yields

$$y^2 + 9 + 40 + 3 = 68$$

i.e., $y^2 = 16$ and $y = \pm 4$.
The additional condition $3y^2 - 9y + \frac{x}{4} = 484$ implies that

$$3y^2 - 9y + 400 = 484$$

i.e.,

$$3y^2 - 9y - 84 = 0 \quad \text{and} \quad 3(y + 4)(y - 7) = 0$$

Therefore, $y = -4$ and $y = 7$. Combining with the above, we see that $y = -4$.

4. How many prime numbers are there whose digits add up to four and are such that none of their digits are equal to zero?

Answer: 3

Solution: As none of the digits can equal zero, the prime can have at most 4 digits.

Since the digits must add up to 4 the only possibilities for the set of digits are $\{1,1,1,1\}$, $\{1,1,2\}$, $\{1,3\}$, $\{2,2\}$, and $\{4\}$.

The numbers 1111, 22, and 4 are not prime ($1111 = 11 \cdot 101$), thus the digits of the prime numbers we are looking for must be either 1,1,2 or 1,3.
Now 112 and 121 are not prime, but 211, 13, and 31 are. So there are exactly three prime numbers of the given form.

5. A square room with dimensions 4m by 4m has a trap door in the very centre of the floor. The trap door is a 1m by 1m square. A round barrel with straight vertical sides and a circular bottom of diameter 1m is placed upright randomly in the room (so the flat round bottom is sitting flat on the floor). If every allowable placement of the barrel in the room is equally likely, what is the probability that some part of the barrel will cover some part of the trap door?

(A) $1/3 + \pi/36$

(B) $1/16$

(C) $\pi - 1/9$

(D) $1/4 + \pi/24$

(E) $4/9 + \pi/18$

Answer: (A) $1/3 + \pi/36$

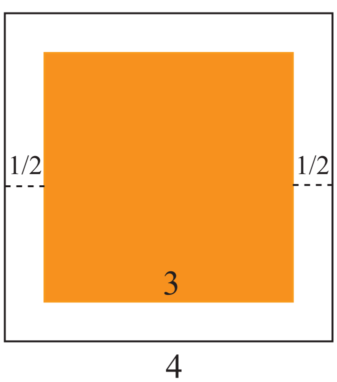
Solution: Consider the position of the centre of the barrel. Since the radius is 1/2 m, the centre of the barrel can be placed anywhere in the room that is at least 1/2 m from the nearest wall. This is a 3m by 3m square, giving 9 square meters of equally likely possible placements (below, left).

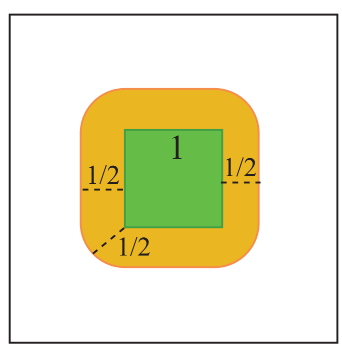
The portion of that area that has the barrel cover at least some part of the trap door is a rounded square around the trap door. The door will be partially covered if the centre of the barrel is (below, right):

(i) on the 1m by 1m trap door,

(ii) within a 1/2 m by 1m rectangle adjacent to any of the four sides of the door, or

(iii) within a quarter circle of radius 1/2 m centred at each of the four vertices of the trap door.





Thus the total area that results in some part of the trap door being covered is

$$1 \cdot 1 + 4 \cdot (1/2) \cdot 1 + \pi \cdot (1/2)^2 = 3 + \frac{\pi}{4}$$

The probability that the trap door is partially covered is

$$\frac{3 + \pi/4}{9} = \frac{1}{3} + \frac{\pi}{36}$$

6. Let $x_n = 1 + 2 + 3 + \cdots + n$. Find the largest value of n such that

$$x_n^3 - 127x_n^2 + 2331x_n - 2205 = 0$$

Answer: 14

Solution: By inspection, we see that $x_n = 1$ is a solution, so we factor. Thus,

$$x_n^3 - 127x_n^2 + 2331x_n - 2205 = (x_n^2 - 126x_n + 2205)(x_n - 1) = 0$$

Factoring the quadratic term, we obtain

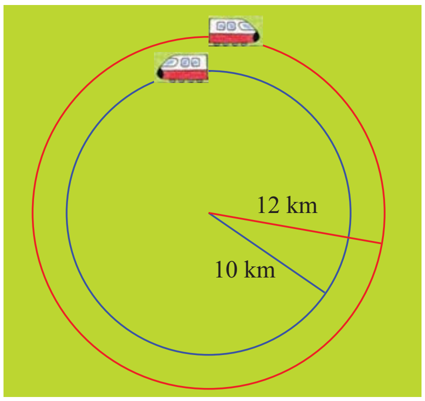
$$(x_n - 105)(x_n - 21)(x_n - 1) = 0$$

and so $x_n = 1, 21$, or 105.
The largest value of x_n will give the largest value of n.
By solving

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = 105$$

for n, we obtain $n^2 + n - 210 = 0$. Discarding the negative solution, we obtain $n = 14$.

7. Two trains travel in circular paths nested inside one another (i.e., their centres are at the same point) Train 1 travels around a circle of radius 12 km at the speed of 150 km/h, and train 2 travels around a circle of radius 10 km at the speed of 120 km/h. If the trains begin at the same point on their respective circles and travel in opposite directions for 4 hours straight, how many times will they pass each other?



(A) 12 times or less

(B) 13

(C) 14

(D) 15

(E) 16 times or more

Answer: (D) 15

Solution: Train 1 must cover 24π km to complete one full circle. Since it travels at 150 km/h, it takes $\frac{24\pi}{150} = \frac{4\pi}{25}$ hours to do it. Thus, in 4 hours, train 1 will complete

$$\frac{4}{\frac{4\pi}{25}} = \frac{25}{\pi}$$

revolutions around its circular path.

Train 2 must travel 20π km to complete one full circle, and since it travels at 120 km/h, it needs $\frac{20\pi}{120} = \frac{\pi}{6}$ hours to do it.

Suppose that the two trains meet. When will they meet again?

If, until they meet again, train 1 travels through the angle α, it will take $\frac{4\pi}{25}\frac{\alpha}{2\pi}$ hours to get there. Train 2 needs to cover the angle of $2\pi - \alpha$, taking $\frac{\pi}{6}\frac{2\pi-\alpha}{2\pi}$ hours. Thus

$$\frac{4\pi}{25} \cdot \frac{\alpha}{2\pi} = \frac{\pi}{6} \cdot \frac{2\pi - \alpha}{2\pi}$$

and so train 1 covers

$$\frac{\alpha}{2\pi} = \frac{\frac{\pi}{6}}{\frac{\pi}{6}+\frac{4\pi}{25}} = \frac{25}{49}$$

of its revolution around its circle.

Since

$$\frac{25}{\pi}/\frac{25}{49} = \frac{49}{\pi} \approx 15.597$$

it follows that the trains will pass each other 15 times.

8. What are the last two digits of the number

$$2^{2^{2019}}$$

(i.e., 2 raised to the power 2^{2019})?

Answer: 56

Solution: Let's try to find a pattern. To start, $2^{2^1} = 2^2 = 4$ and $2^{2^2} = 2^4 = 16$.
Then

$$2^{2^3} = 2^8 = 256$$

and the last 2 digits are 56.
Because

$$2^{2^4} = 2^{2^3} \cdot 2^{2^3}$$

the last two digits of 2^{2^4} are the last two digits of $56^2 = 3136$, thus 36. Likewise,

$$2^{2^5} = 2^{2^4} \cdot 2^{2^4}$$

and the last two digits of 2^{2^5} are the last two digits of $36^2 = 1296$, thus 96. Because

$$2^{2^6} = 2^{2^5} \cdot 2^{2^5}$$

the last two digits of 2^{2^6} are the last two digits of $96^2 = 9216$, thus 16. One more: because

$$2^{2^7} = 2^{2^6} \cdot 2^{2^6}$$

the last two digits of 2^{2^7} are the last two digits of $16^2 = 256$, thus 56.
We see a periodic pattern: $16, 56, 36, 96, 16, 56, 36, 96, \ldots$, where the last two digits of 2^{2^n} with $n \geq 2$ are:

16 if n divided by 4 gives the remainder of 2,

56 if n divided by 4 gives the remainder of 3,

36 if n is divisible by 4, and

96 if n divided by 4 gives the remainder of 1.

Since the remainder when 2019 is divided by 4 is 3, the last two digits of $2^{2^{2019}}$ are 56.

9. Kelly is creating strings from letters A, B, C, or D, with the following rules:

(a) A string must be 9 characters long

(b) Two adjacent characters must be different (e.g., ABCDABCDA is acceptable, but ABCDDACBA and ABBBACABA are not)

(c) No letter can appear more than 4 times (e.g., ABCACACAC and BABCBDBDA are acceptable, but ABACADADA is not).

By following these rules, how many different strings can Kelly make?

Answer: 25,920

Solution: We start by computing the number of strings that satisfy (a) and (b) only.

Kelly needs to build a string nine letters long, picking a letter and then appending letters to the right. There are four choices for the first letter, three for the second letter (any letter except the first letter), three for the third letter (any letter except the second letter), and so on. Thus, there are $4 \cdot 3^8$ strings that satisfy (a) and (b).

Next, Kelly wishes to eliminate all strings for which a single letter appears 5 or more times. Note that it is impossible for a single letter to appear 6 or more times, without appearing in adjacent locations. Thus, the only possibility is a letter appearing 5 times, and in that case, that letter must be in the 1st, 3rd, 5th, 7th and 9th positions.

Thus, the only configuration Kelly needs to consider are strings of the form

XyXyXyXyX

where X is one of A, B, C, or D, and y is any of the three letters distinct from X.
There are 4 choices for X, and with each of these choices, there are 3^4 options to fill the y positions. Thus, there are $4 \cdot 3^4$ strings in which the same letter appears five times.

Consequently, Kelly can make a total of

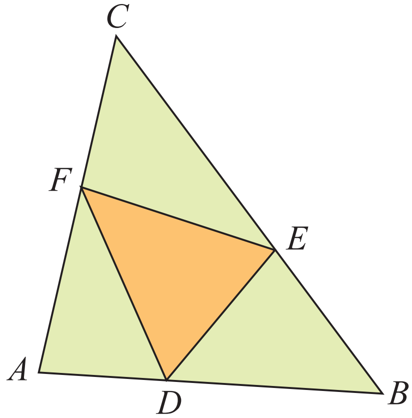
$$4 \cdot 3^8 - 4 \cdot 3^4 = 25920$$

different strings.

10. In the figure below ($\overline{XY}$ denotes the length of the line segment from X to Y)

$$\frac{\overline{AD}}{\overline{DB}} = \frac{\overline{BE}}{\overline{EC}} = \frac{\overline{CF}}{\overline{FA}} = \frac{2}{3}$$

The area of the triangle ABC is 1. What is the area of the triangle DEF?



(A) 1/4

(B) 6/25

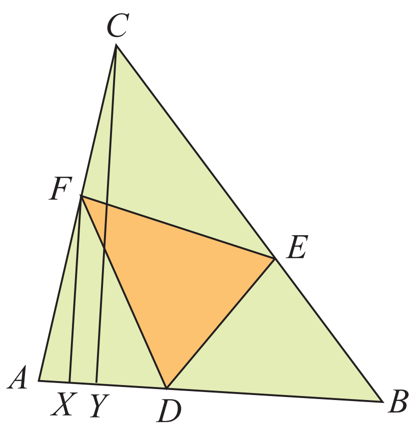
(C) 7/25

(D) 3/16

(E) 5/16

Answer: (C) 7/25

Solution: Draw the altitude FX from F to AB and the altitude CY from C to AB, as shown.



Combining $\frac{\overline{CF}}{\overline{FA}} = \frac{2}{3}$ and $\overline{CF} + \overline{FA} = \overline{CA}$, we obtain

$$\frac{2}{3} = \frac{\overline{CF}}{\overline{FA}} = \frac{\overline{CA} - \overline{FA}}{\overline{FA}} = \frac{\overline{CA}}{\overline{FA}} - 1$$

i.e., $\frac{\overline{CA}}{\overline{FA}} = \frac{5}{3}$ and

$$\overline{FA} = \frac{3}{5}\overline{CA}$$

Since the triangle FAX is similar to the triangle CAY, we conclude that

$$\overline{FX} = \frac{3}{5}\overline{CY}.$$

From $\frac{\overline{AD}}{\overline{DB}} = \frac{2}{3}$ it follows that

$$\frac{\overline{AB}}{\overline{AD}} = \frac{\overline{AD} + \overline{DB}}{\overline{AD}} = 1 + \frac{\overline{DB}}{\overline{AD}} = \frac{5}{2}$$

and

$$\overline{AD} = \frac{2}{5}\overline{AB}.$$

The area of the triangle ADF is

$$\frac{1}{2} \cdot \overline{FX} \cdot \overline{AD} = \frac{1}{2} \cdot \frac{3}{5} \cdot \overline{CY} \cdot \frac{2}{5} \cdot \overline{AB} = \frac{6}{25} \cdot \frac{1}{2} \cdot \overline{CY} \cdot \overline{AB},$$

which is 6/25 of the area of the triangle ABC, i.e., 6/25.

Likewise, each of the triangles EDB and ECF has the area of 6/25, and thus the area of the triangle DEF is $1 - 3(6/25) = 7/25$.