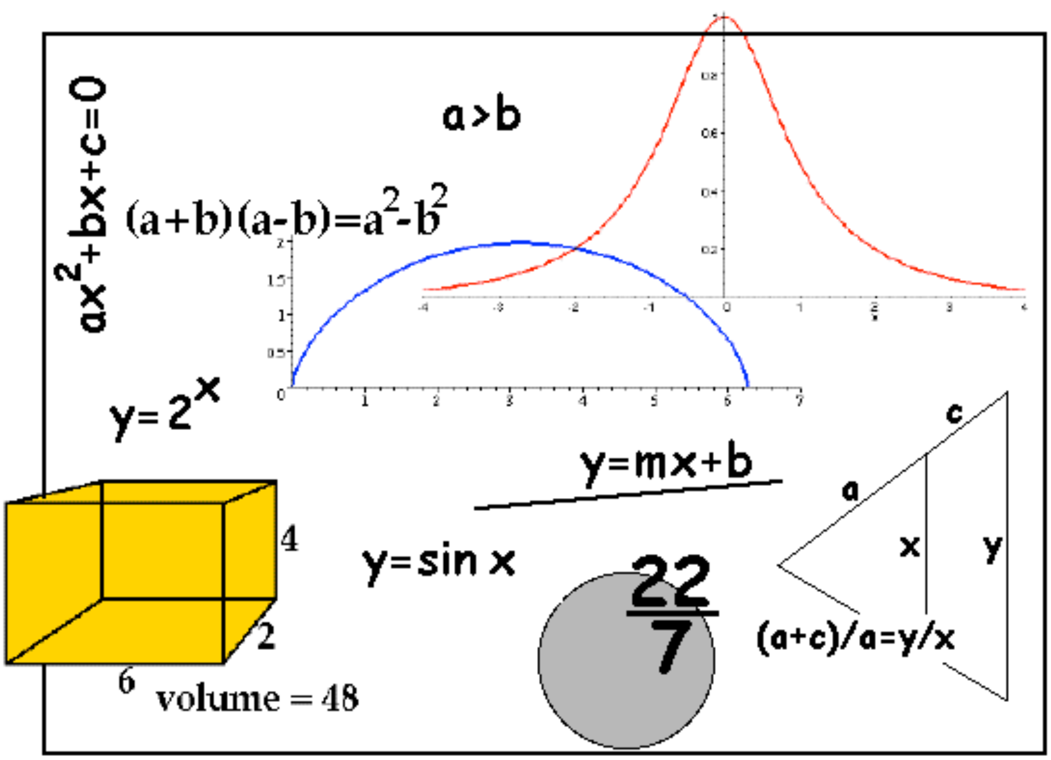
Mathematics Review Manual

With a Brief

First-year Survival Guide

For Students Entering McMaster University



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Expanded version, published as a book, is at
https://ms.mcmaster.ca/lovric/rm.html

written by

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Hamilton 2009-2021

This booklet will help you prepare
for your academic life as a first-year university student,
and, in particular, for the first-year courses in
mathematics that you will take.

If you think that you need more practice, consider
buying an expanded version of this booklet, at
https://ms.mcmaster.ca/lovric/rm.html
[if clicking on this link does not work, copy/paste it into your browser]

Table of Contents

General Information

Mathematics

You have to know and be proficient in the material from the following four chapters. Very little of it will be reviewed in class.

The material from the three chapters below will be taught and discussed in your first-year calculus course. However, you will have to spend extra time working on these areas, to gain technical proficiency and confidence with the material.

The material from the chapter below will be covered in depth in your first-year calculus course. If you decide to skip something in this manual, then skip this chapter.

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This manual is also available online at (free download)
www.math.mcmaster.ca/lovric/rm.html

What's this manual and survival guide about?

A leap from secondary education to university environment will be, without doubt, one of the most challenging and stressful events in your life. It is a true rite of passage, with all of its anxieties, pains, hopes, frustrations, joys and rewards. You have probably created a mental image of the new environment you will be encountering soon - but it is blurry, lots of fine detail is missing. The better prepared you are, the easier it will be to adjust to new situations, demands and expectations that university life will place on you.

No matter which high school you came from, you have certain strengths and certain weaknesses. There are things that you learnt well in high school, things you know and are comfortable with. But, there are things that you forgot, or you don't know about or have very little experience with. In high school you acquired certain skills, but need to brush up on some others.

This manual will tell you where you are; it will help you identify those areas of mathematics that you are good at, and those areas that you need to learn, review and work on. All you need is a little dedication, a pencil and paper, and about an hour of your (uninterrupted) time per day (say, during the last three weeks of August). Unplug the TV, turn off your cell phone, kick your sibling(s) and/or your parent(s) out of your room, and tell them that you need to work on something really important.

This manual has two parts. The first part is about the things you have thought a lot lately. How is life in university different from high school? What should I expect from my fist-year classes? How is university math different from math in high school? Read, and reflect on the issues raised ... discuss it with your parents, friends, teachers, or older colleagues. Nobody can give you detailed and precise answers to all questions that you have, but at least, you will get a good feeling about the academic side of your first-year university experience.

What will my first-year professors assume that I know about mathematics? The big part of this booklet is dedicated to answering this question. Look at the table of contents to see what areas of mathematics are covered.

As I said, have a pencil and paper handy. I suggest that you start with the first section, and work from there, without skipping sections. Read the material presented in a section slowly, with understanding; make notes and try to solve exercises as you encounter them (answers to all exercises are in the back of this manual). Even if the material in some section looks easy, do not skip the whole section – select several problems and test your knowledge.

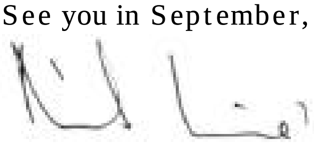
If you realize that you have **problems with certain material**, read the section carefully, twice or three times if needed. Work on problems slowly, making sure that you understand what is going on. If needed, consult your high school textbooks, or go to a local library and find a reference. Ask somebody who knows the stuff to discuss it with you; if you prefer, hire a private math tutor for a few sessions.

I know that doing math is not the coolest thing to do in summer - BUT think a bit about the future. Change from high school to university is a big change; the better prepared you are, the easier it will be for you to adjust successfully to your new life as a university student. Student life is a busy life. It will be quite difficult for you (I did not say impossible!) to find time to do two things: learn new material presented in a lecture and, at the same time, review background material that you are assumed to know and be comfortable with. Not to mention that, without adequate preparation, you will have difficulties following lectures. **Review your math now, while you have lots of free time on your hands!**

One thing is certain: **the more math you do, the easier it gets** - experience helps! **Do as many problems as you can, don't give up because the stuff looks difficult or you feel bored with it.** Little investment of your time now, in summer, will make studying mathematics in the fall a whole lot easier.

Good luck!

See you in September,



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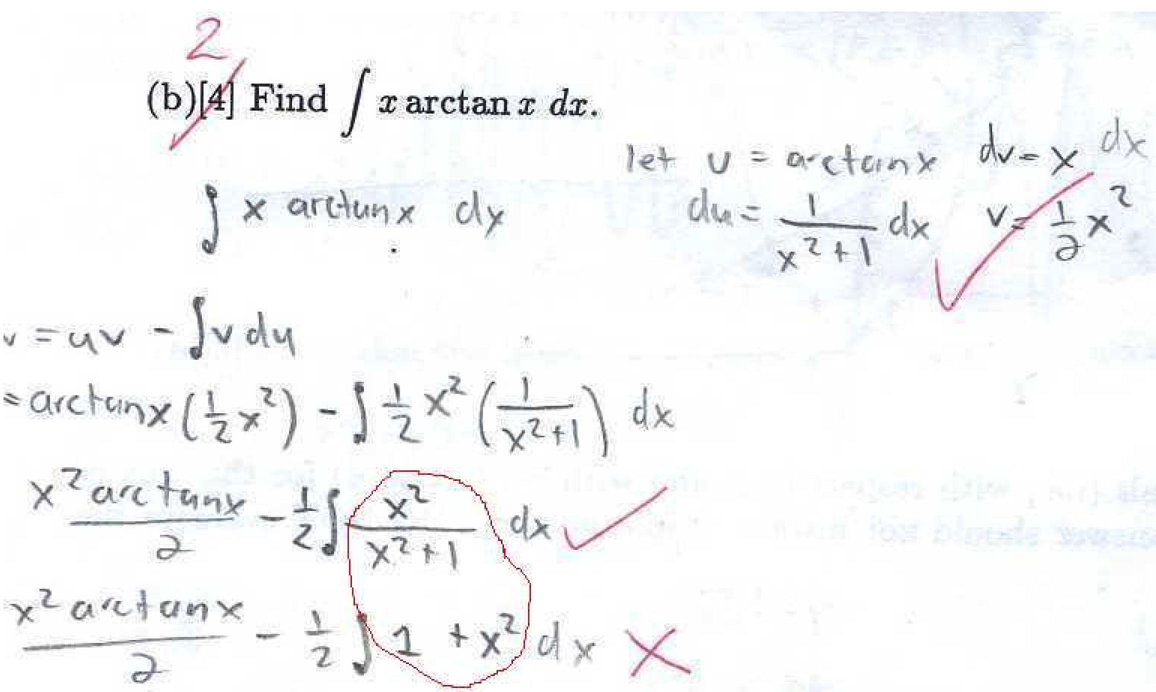
Why Background Knowledge Matters

Mathematics is cumulative, **new material builds upon the previously covered (i.e., understood, learnt) material.** It is not possible to truly understand and apply an advanced concept (say, derivatives) without understanding all basic concepts that are used to define it (fractions, limits, graphs, etc.).

Many times, **the reason why students lose marks on tests in first-year Calculus (and other math courses!) is due to a problem with something elementary**, such as fractions, simplifying, solving equations, or recalling basic properties of exponents, trigonometric functions. etc. Let us look at a few samples of actual test solutions.

In the case below, the student chose the appropriate integration method (which is taught in the first-year Calculus course), but then did not simplify correctly the fraction in the integral (see the last two lines). This error cost the student 50% of the credit for the question.



2

(b)[~~4~~] Find $\int x \arctan x \, dx$.

let $u = \arctan x$ $\quad dv = x\,dx$

$du = \frac{1}{x^2+1}dx$ $\quad v = \frac{1}{2}x^2$

$\int x \arctan x \; dx$

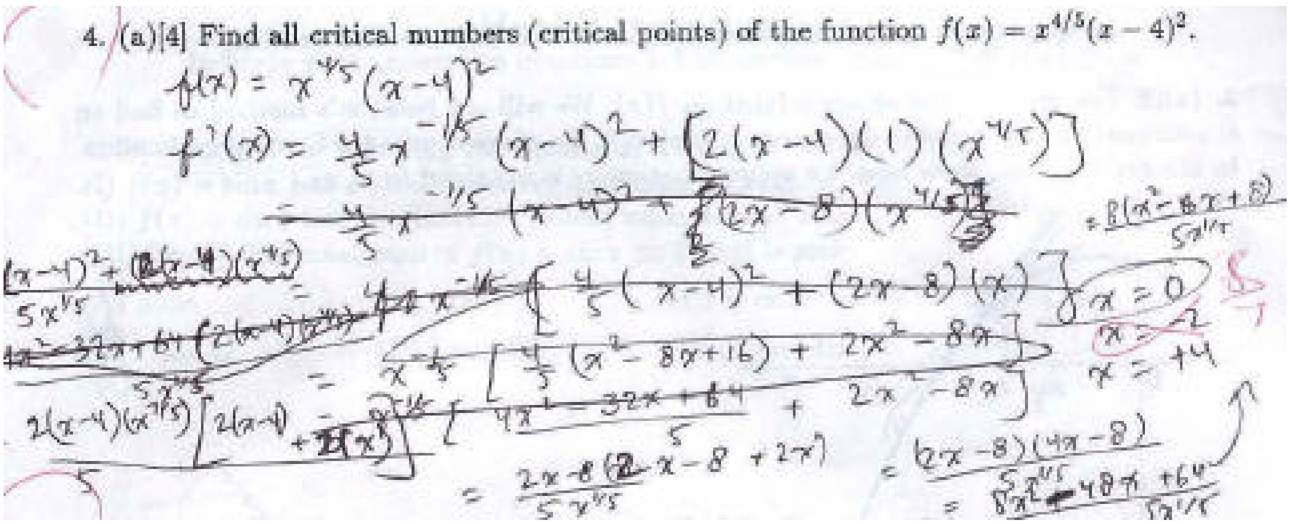
$= uv - \int v\,du$

$= \arctan x\left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2\left(\frac{1}{x^2+1}\right)dx$

$\frac{x^2 \arctan x}{2} - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx$

$\frac{x^2 \arctan x}{2} - \frac{1}{2}\int 1 + x^2\,dx$

Look how mach effort was put into simplifying the expression for f'(x) below – not to mention **how much valuable time was lost!** Moreover, the student made a mistake in simplifying and got two (of three) correct values for x. Penalty for this mistake: 25% of the credit for the question. As in the previous case, **note that the credit lost was not due to a new concept learnt in the university Calculus course, but due to errors related to high school material.**


4. (a)[4] Find all critical numbers (critical points) of the function $f(x) = x^{4/5}(x-4)^2$.

$f(x) = x^{4/5}(x-4)^2$

$f'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + [2(x-4)(1)(x^{4/5})]$

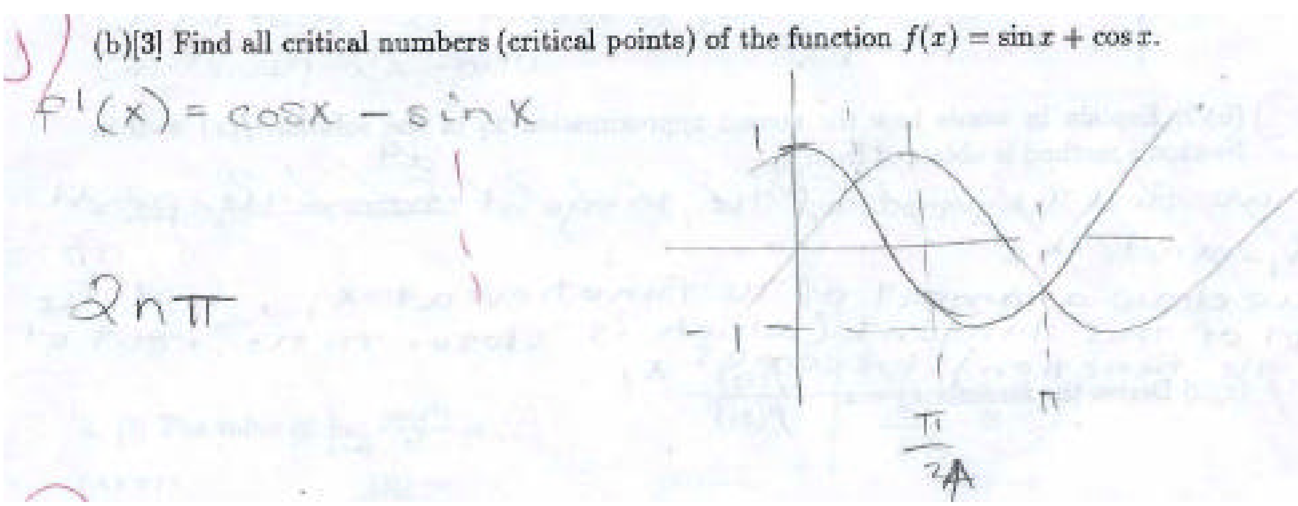
$x = 0$

$x = 2$

$x = +4$

Most-often-heard comment about a test is that **'there was not enough time.' Certainly, if it takes you more than 5 minutes to do this question, you will not have enough time to complete the test.**

In the case below, the student tried to analyze the expression for f'(x) by looking at the graphs of sin x and cos x (excellent idea!). However, the graphs of the two functions are incorrect, and the answer does not make sense. The student lost all credit for the question.


(b)[3] Find all critical numbers (critical points) of the function $f(x) = \sin x + \cos x$.

$f'(x) = \cos x - \sin x$

$2n\pi$

TRANSITION FROM HIGH SCHOOL TO UNIVERSITY

What is new and different in university? Well, almost everything: **new people** (your peers/colleagues, teaching and lab assistants, teachers, administrators, etc.), **new environment, new social contexts, new norms,** and – very important - **new demands and expectations. Think about the issues raised below. How do you plan to deal with it? Read tips and suggestions, and try to devise your own strategies.**

First-year lectures are large – you will find yourself in a huge auditorium, surrounded by 300, 400, or perhaps even more students. Large classes create **intimidating situations**. You listen to a professor lecturing, and hear something that you do not understand. Do you have enough courage to rise your hand and ask the lecturer to clarify the point? Keep in mind that you are not alone – other students feel the same way you do. It's hard to break the ice, but you have to try. Other students will be grateful that you asked the question – you can be sure that lots of them had exactly the same question in mind.

Remember, **learning is your responsibility**. Come to classes regularly, be active, take notes, ask questions. Find a quiet place to study. Use all resources available to you. Discuss material with your colleagues, teaching and lab assistants, and/or professors.

Courses have different requirements and restrictions with regards to **calculators and computer software**. You will find the information about it in the course syllabus that will be given to you (usually) in the first lecture of a course.

The **amount of personal attention** you get from your teachers, compared to high school, is drastically lower. If you have a question, or a problem, you will have to make an effort to talk to your lecturer, or to contact the most appropriate person.

Consider taking courses that help you develop research skills (such as: critical use of electronic resources, logical and critical thinking, library search skills, communication and presentation skills etc.). Have you heard of **inquiry courses**?

Good time management is essential. Do not leave everything for the last moment. Can you complete three assignments in one evening? Or write a major essay and prepare for a test in one weekend? Plan your study time carefully. Eat well, exercise regularly, plan social activities - have a life! **Amount of material covered** in a unit of time increases at least three-fold in university courses, compared to high school. This means that things happen very quickly. If you miss classes and do not study regularly you will get behind in your courses. Trying to catch up is not easy. **For each hour of lecture** plan to spend (at least) three hours studying, reviewing, doing assignments, etc.

Inquire about **learning resources** available to you. Do you know where the science (engineering, humanities) library is? When are computer labs open? Do you know how the Centre for Student Development can help you deal with academic issues? Before coming to Mac, browse through its internet site. Bookmark the sites that link to learning resources.

'**In university grades drop by 30%**.' Not necessarily. Study regularly (do you know how to study math? Chemistry? Physics? Why not discuss it with your lecturer?). Most probably, you will have to adjust/modify your present study habits. One thing is certain: the amount of work that earned you good marks in high school will not suffice to keep those marks in university.

It is possible to **study hard and still fail a test**. If you fail a test, react immediately. Identify reasons for your poor performance. Visit your professor during her/his office hours, bring your test and discuss it. Be ready to re-examine and modify your study strategies. Do not get discouraged by initial bad marks that you might get.

If you have **problems**, react and deal with it immediately. Ask your professor for advice. Talk to an adviser in your faculty office.

What constitutes **academic dishonesty**? Copying stuff from internet and pasting into your assignment could be considered academic dishonesty. If you are caught, you might fail the course. What other practices are considered academic dishonesty? Be informed about it, so that you don't get in trouble. The course syllabus (for any of your courses) will provide information and links to McMaster policy regarding academic dishonesty.

HOW IS MATH AT UNIVERSITY DIFFERENT FROM HIGH SCHOOL MATH?

Lectures move at a faster pace. Usually, one lecture covers one section from your textbook. Although lectures provide necessary theoretical material, they rarely present sufficient number of worked examples and problems. You have to do those on your own.

Certain topics (trigonometry, exponential and logarithm functions, vectors, matrices, etc.) **will be taught and/or reviewed** in your first-year calculus and linear algebra courses. However, the time spent reviewing in lectures will not suffice to cover all details, or to provide sufficient number of routine exercises – you are expected to do it on your own. Use this manual! Don't leave it at home, bring it with you to McMaster.

You have to know and be proficient with the material covered in the first four chapters of this booklet:

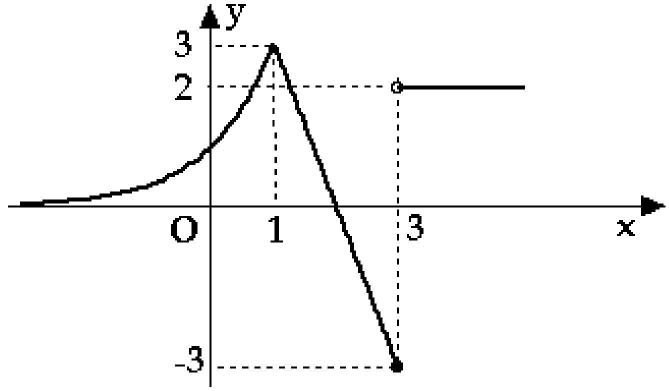
- Basic Algebra
- Basic Formulas from Geometry
- Equations and Inequalities
- Elements of Analytic Geometry.

For instance, computing common denominators, solving equations involving fractions, graphing the parabola $y=x^2$, or solving a quadratic equation will not be reviewed in lectures.

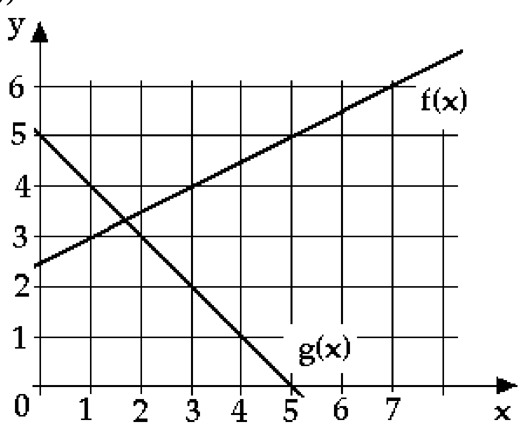
In university, there is **more emphasis on understanding** than on technical aspects. For instance, your math tests and exams will include questions that will ask you to quote a definition, or to explain a theorem, or answer a 'theoretical question.' Here is a sample of questions that appeared on past exams and tests in the first-year calculus course:

- Is it true that $f'(x)=g'(x)$ implies $f(x)=g(x)$? Answering 'yes' or 'no' only will not suffice. You must explain your answer.
- State the definition of a horizontal asymptote.
- Given the graph of $1/x$, explain how to construct the graph of $1+1/(x-2)$.

- Using the definition, compute the derivative of $f(x)=(x-2)^{-1}$.
- Can a polynomial of degree 3 have two inflection points? You must explain your answer to get full credit.
- Given below is the graph of the function f(x). Make a rough sketch of the graph of its derivative f'(x).



- Below are the graphs of two functions, f(x) and g(x). Compute the composition g(f(3)).



You will be allowed - and encouraged - to use your **(graphing) calculator** and/or **computer software** (such as Maple) to study mathematics, to do homework assignments and computer labs. On tests and exams, either no calculator will be allowed (would you really need a calculator to answer any of the above test/exam questions?), or you will be asked to use the calculator that McMaster chose as a standard (this way, everybody uses the same calculator). Calculators and software are an aid, but not a replacement for your brain, and you should treat them as such. If a calculator says something, it is not necessarily a correct answer.

Mathematics is not just formulas, rules and calculations. In university courses, you will study definitions, theorems, and other pieces of 'theory.' Proofs are integral parts of mathematics, and you will meet some in your first-year courses. You will learn how to approach learning 'theory,' how to think about proofs, how to use theorems, etc.

Layperson-like attitude towards mathematics (and other disciplines!) - accepting facts, formulas, statements, etc. at face value - is no longer acceptable in university. **Thinking (critical thinking!)** must be (and will be) integral part of your student life. In that sense, you must accept the fact that proofs and definitions are as much parts of mathematics as are computations of derivatives and operations with matrices.

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Have you given any thought to **Mathematics as a career?** Attend information sessions organized by Mathematics and Statistics Department (will be advertised in lectures), learn about programs and careers in mathematics and statistics. Get informed, keep your options open!

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If, for some reason, you developed negative attitude and feelings towards mathematics in high school, then leave them there! You will have a chance to start fresh at a university. First-year math courses at McMaster start at a level that is appropriate for most high school graduates. Use this manual, read it from cover to cover, get prepared!

LEARNING MATHEMATICS

Learning mathematics requires seriousness, dedication, discipline, concentration, significant amount of time and hard work. Your teachers will help you learn how to learn mathematics.

To learn mathematics means to understand AND to memorize.

To understand something means to be able to correctly and effectively communicate it to somebody else, in writing and orally; to be able to answer questions about it, and to be able to relate it to known mathematics material. Understanding is a result of a thinking process. It is not a mere transfer from the one who understands (your lecturer) to the one who is supposed to understand (you).

How do you make yourself understand math? Ask questions about the material and answer them (either by yourself, or with the help of your colleague, teaching assistant or lecturer). Approach material from various perspectives, study solved problems and work on your own on problems and exercises. Make connections with previously taught material and apply what you just learnt to new situations.

It is necessary to memorize certain mathematics facts, formulas and algorithms. Memorizing is accomplished by exposure: by doing drill exercises, by using formulas and algorithms to solve exercises, by using mathematics facts in solving problems.

The only way to master basic technical and computational skills is to solve a large number of exercises. Drill.

It is impossible to understand new mathematics unless one has mastered (to a certain extent) the required background material.

right approach to learning math:

understand so that you won't have to memorize

wrong approach to learning math:
memorize so that you won't have to understand

think about it!!!

IMPORTANT LITTLE BITS

Come to classes, tutorials and labs regularly, be active! Think, ask questions in class, give feedback to your lecturer.

Lecture by itself will not suffice. You need to spend time on your own doing math: studying, working on assignments, preparing for tests and exams, etc. Rule of thumb: three hours on your own for each hour of lecture.

Plan your study time carefully. Don't underestimate the amount of time you need to prepare for a test, or to work on an assignment; try not to do everything in the last minute (the fact that your hard drive crashed night before your assignment is due is not an acceptable excuse for a late assignment).

Make sure that you are aware of (and use!) **learning resources** available to you. Here are some of them:

- Lectures, tutorials, and review sessions
- Your lecturer's office hours
- Your teaching assistant's office hours
- E-mail and course internet page
- Mathematics help centre in Hamilton Hall
- Thode Library (Science Library)
- Centre for Student Development

Always learn by understanding. Memorizing will not get you too far. Think, do not just read; highlighting every other sentence in your textbook is not studying!

If you are able to explain something to a colleague and answer their questions about it, **then you have learnt it!**

Drill is essential for a success (not just in math!). It's boring, but it works! Solving hundreds of problems will help you gain routine and build confidence you need (together with a few other things) to write good exams.

You will be allowed to use a **(graphing) calculator and/or computer software** for assignments and labs. On tests and exams, you might not be allowed to use a calculator, or will have to use a model that is accepted as a standard at McMaster. Your instructor will give you detailed information about this.

Note about your lecture notes

Your lecture notes will be your most valuable resource. You will refer to them when you do homework, a computer lab, or prepare for a test or an exam. So:

- during a lecture, take notes
- later, read the notes; make sure that you have correct statements of all definitions, theorems, and other important facts; make sure that all formulas and algorithms are correct, and illustrated by examples
- fill in the gaps in your notes, fix mistakes; supplement with additional examples, if needed
- add your comments, interpret definitions in your own words; restate theorems in your own words and pick exercises that illustrate their use
- write down your questions, and attempts at answering them; discuss your questions with your colleague, lecturer or teaching assistant, write down the answers
- it is a waste of time to try again and yet again to understand a concept; so, once you understood it, write it down correctly, in a way that you will be able to understand later; this way, studying for an exam consists of re-calling and not re-learning; re-calling takes less time, and is easier than re-learning
- keep your notes for future reference: you might need to recall a formula, an algorithm or a definition in another mathematics course.

1. Basic Algebra

This section contains review material on:

- Real numbers, intervals and absolute value
- Polynomials
- Radicals and rational expressions
- Fractional expressions

Real Numbers and Its Famous Subsets. In Calculus, we deal with the set of real numbers and its subsets. One important subset is the set of integers

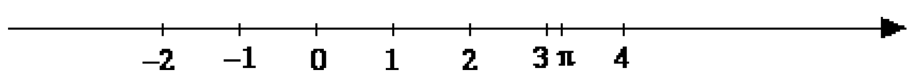
$$\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

The numbers $0, 1, 2, 3, 4, \ldots$ are called non-negative integers; the set of natural numbers consists of the numbers $1, 2, 3, 4, \ldots$ The quotient $\frac{p}{q}$ of two integers p and q (where $q \neq 0$) is called a rational number. For example, $\frac{1}{3}, \frac{-4}{11}, 0.47 = \frac{47}{100}, -32 = \frac{-32}{1}, 13 = \frac{13}{1}$ are rational numbers. (Looking at the last two examples, we see that integers belong to the set of rational numbers.) It is important to remember that division by zero is not allowed. Expressions such that $\frac{7}{0}$ or $\frac{0}{0}$ are not defined. However, zero divided by a non-zero number is zero; for example, $\frac{0}{7} = 0$ or $\frac{0}{-11} = 0$.

The numbers that cannot be represented as quotients of rational numbers are called irrational numbers. Numbers such as $\sqrt{2}$, π, $\sqrt{5}$ and $\log_{10} 2$ are irrational. The set of real numbers consists of rational and irrational numbers. It is usually denoted by $\mathbb{R}$. We can use decimal notation to express real numbers. Rational numbers have a repeating decimal representation; for example, $\frac{1}{2} = 0.50000 = 0.5\overline{0}$, $\frac{1}{3} = 0.33333 = 0.\overline{3}$, $\frac{13}{44} = 0.295454545 = 0.295\overline{45}$, etc. Non-repeating decimal expressions represent irrational numbers: $\pi = 3.1415926535\ldots$, $\sqrt{2} = 1.414213562\ldots$, etc.

Real numbers are ordered. $a < b$ means that "a is less than b," $a > b$ means that "a is greater than b," $a \leq b$ means that "a is less than b or a is equal to b," and $a \geq b$ means that "a is greater than b or a is equal to b." For example, $\sqrt{2} < 1.42$, $\frac{3}{5} \geq 0.6$, $4 \leq 4$, $\pi > 3.1415$, etc.

Number Line and Intervals. Real numbers can be represented visually as points on a number line, see the figure below.



We choose an arbitrary point on the line to be the origin (denoted by O, it represents the number zero), and indicate the direction in which the numbers increase by an arrow. For example, $a < b$ means that a lies to the left of b on the number line.

An open interval (a, b) consists of all real numbers between a and b, not including a and not including b. In symbols, (a, b) consists of all real numbers x such that $a < x < b$. If we need to include endpoints, we use square brackets. For example, $[a, b)$ represents all numbers x such that $a \leq x < b$. In words, the interval $[a, b)$ contains all real numbers between a and b, including a but not including b.

To denote the set of all numbers that are greater than a, we use (a, ∞). If we need all numbers that are greater than or equal to a, we use $[a, \infty)$. The numbers smaller than b form the interval $(-\infty, b)$. Remember that ∞ and $-\infty$ are not real numbers (thus, when infinity is involved, we always use round parentheses).

We could also use set-theoretic notation. For example, $[a, b) = \{x \in \mathbb{R} \big| \ a \leq x < b\}$, $(-\infty, b] = \{x \in \mathbb{R} \big| \ x \leq b\}$ or $(a, \infty) = \{x \in \mathbb{R} \big| \ x > a\}$.

Example 1. Describe the following sets of real numbers in interval notation:
(a) $\{x \in \mathbb{R} \big| \ 4 \leq x \leq 7\}$ (b) all real numbers smaller than 3
(c) $\{x \in \mathbb{R} \big| \ x \leq -31\}$ (d) all real numbers.

Solution

(a) $[4, 7]$ (b) $(-\infty, 3)$ (c) $(-\infty, -31]$ (d) $(-\infty, \infty)$. ∎

Absolute Value. The absolute value of a real number a is defined as

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Thus, $|6| = 6$, $|0| = 0$ and $|-13| = -(-13) = 13$.

The distance between two numbers a and b on a number line is given by $|b - a|$.

Exercise 1. (a) Compute $|\pi|$, $-|-1/2|$ and $|-0.33|$.

(b) Find the distance between -3 and -4 on the number line.

____________________ ♣

Properties of absolute value		
$\lvert ab\rvert = \lvert a\rvert \ \lvert b\rvert$	$\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a\rvert}{\lvert b\rvert}$	$\lvert a^n\rvert = \lvert a\rvert^n$

Integers as Exponents. If a is a real number and $n = 1, 2, 3, \ldots$ a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{\text{n factors}}.$$

By definition, $a^0 = 1$ (for $a \neq 0$). We usually drop the exponent 1, and write a instead of a^1. If $a \neq 0$ and $n = 1, 2, 3, \ldots$, then we define

$$a^{-n} = \frac{1}{a^n}.$$

Rules for exponents

$$a^m \cdot a^n = a^{m+n} \qquad (ab)^n = a^n b^n \qquad (a^m)^n = a^{mn}$$
$$\frac{a^m}{a^n} = a^{m-n} \qquad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

Exercise 2. Compute the following:
(a) $(-3)^4$ (b) 4^{-3} (c) 0^{-3} (d) $(1/2)^{-3}$.

♣

Polynomials. A term (or a monomial) is either a real number or a product of a real number and a positive integer power of one (or more) variables. Examples of terms are: 0.4, $3x^3$, $-4.5y^5$, $2x^3y^4$, etc. A polynomial is a sum or a difference of terms (monomials). The expressions $0.5+2x-4x^3+x^5$ and $4x^2y-3xy^3z-12xyz$ are examples of polynomials. The former is a polynomial in one variable, and the latter is a polynomial in three variables.

A polynomial with two terms is also called a binomial. If a polynomial contains three terms, it is called a trinomial. Polynomials are added/subtracted by adding/subtracting the like terms. For example, the sum of the binomial $5x^2y^3 - x^3y^3$ and the trinomial $x - 2x^2y^3 + x^3y^3$ is equal to $x+3x^2y^3$. Their difference is $(5x^2y^3-x^3y^3)-(x-2x^2y^3+x^3y^3) = 7x^2y^3-2x^3y^3-x$. To multiply two polynomials, we multiply each term of the first polynomial with each term of the second polynomial. In case of two binomials, we can use (a, b, c and d are the terms)

$$\boxed{(a+b)(c+d) = ac+ad+bc+bd}$$

This rule is sometimes referred to as the FOIL method.

Special products

$$(a+b)^2 = a^2+2ab+b^2$$
$$(a-b)^2 = a^2-2ab+b^2$$
$$(a+b)(a-b) = a^2-b^2$$

Example 2. Determine the following:
(a) $(x-4)(2x^3+x)$ (b) $(y-4)^2$ (c) $(x^3+y^2)^2$
(d) $(x^2-12)(x^2+12)$ (e) $(x+1)^3$.

Solution

(a) Using the FOIL method, we get

$(x-4)(2x^3+x) = (x)(2x^3)+(x)(x)+(-4)(2x^3)+(-4)(x) = 2x^4-8x^3+x^2-4x.$

(b) $(y-4)^2 = (y)^2 - 2(y)(4) + (4)^2 = y^2 - 8y + 16$.

(c) $(x^3 + y^2)^2 = (x^3)^2 + 2(x^3)(y^2) + (y^2)^2 = x^6 + 2x^3y^2 + y^4$.

(d) using the difference of squares formula, we get $(x^2 - 12)(x^2 + 12) = (x^2)^2 - (12)^2 = x^4 - 144$.

(e) $(x+1)^3 = (x+1)^2(x+1) = (x^2+2x+1)(x+1) = x^3+2x^2+x+x^2+2x+1 = x^3+3x^2+3x+1$.

In (e), we could have used the ready-made formula for $(a+b)^3$, see the box below. ∎

$$\boxed{(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3}$$

Exercise 3. Compute the following:

(a) $(x^2 - x + 1)(3x - 4)$

(b) $(2xy - y^2)^2$

(c) $(x^2 - 0.2)^2$

(d) $x(x^3 - 1)(x^3 + 1)$.

(e) $(2x - 1)^3$.

♣

Factoring. In a number of situations, it is useful to rewrite a given polynomial as a product. Remember that every polynomial can be factored into a product of linear factors (i.e., polynomials of degree one) and quadratic factors (i.e., polynomials of degree two that cannot be broken further into linear factors).

There are several ways of doing this.

We can factor out a common expression; for example,

$$16x^4 - 4x^3 - 4x = 4x(4x^3 - x^2 - 1).$$

Alternatively, one of the following formulas might be useful:

difference of squares:	$a^2 - b^2 = (a+b)(a-b)$
difference of cubes:	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
sum of cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

The sum of the squares $(a^2 + b^2)$, cannot be factored.

We can try factoring by grouping, such as in

$$x^3 - 4x^2 + 2x - 8 = x^2(x-4) + 2(x-4) = (x^2+2)(x-4).$$

Finally, we could try to factor a trinomial by trial and error. Probably the most common case is a trinomial $x^2 + mx + n$; if it can be factored as $(x+a)(x+b)$, then the sum $a+b$ must be equal to m and the product ab must be n.

Example 3. Factor the following expressions:

(a) $6x^6 - 24x^2 + 12x$

(b) $4y^5 - 3y^3 - 4y^2 + 3$

(c) $4a^4 - 0.25$

(d) $8 + 64x^9$ (e) $x^2 + 7x + 10$.

Solution

(a) $6x^6 - 24x^2 + 12x = 6x(x^5 - 4x + 2)$.

(b) Grouping the first two and the last two terms, we get

$$4y^5 - 3y^3 - 4y^2 + 3 = y^3(4y^2 - 3) - (4y^2 - 3) = (y^3 - 1)(4y^2 - 3).$$

We can factor further:

$$(y^3 - 1)(4y^2 - 3) = (y - 1)(y^2 + y + 1)(2y - \sqrt{3})(2y + \sqrt{3}).$$

(c) Use the difference of squares formula:

$$4a^4 - 0.25 = (2a^2)^2 - (0.5)^2 = (2a^2 - 0.5)(2a^2 + 0.5).$$

We can further factor $2a^2 - 0.5$, but that's not really useful.

(d) Use the sum of cubes formula:

$$\begin{aligned} 8 + 64x^9 = (2)^3 + (4x^3)^3 &= (2 + 4x^3)((2)^2 - (2)(4x^3) + (4x^3)^2) \\ &= (2 + 4x^3)(4 - 8x^3 + 16x^6). \end{aligned}$$

(e) The two numbers whose sum is 7 and whose product is 10 are 2 and 5. Thus,

$$x^2 + 7x + 10 = (x + 2)(x + 5).$$ ∎

Exercise 4. Factor the following expressions:

(a) $x^2 + 3x - 10$ (b) $2x^2 + 7x - 4$

(c) $2x^2 + 2x - 24$ (d) $x^6 - 1$

♣

For additional practice, do the following exercise.

Exercise 5. Factor the following expressions:

(a) $2x^2y + y^2 - 4x^2 - 2y$ (b) $x^4 - x^2$

(c) $27a^2 - 75b^2$ (d) $x^5 - x$.

♣

Radicals and Rational Exponents. The expression $a^n = b$ $(n = 2, 3, 4, \ldots)$ can also be written as $a = \sqrt[n]{b}$. If $n = 2$, then $\sqrt[2]{b}$ is denoted by $\sqrt{b}$.

For example, since $4^3 = 64$, we conclude that $\sqrt[3]{64} = 4$; similarly, $\sqrt[5]{-32} = -2$, since $(-2)^5 = -32$. In case of even values of n, there are two possibilities: $\sqrt{16}$ could be 4 or -4, since $4^2 = 16$ and $(-4)^2 = 16$. To avoid ambiguity, we define $\sqrt[n]{b}$ for even n to be the positive n-th root of b. Thus, $\sqrt{16} = 4$, $\sqrt[4]{16} = 2$, etc. Note that $\sqrt[n]{0} = 0$ for all $n = 2, 3, 4, \ldots$

If n is even and $b < 0$, then $\sqrt[n]{b}$ is not defined. For example, $a = \sqrt{-25}$ would imply that $a^2 = -25$; but a square of a real number cannot be negative.

Example 4. Evaluate, if possible, the following expressions: $\sqrt[3]{-125}$, $\sqrt[5]{0}$, $-\sqrt[3]{64}$, $\sqrt{-32}$ and $\sqrt{9/4}$.

Solution

$\sqrt[3]{-125} = -5$, since $(-5)^3 = -125$. $\sqrt[5]{0} = 0$. $-\sqrt[3]{64} = -4$, since $4^3 = 64$. $\sqrt{-32}$ is not defined. Since $(3/2)^2 = 9/4$, we get $\sqrt{9/4} = 3/2$. ∎

Remember that

$$\sqrt[n]{a^n} = |a|, \qquad \text{if } n \text{ is even}$$

(the absolute value guarantees that the right side is positive; the left side is positive by the definition of the n-th root for even n). On the other hand,

$$\sqrt[n]{a^n} = a, \qquad \text{if } n \text{ is odd.}$$

Thus, $\sqrt[3]{(-5)^3} = -5$ and $\sqrt[4]{(-5)^4} = |-5| = 5$.

Rules for radicals

$$\sqrt[n]{a^m} = a^{m/n} \qquad \sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Important warning: expression $\sqrt{a+b}$ (and also $\sqrt{a-b}$) cannot be simplified! Formulas such as $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ are wrong! You can easily convince yourself that this is so: take $a = 16$ and $b = 9$; then $\sqrt{a+b} = \sqrt{25} = 5$, whereas $\sqrt{a} + \sqrt{b} = \sqrt{16} + \sqrt{9} = 7$.

Example 5. Simplify or evaluate each of the following:

(a) $x\sqrt{x}\sqrt[3]{x^2}$ (b) $4 \cdot 4^{3/2}$ (c) $\sqrt{12}/\sqrt{27}$

(d) $(3+\sqrt{7})(4-\sqrt{7})$ (e) $(9/16)^{-3/2}$

Solution

(a) $x\sqrt{x}\sqrt[3]{x^2} = x^1 x^{1/2} x^{2/3} = x^{1+1/2+2/3} = x^{13/6}$.

(b) $4 \cdot 4^{3/2} = 4^1 4^{3/2} = 4^{5/2} = (2^2)^{5/2} = 2^5 = 32$.

(c) We simplify the terms under the square roots and then cancel:

$$\frac{\sqrt{12}}{\sqrt{27}} = \frac{\sqrt{4 \cdot 3}}{\sqrt{9 \cdot 3}} = \frac{2\sqrt{3}}{3\sqrt{3}} = \frac{2}{3}.$$

(d) Multiplying the two binomials, we get

$$(3+\sqrt{7})(4-\sqrt{7}) = 12 - 3\sqrt{7} + 4\sqrt{7} - \sqrt{7}\sqrt{7} = 12 + \sqrt{7} - 7 = 5 + \sqrt{7}.$$

(e) $\left(\frac{9}{16}\right)^{-3/2} = \frac{1}{\left(\frac{9}{16}\right)^{3/2}} = \frac{1}{\left(\frac{9^{3/2}}{16^{3/2}}\right)} = \frac{16^{3/2}}{9^{3/2}} = \frac{(4^2)^{3/2}}{(3^2)^{3/2}} = \frac{4^3}{3^3} = \frac{64}{27}$. In the first step, we used the formula $a^{-n} = 1/a^n$. ∎

Exercise 6. Simplify or evaluate each of the following:

(a) $\sqrt[3]{-125/64}$ (b) $\sqrt{100}/\sqrt{200}$ (c) $(\sqrt{2}+\sqrt{3})^2$

(d) $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ (e) $(4/9)^{-3/2}$

♣

Fractional Expressions. In this part, we review operations with fractions.

Example 6. Simplify $\dfrac{2x^2-3x-2}{x^2+2x-8}$.

Solution

Factor both the numerator and the denominator and cancel:

$$\frac{2x^2-3x-2}{x^2+2x-8}=\frac{(2x+1)(x-2)}{(x+4)(x-2)}=\frac{2x+1}{x+4}.$$

∎

Exercise 7. Simplify

(a) $\dfrac{x^3-27}{x^2-9}$ (b) $\dfrac{x^4-x^2}{2x^2+3x+1}$

♣

Example 7. Simplify $\dfrac{\frac{x+4}{x-3}}{\frac{x^2-16}{x^2-2x-3}}$.

Solution

Get rid of the double fraction first, then factor and cancel:

$$\frac{\frac{x+4}{x-3}}{\frac{x^2-16}{x^2-2x-3}}=\frac{x+4}{x-3}\cdot\frac{x^2-2x-3}{x^2-16}=\frac{x+4}{x-3}\cdot\frac{(x+1)(x-3)}{(x-4)(x+4)}=\frac{x+1}{x-4}.$$

To factor x^2-16 we used the difference of squares formula, and to factor x^2-2x-3 as a product of linear factors we had to find the two numbers whose sum is -2 and whose product is -3. ∎

Exercise 8. Simplify $\dfrac{1-\frac{3}{x}}{1-\frac{6}{x}+\frac{9}{x^2}}$.

♣

Important warning: remember that the formula $\frac{1}{a+b}=\frac{1}{a}+\frac{1}{b}$ is wrong! To convince yourself that this is so, let $a=b=1$; then $\frac{1}{a+b}=\frac{1}{2}$, whereas $\frac{1}{a}+\frac{1}{b}=\frac{1}{1}+\frac{1}{1}=2$.

Example 8. Rationalize the denominator in $\dfrac{1}{\sqrt{5}+\sqrt{6}}$.

Solution

Multiply the given fraction by 1, written in the form $(\sqrt{5}-\sqrt{6})/(\sqrt{5}-\sqrt{6})$:

$$\frac{1}{\sqrt{5}+\sqrt{6}}=\frac{1}{\sqrt{5}+\sqrt{6}}\,\frac{\sqrt{5}-\sqrt{6}}{\sqrt{5}-\sqrt{6}}=\frac{\sqrt{5}-\sqrt{6}}{5-6}=\sqrt{6}-\sqrt{5}.$$

To compute $(\sqrt{5}+\sqrt{6})(\sqrt{5}-\sqrt{6})$, we used the difference of squares formula. ∎

Exercise 9. Rationalize the denominator in $\dfrac{3}{\sqrt{x}-\sqrt{2}}$.

♣

For extra practice, solve the following exercise.

Exercise 10. (a) Rationalize the denominator in $\dfrac{4-\sqrt{x}}{2+\sqrt{x}}$.

(b) Simplify $\dfrac{1+\frac{1}{x-1}}{1-\frac{1}{x-1}}$.

(c) Simplify (find the common denominator) $\dfrac{3}{x}+\dfrac{4}{x-1}+\dfrac{5}{x+1}$.

♣

In the next section, we will review solving equations in one variable, inequalities and systems of two equations.

2. Basic Formulas from Geometry

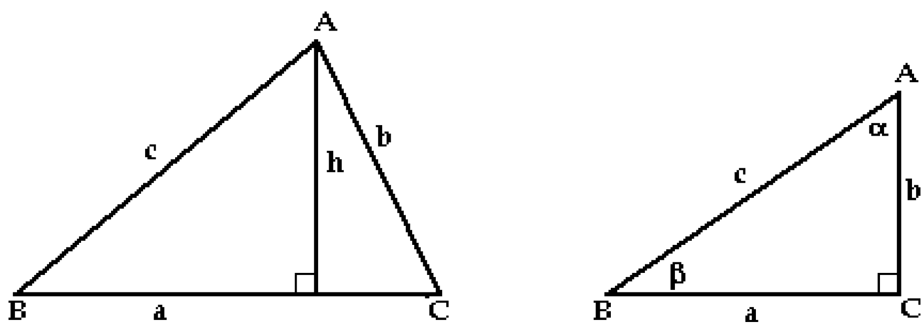
This section contains review material on:

- Plain geometry
- Geometry in three-dimensional space.

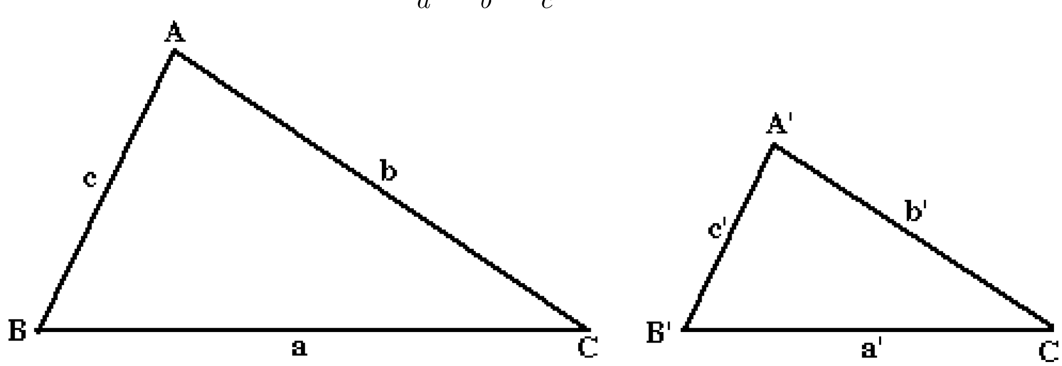
Plane Geometry. Usually, we use lowercase letters a, b, $c, \ldots$ to denote the sides (edges), and the uppercase letters A, B, $C, \ldots$ to denote the vertices of a polygon. Lowercase greek letters α, β, $\gamma, \ldots$ are used for angles. Sometimes, we use $\angle A$ to denote the angle at vertex A, $\angle B$ to denote the angle at vertex B, etc.

Triangle (below, left). Sum of angles $= \angle A + \angle B + \angle C = 180^0$ or π radians (see the section on trigonometry for conversion between degrees and radians). Perimeter $= a + b + c$. Area $= \frac{1}{2}ah$ (h is the height of the triangle from vertex A).

Right triangle (below, right) $\angle C = 90^0$ (or $\pi/2$ radians). Pythagorean Theorem states that $a^2 + b^2 = c^2$. Area $= \frac{1}{2}ab$.

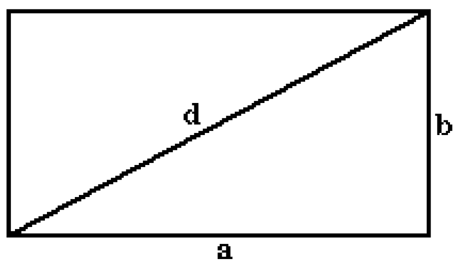


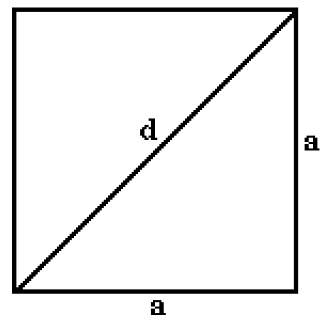
Two triangles are called similar if the corresponding angles are equal (in the figure below: $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$). In that case, $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.



Rectangle with sides a and b (below, left). Perimeter $= 2(a+b)$. Area $= ab$. Diagonal $= d = \sqrt{a^2 + b^2}$.

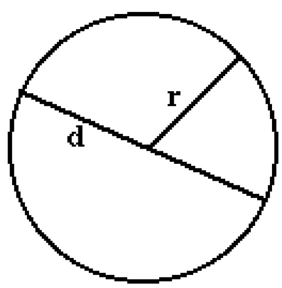
Square with side a (below, right). Perimeter $= 4a$. Area $= a^2$. Diagonal $= d = a\sqrt{2}$.

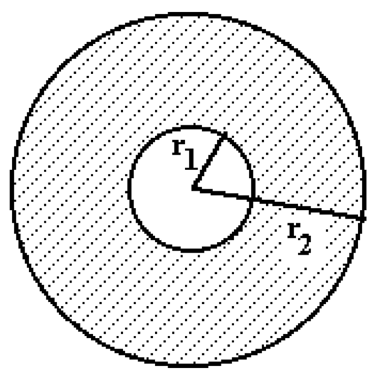




Disk of radius r (below, left). Circumference $= 2\pi r$. Area $= \pi r^2$. Diameter $= d = 2r$.

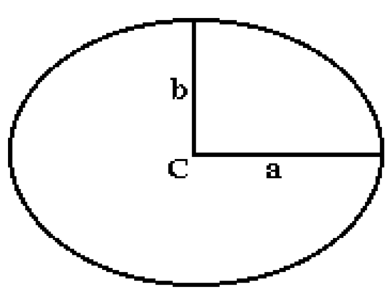
Annular region bounded by the circles of radius r_1 and r_2, $r_1 < r_2$ (shaded area in the figure below, right). Area $= \pi r_2^2 - \pi r_1^2 = \pi(r_2^2 - r_1^2)$.

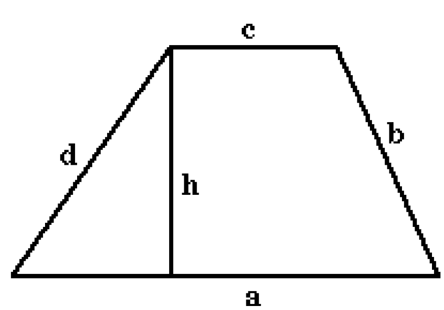




Ellipse with semi-axes a and b (below, left). C is the centre of the ellipse. Area $= \pi ab$.

Trapezoid with sides a, b, c and d and height h. (below, right). Sides a and c are parallel. Perimeter $= a + b + c + d$. Area $= \frac{1}{2}(a+c)h$.

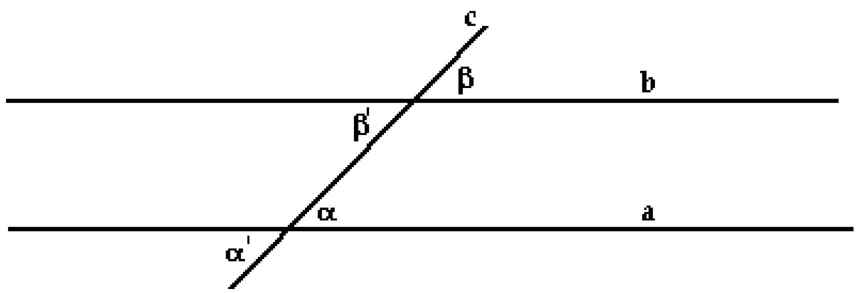




Miscellaneous facts.

The sum of angles in a polygon with n sides is $(n-2)\pi$ radians or $180(n-2)$ degrees. Thus, the sum of angles in a traingle is π radians or 180^0. In a rectangle, square, trapeze, or any other polygon with four sides, the sum of angles is 2π radians or 360^0. Sum of angles in a pentagon is 3π (540^0), etc.

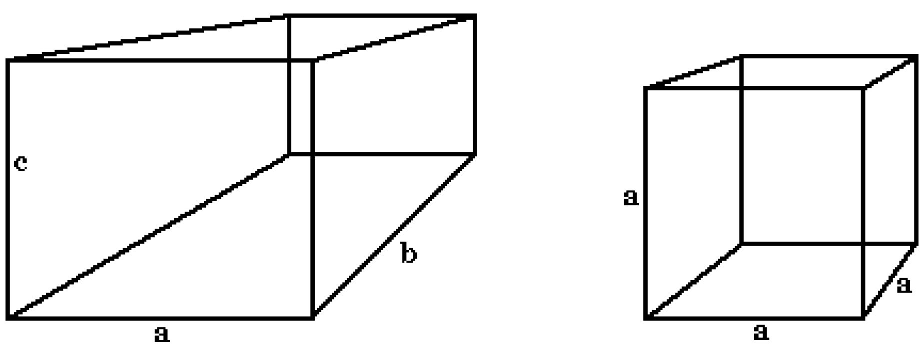
Assume that a line c intersects two paralell lines a and b (figure below). Then $\alpha = \beta$. Since $\alpha = \alpha'$ and $\beta = \beta'$, it follows that $\alpha = \alpha' = \beta = \beta'$.



Geometry in Three-dimensional Space.

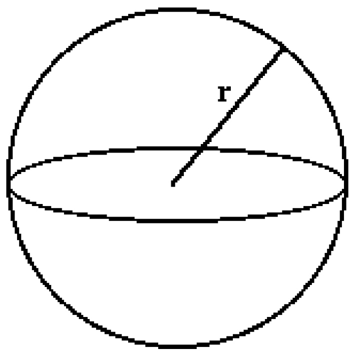
Parallelepiped (rectangular box) of length a, width b, and height c (below, left). Volume $= abc$. Surface area $= 2(ab + ac + bc)$.

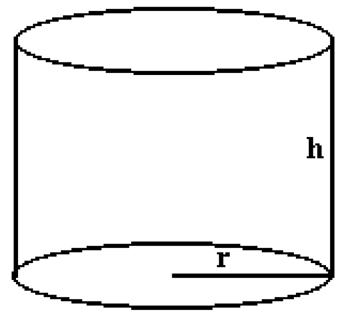
Cube of side a (below, right). Volume $= a^3$. Surface area $= 6a^2$.



Sphere of radius r (below, left). Volume $= \frac{4}{3}\pi r^3$. Surface area $= 4\pi r^2$.

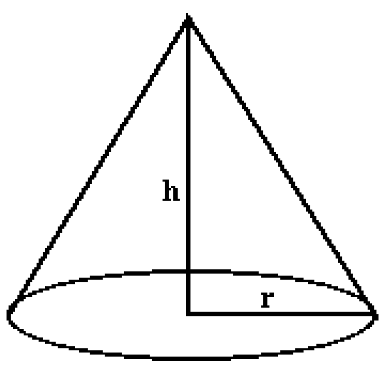
Cylinder of radius r and height h (below, right). Volume $= \pi r^2 h$. Surface area $= 2\pi rh + 2\pi r^2 = 2\pi r(r + h)$.

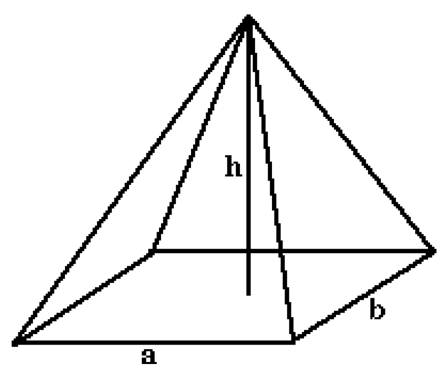




Cone of radius r and height h (below, left). Volume $= \frac{1}{3}\pi r^2 h$. I.e., the volume of the cone is 1/3 of the volume of the cylinder with the same radius and the same height.

Pyramid of height h whose base is a rectangle with sides a and b (below, right). Volume $= \frac{1}{3}abh$. I.e., the volume of the pyramid is 1/3 of the volume of the parallelepiped with the same base and the same height.





3. Equations and Inequalities

This section contains review material on:

- Solving linear equations in one variable
- Solving inequalities in one variable
- Solving equations and inequalities involving absolute value
- Systems of two equations with two unknowns

To "solve an equation" means to find all values of unknown variable(s) (these values are called solutions) that satisfy a given equation. In other words, a solution is a real number that, when substituted into the equation, gives the identity. For example, $x = 3$ is a solution of the equation $x^2 - 4x = 6 - x^2$, since $x^2 - 4x = (3)^2 - 4(3) = -3$ and $6 - x^2 = 6 - (3)^2 = -3$; i.e., both sides are equal to -3. On the other hand, $x = 4$ is not a solution of the above equation; the left side is $x^2 - 4x = (4)^2 - 4(4) = 0$, whereas the right side is $6 - x^2 = 6 - (4)^2 = -10$.

Solving Linear Equations. A linear equation $ax + b = 0$ (assume that $a \neq 0$) has only one solution, namely $x = -b/a$. (If $a = 0$, we don't really have an equation!)

Example 1. Solve the equation $3(x + 4) = -4(2 - 2x)$.

Solution

We simplify first

$$\begin{aligned} 3(x+4) &= -4(2-2x) \\ 3x + 12 &= -8 + 8x \end{aligned}$$

and then gather the like terms together

$$\begin{aligned} 3x - 8x &= -8 - 12 \\ -5x &= -20. \end{aligned}$$

Dividing by (-5), we get $x = 4$. ∎

Example 2. Solve the equation $\dfrac{4x-1}{3} + \dfrac{x}{4} = -2$.

Solution

Multiplying the equation by 12, we get

$$\begin{aligned} 4(4x-1) + 3(x) &= -2(12) \\ 16x - 4 + 3x &= -24. \end{aligned}$$

Thus,

$$\begin{aligned} 19x &= -20 \\ x &= \frac{-20}{19}. \end{aligned}$$ ∎

Exercise 1. Solve the equation $4(3x - 0.5) + 11 = -3(x + 2)$.

♣

Example 3. Solve the equation $\dfrac{4}{x-3} = \dfrac{5}{x+2}$.

Solution

Multiplying the given equation by $(x - 3)(x + 2)$, we get

$$\frac{4(x-3)(x+2)}{x-3} = \frac{5(x-3)(x+2)}{x+2}$$
$$4(x+2) = 5(x-3)$$

(in other words, we "cross-multiplied" the equation). Thus,

$$4x + 8 = 5x - 15$$
$$x = 23.$$

We can check that our solution is correct. Substituting $x = 23$ into the left side, we get $\frac{4}{x-3} = \frac{4}{23-3} = \frac{4}{20} = \frac{1}{5}$; substituting it into the right side, we get $\frac{5}{x+2} = \frac{5}{23+2} = \frac{5}{25} = \frac{1}{5}$. ∎

Exercise 2. Solve the equation $\dfrac{4x-2}{5} - \dfrac{3x-11}{4} = 0$.

♣

It is important to check that what we compute is really a solution. The following example serves as a warning to that effect.

Example 4. Solve the equation $\dfrac{x}{x+2} - 1 = \dfrac{8}{x^2-4}$.

Solution

Multiplying the given equation by $x^2 - 4 = (x - 2)(x + 2)$, we get

$$\frac{x}{x+2}(x-2)(x+2) - 1(x^2-4) = \frac{8}{x^2-4}(x^2-4)$$
$$x^2 - 2x - x^2 + 4 = 8$$
$$-2x = 4$$
$$x = -2.$$

However, $x = -2$ is not a solution, since neither of the two fractions is defined in that case. Thus, the given equation does not have a solution. ∎

Exercise 3. Solve the following equations.

(a) $\dfrac{2}{x} - \dfrac{5}{2x} + \dfrac{4}{3x} = \dfrac{1}{2}$.

(b) $\dfrac{x}{x-3} = \dfrac{x+3}{x+5}$.

♣

Solving Quadratic Equations. A quadratic equation $ax^2+bx+c=0$ can be solved by factoring, by completing the square or by using the quadratic formula.

$$\boxed{\text{The solutions of } ax^2+bx+c=0 \text{ are given by } \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

The expression $D=b^2-4ac$ is called the discriminant of the quadratic equation. If $D>0$, the equation has two distinct real solutions; if $D=0$, it has one real solution, and if $D<0$, the equation has no real solutions.

Example 5. Solve the equations

(a) $x^2+5x-24=0$ (b) $x^2+2x-2=0$

Solution

(a) From $x^2+5x-24=(x-3)(x+8)=0$, it follows that either $x-3=0$ (in which case $x=3$), or $x+8=0$ (and thus $x=-8$). So, the solutions are $x=-8$ and $x=3$.

(b) Using the quadratic formula, we get

$$x=\frac{-2\pm\sqrt{4+8}}{2}=\frac{-2\pm\sqrt{12}}{2}=\frac{-2\pm2\sqrt{3}}{2}=-1\pm\sqrt{3}.$$

In the above computation, we simplified $\sqrt{12}$ using $\sqrt{12}=\sqrt{4\cdot3}=2\sqrt{3}$. ∎

Example 6. Solve the equation $1+\sqrt{2-x}=2x$.

Solution

Rearrange the terms first: $\sqrt{2-x}=2x-1$. Squaring both sides, we get

$$2-x=4x^2-4x+1$$
$$4x^2-3x-1=0.$$

Thus,

$$x=\frac{3\pm\sqrt{9+16}}{8}=\frac{3\pm5}{8}.$$

So, $x=(3+5)/8=1$ and $x=(3-5)/8=-1/4$ are candidates for solutions. We have to check whether they really give solutions. Substituting $x=1$ into the given equation, we get $1+\sqrt{2-1}=2(1)$; i.e., $2=2$; thus, $x=1$ is a solution. Substituting $x=-1/4$ into the given equation, we get $1+\sqrt{2+1/4}=2(-1/4)$; i.e., $1+3/2=-1/2$; consequently, $x=-1/4$ is not a solution. ∎

Exercise 4. Solve the following equations.

(a) $x^2-18x+81=0$

(b) $\dfrac{2}{x^2-9}=\dfrac{1}{x+3}+\dfrac{3}{x-3}$

(c) $\sqrt{2x+4}=\sqrt{6x+1}-1.$

(d) $\dfrac{3}{x+1}-\dfrac{2}{x-1}=-1$

♣

Inequalities. Rules for inequalities state that we can apply the same rules that we use for equations, except when we have to multiply or divide an inequality by a negative number. In that case, we reverse the direction of the inequality. For example, the inequality $3 < 4$, multiplied by (-5), gives $-15 > -20$. Likewise, dividing $-4 \leq -2x < 6$ by (-2), we get $2 \geq x > -3$.

To solve an inequality means to find all values of the unknown variable that satisfy it. In case on inequalities, the solution usually belongs to an interval (or intervals) of real numbers.

Example 7. Solve the inequality $-(3+x) < 2(3x+2)$.

Solution

We simplify first

$$\begin{aligned} -(3+x) &< 2(3x+2) \\ -3-x &< 6x+4 \\ -x-6x &< 4+3 \\ -7x &< 7. \end{aligned}$$

Dividing by (-7), we get $x > -1$. Using interval notation, we write the solution as $(-1, \infty)$. ∎

Example 8. Solve the inequalities $-11 < 3x + 4 \leq 7$.

Solution

We can work with both inequalities at the same time. Adding (-4) to both sides and dividing by 3, we get

$$\begin{aligned} -11 < 3x + 4 \leq 7 \\ -15 < 3x \leq 3 \\ -5 < x \leq 1. \end{aligned}$$

Thus, the solution is in the interval $(-5, 1]$. ∎

Sometimes, we have to solve inequalities separately, as in the following example.

Example 9. Solve the inequalities $3x + 1 \geq x - 5 \geq 1 + 4x$.

Solution

Solving $3x + 1 \geq x - 5$, we get $2x \geq -6$ and $x \geq -3$. Solving $x - 5 \geq 1 + 4x$, we get $-3x \geq 6$ and $x \leq -2$. It follows that $x \geq -3$ and $x \leq -2$. Thus, the solution lies in the interval $[-3, -2]$. ∎

Exercise 5. Solve the following inequalities.
(a) $13x - 17 > 4x + 1$ (b) $1 - x \geq 3 - 2x \geq x - 6$ (c) $3 \leq 3x - 2 \leq 4$.

♣

We have to be careful when working with reciprocals.

$$\boxed{\begin{array}{l} \text{If } 0 < a < b, \text{ then } \dfrac{1}{a} > \dfrac{1}{b} \\ \text{If } a < b < 0, \text{ then } \dfrac{1}{a} > \dfrac{1}{b} \end{array}}$$

The above formulas do not work if one number is positive and the other is negative. For example, $-2 < 4$, but $-1/2$ is not greater than $1/4$.

Some methods involving more complicated inequalities are reviewed in the next two examples.

Example 10. Solve the inequality $x^2 + 6x - 7 \geq 0$.

Solution

Factoring the left side, we get $(x-1)(x+7) \geq 0$. The solutions of the equation $(x-1)(x+7) = 0$ are $x = 1$ and $x = -7$. The numbers -7 and 1 divide the number line into three intervals: $(-\infty, -7)$, $(-7, 1)$ and $(1, \infty)$. Since the expression $x^2 + 6x - 7 = (x-1)(x+7)$ can change its sign only at -7 and 1, it follows that its sign on each of the intervals is constant.

We check the sign of each factor on each interval and record it in the table below. For example, if x is in $(-7, 1)$, then $x > -7$ and so $x+7 > 0$; that is why we put the plus sign in the row corresponding to $x + 7$ and in the column for the interval $(-7, 1)$.

	$(-\infty, -7)$	$(-7, 1)$	$(1, \infty)$
$x-1$	$-$	$-$	$+$
$x+7$	$-$	$+$	$+$
$(x-1)(x+7)$	$+$	$-$	$+$

It follows that the solution consists of the intervals $(-\infty, -7]$ and $[1, \infty)$, since the value 0 is allowed. Note that if the inequality were a strict inequality, i.e., $(x-1)(x+7) > 0$, then the solution would have been $(-\infty, -7)$ and $(1, \infty)$.

Alternatively, we can use test values for each interval. The rationale is that the polynomial does not change its sign inside each interval. So, if it is positive/negative at one point in the interval, then it is positive/negative in the whole interval. For example, to test the sign of $x + 7$ in the interval $(-7, 1)$, we can use any number in $(-7, 1)$; say, $x = -4$. Since $x + 7 = -4 + 7 = 3 > 0$, we conclude that $x + 7$ is positive in the interval $(-7, 1)$.

We can use test values to check the sign of the whole polynomial at once. Here is how it works. Test the interval $(-\infty, -7)$: take, for example $x = -10$; then $x^2 + 6x - 7 = (x-1)(x+7) = (-11)(-3) = 33 > 0$. It follows that $(-\infty, -7)$ is a part of the solution. Next, take $x = 0$ to test the interval $(-7, 1)$: it follows that $x^2 + 6x - 7 = (x-1)(x+7) = (-1)(7) = -7 < 0$; thus, $(-7, 1)$ is not a part of the solution. Analogously, we check the interval $(1, \infty)$. In the end, we include -7 and 1, since they make the expression $x^2 + 6x - 7$ equal to 0, and the given inequality allows for that possibility to happen. ∎

Example 11. Solve the inequality $\dfrac{4}{3x-2} \leq 2$.

Solution

It is convenient to have an inequality that involves zero. So, we move 2 to the left side and compute the common denominator:

$$\begin{aligned} \frac{4}{3x-2} &\leq 2 \\ \frac{4}{3x-2} - 2 &\leq 0 \\ \frac{4}{3x-2} - \frac{2(3x-2)}{3x-2} &\leq 0 \\ \frac{-6x+8}{3x-2} &\leq 0 \end{aligned}$$

The sign of the fraction depends on the signs of both the numerator and the denominator. From $-6x+8=0$, we get $x = -8/(-6) = 4/3$. From $3x-2=0$, we get $x = 2/3$. Thus, we have to check the intervals $(-\infty, 2/3)$, $(2/3, 4/3)$ and $(4/3, \infty)$. On each interval, we use test values.

Testing the interval $(-\infty, 2/3)$: when $x = 0$, $\frac{-6x+8}{3x-2} = \frac{8}{-2} = -4 < 0$. Testing the interval $(2/3, 4/3)$: when $x = 1$, $\frac{-6x+8}{3x-2} = \frac{2}{1} = 2 > 0$. Testing the interval $(4/3, \infty)$: when $x = 2$, $\frac{-6x+8}{3x-2} = \frac{-4}{4} = -1 < 0$. It follows that the solution consists of the intervals $(-\infty, 2/3)$, and $[4/3, \infty)$. The reason why we included $x = 4/3$ is that it makes the fraction equal to zero (since its numerator is zero), and the given inequality allows for that possibility to occur. ∎

Exercise 6. Solve the following inequalities.

(a) $x^2 > 21 - 4x$ (b) $\dfrac{6}{x-5} \leq 2$ (c) $x^3 - x \leq 0$.

♣

Equations and Inequalities Involving Absolute Value. To recall the definition of the absolute value, we consider the following example.

Example 12. Rewrite $|5x-2|$ without the use of the absolute value signs.

Solution

By definition,

$$|5x-2| = \begin{cases} 5x-2 & \text{if } 5x-2 \geq 0 \\ -(5x-2) & \text{if } 5x-2 < 0 \end{cases} = \begin{cases} 5x-2 & \text{if } x \geq 2/5 \\ -5x+2 & \text{if } x < 2/5. \end{cases}$$ ∎

In working with absolute values, the following statements might be helpful.

Let A be any expression
$\lvert A\rvert = a$ if and only if $A = a$ and $A = -a$
$\lvert A\rvert < a$ if and only if $-a < A < a$
$\lvert A\rvert > a$ if and only if $A > a$ or $A < -a$

A convenient way to visualize $|A|$ is to think of it as $|A| = |A-0|$; i.e., to interpret it as the distance between A and the origin. Now $|A| < a$ means that we need all numbers A whose distance from the origin is smaller than a; thus, $-a < A < a$. The inequality $|A| > a$ is interpreted similarly.

Example 13. Solve $|4x - 3| = 2$.

Solution

By definition,

$$|4x-3| = \begin{cases} 4x-3 & \text{if } 4x-3 \geq 0 \\ \\ -(4x-3) & \text{if } 4x-3 < 0 \end{cases} = \begin{cases} 4x-3 & \text{if } x \geq 3/4 \\ \\ -4x+3 & \text{if } x < 3/4. \end{cases}$$

Thus, the given equation breaks up into two equations. If $x \geq 3/4$, it reads $4x - 3 = 2$ (and the solution is $x = 5/4$). If $x < 3/4$, it reads $-4x + 3 = 2$ (and the solution is $x = 1/4$). Consequently, there are two solutions, $x = 5/4$ and $x = 1/4$. ∎

Exercise 7. Solve the equation $|2x - 3| = 7$.

______________________________ ♣

Example 14. Solve the following inequalities.

(a) $|2x + 1| \leq 4$ (b) $|3x - 4| > 1$.

Solution

(a) The given inequality is equivalent to

$$-4 \leq 2x + 1 \leq 4$$
$$-5 \leq 2x \leq 3$$
$$-5/2 \leq x \leq 3/2.$$

(b) The given inequality is equivalent to $3x - 4 > 1$ or $3x - 4 < -1$. Solving $3x - 4 > 1$, we get $3x > 5$ and $x > 5/3$. Solving $3x - 4 < -1$, we get $3x < 3$ and $x < 1$. Thus, the solution consists of the intervals $(-\infty, 1)$ or $(5/3, \infty)$. ∎

Exercise 8. Solve the following inequalities.

(a) $|2x-3|<2$ (b) $|3x+4|\geq 4$.

♣

Systems of Equations. In the two examples below, we review most common methods of soving systems of two equations with two unknowns.

Example 15. Solve the system $2x+y=10$, $4x-y=2$.

Solution

We use the substitution method (also called method of elimination). The idea is to eliminate one variable, so that we end up with one equation and one unknown. Computing y from $2x+y=10$, we get $y=10-2x$. Substituting y into the second equation $4x-y=2$, we get

$$\begin{aligned}4x-y&=2\\4x-(10-2x)&=2\\6x&=12\\x&=2.\end{aligned}$$

The corresponding solution for y is $y=10-2x=10-4=6$.

Alternatively, adding up the two equations, we get

$$(2x+y)+(4x-y)=10+2.$$

Thus, $6x=12$ and $x=2$. Using any of the two equations, we get $y=6$. ∎

Example 16. Solve the system $2x-y=-5$, $y=x^2+2$.

Solution

From $2x-y=-5$, we get $y=2x+5$. Substituting y into the second equation, we get

$$\begin{aligned}y&=x^2+2\\2x+5&=x^2+2\\x^2-2x-3&=0\\(x-3)(x+1)&=0.\end{aligned}$$

Consequently, there are two solutions for x, $x=-1$ and $x=3$. When $x=-1$, $y=2(-1)+5=3$. When $x=3$, $y=2(3)+5=11$. Thus, there are two solutions, $x=-1$, $y=3$ and $x=3$, $y=11$. ∎

Exercise 9. Solve the following systems of equations.

(a) $2x-y=13$, $x+2y=-11$

(b) $y=x^2+2x+2$, $x-y+4=0$.

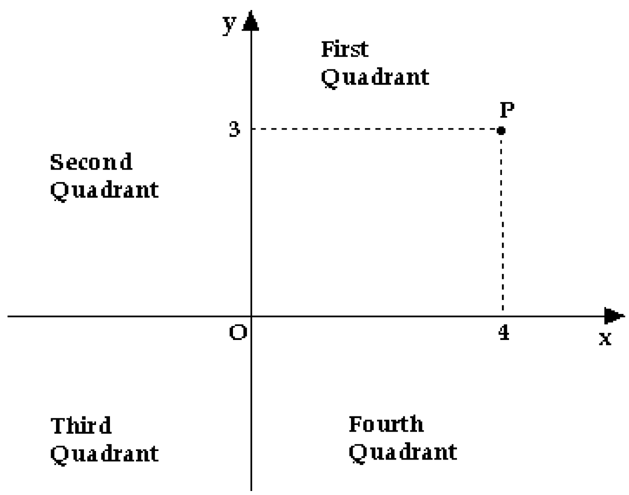
♣

4. Elements of Analytic Geometry

This section contains review material on:

- Cartesian coordinate system and distance between points
- Equations of a line
- Graphs of second degree equations: circle, ellipse, parabola and hyperbola

Cartesian Coordinate System. The Cartesian coordinate system consists of two number lines that are perpendicular to each other and are placed so that they intersect at a point that represents the number zero for both of them. The number lines are named x-axis and y-axis, and are usually visualized as in the figure below. Their intersection is called the origin (and is denoted by O).



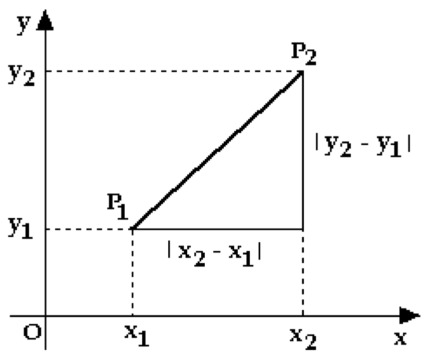
The position of a point is uniquely determined by specifying an ordered pair (x, y), where x is the x-coordinate and y is the y-coordinate of the point. The coordinates of the point P in the figure above are $P(4, 3)$. The points whose x-coordinate is zero lie on the y-axis. The points whose y-coordinate is zero lie on the x-axis. The origin has coordinates $(0, 0)$.

The x-axis and the y-axis divide the plane into four quadrants. The first quadrant consists of points (x, y) whose coordinates satisfy $x > 0$ and $y > 0$. See the figure above for the location of the second, third and fourth quadrants.

Distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance formula is obtained using the Pythagorean theorem. The figure below shows how (recall that the distance between two numbers on a number line is equal to the absolute value of their difference).



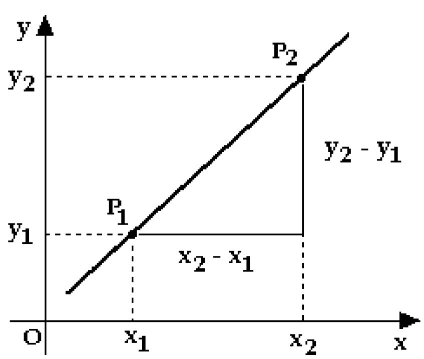
Exercise 1. Find the distance between the points whose coordinates are $(-3, 4)$ and $(4, -2)$.

♣

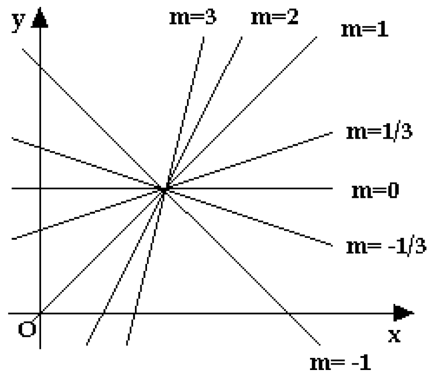
Line. A line is uniquely determined by a point that lies on it and by its slope. The slope of a line is the ratio of the change in y to the change in x ("rise over run"). The line is characterized by the property that that ratio is constant (i.e., it is the same, no matter where on the line we measure it). If a line passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, then its slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

see the figure below.



The slope of a vertical line is not defined. A line parallel to the x-axis (i.e., a horizontal line) has slope zero. Lines with positive slopes slant upward: the bigger the slope, the more slanted the line is. Lines with negative slopes slant downward, see the figure below.



The equation below is known as the "point-slope" equation of a line.

Equation of a line with slope m that passes through $P(x_0, y_0)$ is $y - y_0 = m(x - x_0)$

Example 1. Find an equation of the line that contains the points $(0, 4)$ and $(3, 3)$.

Solution

The slope is $m = \frac{4-3}{0-3} = -\frac{1}{3}$. Thus, the desired equation is

$$y - 4 = -\frac{1}{3}(x - 0);$$

i.e., $y = -x/3 + 4$. Of course, it does not matter which of the two given points we use for $P(x_0, y_0)$ in the "point-slope" equation. ∎

Exercise 2. Find an equation of the line that contains the points $(1, 3)$ and $(2, -4)$.

________________ ♣

The equation of a line $y - y_0 = m(x - x_0)$ can be written in the form $y = mx + y_0 - mx_0$, or in the form $y = mx + b$, where $b = y_0 - mx_0$ is the y-intercept.

A horizontal line that crosses the y-axis at b has the equation $y = b$. A vertical line that crosses the x-axis at a cannot be written in the form $y = mx + b$. Its equation is $x = a$.

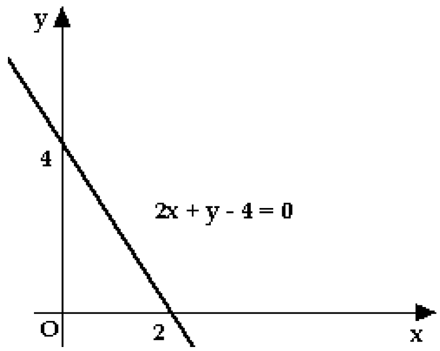
Remember that an equation of the form $ax + by + c = 0$ always represents a line (assuming that at least one of a or b is not zero). This equation is called the implicit equation of a line.

The equation $y = mx + b$ is called the explicit equation of the line.

Example 2. Sketch the graph of the equation $2x + y - 4 = 0$.

Solution

Solving for y, we get $y = -2x + 4$. Thus, the graph is a line of slope -2 and y-intercept 4.



∎

Exercise 3. Sketch the graph of the equation $-x - 2y = 0$.

♣

Parallel lines have equal slopes. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_2 = -1/m_1$.

Example 3. Find an equation of a line that is

(a) parallel to the line $6x + 2y - 4 = 0$ and goes through the origin

(b) perpendicular to the line $6x + 2y - 4 = 0$ and goes through $(3, -2)$.

Solution

(a) Rewriting $6x + 2y - 4 = 0$, we get $y = -3x + 2$. Thus, the slope of the given line (and, consequently, the slope of the line we are looking for) is -3. Using the point-slope equation, we get $y - 0 = -3(x - 0)$; i.e., $y = -3x$.

(b) The slope of the given line is -3 (we know that from (a)). Thus, the slope of a perpendicular line is $1/3$. It follows that the equation of the desired line is $y + 2 = (1/3)(x - 3)$; i.e., $y = \frac{1}{3}x - 3$. ∎

Exercise 4. Answer the following questions.

(a) Are the lines $x + 2y + 4 = 0$ and $y = -3x + 4$ parallel?

(b) Find an equation of a line perpendicular to the line $2x + y - 4 = 0$ that goes through the point $(1, -2)$.

(c) Show that the lines $3x - 2y = 6$ and $2x + 3y - 12 = 0$ are perpendicular.

♣

Circle. The equation of a circle of radius r with centre at (p, q) is given by

$$(x-p)^2 + (y-q)^2 = r^2.$$

In particular, the equation $x^2 + y^2 = r^2$ represents a circle of radius r centred at the origin.

Example 4. Show that the equation $x^2 + y^2 + 4x - 6y + 3 = 0$ represents a circle.

Solution

We first group the x terms and the y terms together

$$x^2 + 4x + y^2 - 6y = -3,$$

and then complete the square:

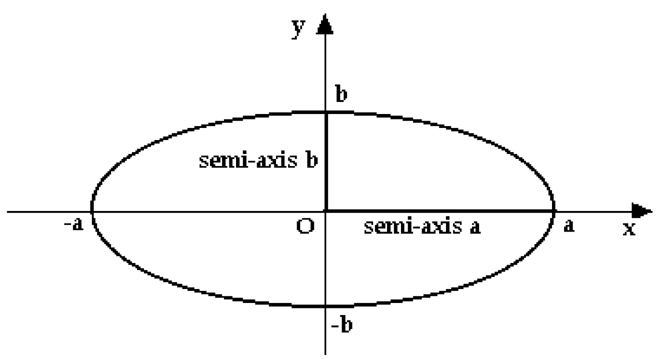
$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = -3 + 4 + 9$$
$$(x+2)^2 + (y-3)^2 = 10.$$

Thus, the given equation represents the circle of radius $\sqrt{10}$ centered at $(-2, 3)$. ∎

Ellipse. The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(where $a > 0$ and $b > 0$) represents an ellipse with semi-axes a and b, see the figure below. It is symmetric with respect to the x-axis and with respect to the y-axis (thus, it is also symmetric with respect to the origin).



If $a = b$, then $x^2/a^2 + y^2/a^2 = 1$ gives $x^2 + y^2 = a^2$; i.e., we get a circle of radius a.

Substituting $x = 0$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we get $y^2 = b^2$ and $y = \pm b$. In words, the y-intercepts are b and $-b$. Similarly, $y = 0$ implies $x^2/a^2 = 1$ and $x^2 = a^2$; thus $x = a$ and $x = -a$ are the x-intercepts.

Example 5. Sketch the graph of the equation $4x^2 + 2y^2 = 8$.

Solution

Dividing the given equation by 8, we get

$$x^2/2 + y^2/4 = 1.$$

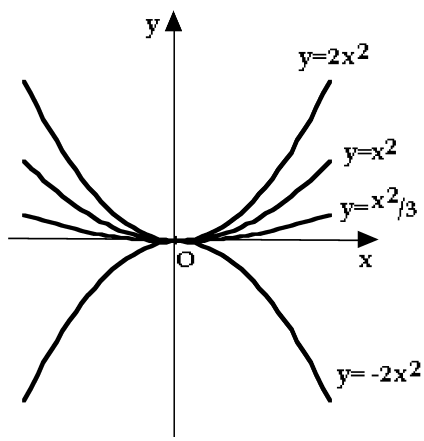
Thus, the given equation represents an ellipse whose semi-axes are $\sqrt{2}$ and $\sqrt{4} = 2$. Its x-intercepts are $\pm\sqrt{2}$, and its y-intercepts are ± 2. ∎

Exercise 5. Describe in words the curves described by the following equations.
(a) $x^2 + 3y^2 = 9$ (b) $x^2 + y^2 - 2x - 6y = 0$.

♣

Parabola. The graph of the equation $y = ax^2 + bx + c$ is a parabola.

Consider a special case $y = ax^2$ first. By plotting points, we obtain the following picture, that shows the graphs of $y = ax^2$ for $a = 2$, $a = 1$, $a = 1/3$ and $a = -2$.



The parabola $y = ax^2$ goes through the origin. It opens upward if $a > 0$ and it opens downward if $a < 0$. It is symmetric with respect to the y-axis. The point which is the lowest point or the highest point on a parabola is called its vertex. The vertex of a parabola $y = ax^2$ is at the origin.

Now consider a general parabola $y = ax^2+bx+c$. The x-intercepts (if they exist) are given as solutions of the quadratic equation $y = ax^2 + bx + c = 0$. The x-coordinate of the vertex is $x = -b/2a$; it is located half-way between the x-intercepts (if they exist).

Example 6. Sketch the graphs of
(a) $y = -x^2 + 5x - 6$ (b) $y = x^2 + x + 2$.

Solution

(a) Since $a = -1$, the parabola opens downward. Solving

$$-x^2 + 5x - 6 = -(x^2 - 5x + 6) = -(x - 2)(x - 3) = 0,$$

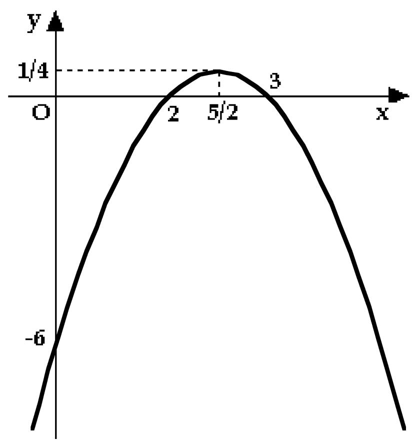
we get $x = 2$ and $x = 3$ for the x-intercepts. The x-coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-5}{-2} = \frac{5}{2}$$

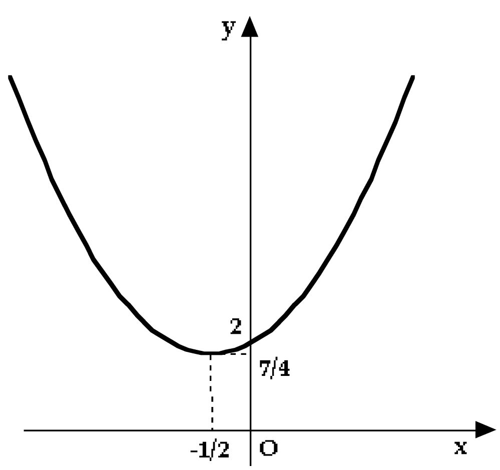
(remember, must be half way between 2 and 3!). The y-coordinate of the vertex is

$$y = -x^2 + 5x - 6 = -\left(\frac{5}{2}\right)^2 + 5\left(\frac{5}{2}\right) - 6 = \frac{1}{4}.$$

The y-intercept (substitute $x = 0$ into $y = -x^2 + 5x - 6$) is $y = -6$.



(b) The parabola $y = x^2 + x + 2$ opens upward. Since the discriminant of $y = x^2 + x + 2 = 0$ is $D = 1 - 8 = -7 < 0$, it follows that the equation has no real solutions. Thus, there are no x-intercepts; since the parabola opens upward, it must lie above the x-axis. The x-coordinate of the vertex is $x = -b/2a = -1/2$. The y-coordinate of the vertex is $y = x^2 + x + 2 = (-1/2)^2 + (-1/2) + 2 = 7/4$. The y-intercept is 2.



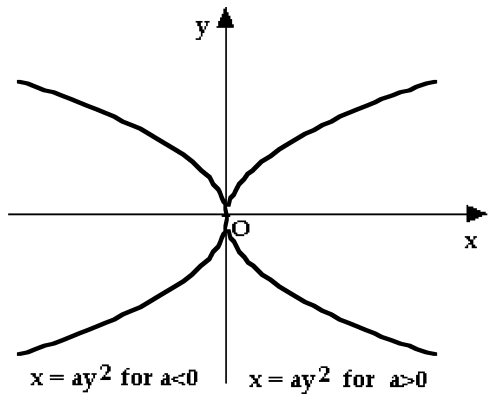
∎

Exercise 6. Find the x-intercepts (if they exist), the y-intercept, coordinates of the vertex and sketch the given parabola

(a) $y = x^2 + 6x + 3$ (b) $y = x^2 - 4x + 4$ (c) $y = -x^2 - 1$.

♣

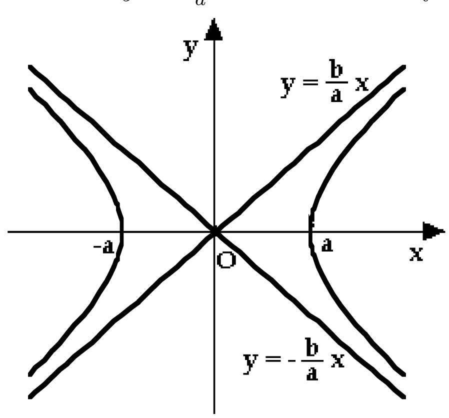
Switching x and y axis (i.e., reflecting the graph with respect to the line $y = x$) we obtain the graphs of the parabolas $x = ay^2$. See the figure below.



Hyperbola. The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

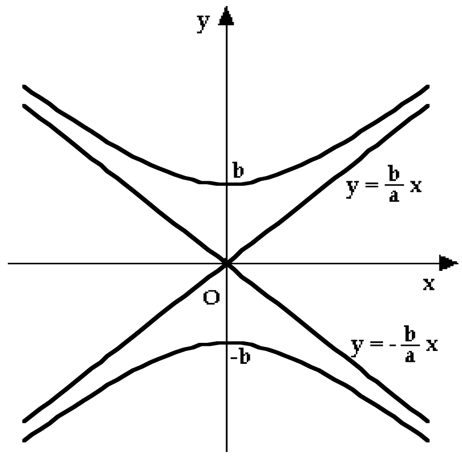
(where $a > 0$ and $b > 0$) represents a hyperbola. Its x-intercepts are given by (substitute $y = 0$ into $x^2/a^2 - y^2/b^2 = 1$) $x^2/a^2 = 1$; i.e., $x = \pm a$. There are no y-intercepts, since $-y^2/b^2 = 1$ implies $y^2 = -b^2 < 0$. The hyperbola consists of two branches, that approach the lines $y = \pm \frac{b}{a}x$. See the figure below. The lines $y = \pm \frac{b}{a}x$ are called the asymptotes.



The equation

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is also a hyperbola. This time, there are no x-intercepts (since $-x^2/a^2 = 1$ implies that $x^2 = -a^2 < 0$). The y-intercepts are $y = \pm b$. The asymptotes are given by $y = \pm \frac{b}{a} x$. See the figure below.



We will study more graphs in the sections that follow.

5. Functions

This section contains review material on:

- Definition of a function, domain, range
- Graph of a function
- Important graphs
- Creating new functions from old

Function. A function f is a rule that assigns, to each real number x in a set A a unique real number $f(x)$ in a set B. The set A is called the domain of f, and the set of all values $f(x)$ for all x in A is called the range of f.

We use the notation D for the domain and R for the range.

If a domain of f is not specified explicitly, and f is given by a formula (as in the next example), it is assumed that the domain of f consists of all real numbers for which the formula for f makes sense (i.e., defines a real number).

The value x is called the value of the independent variable, and $f(x)$ is called the corresponding value of the function f. The value $f(x)$ is also denoted by y; hence the usual notation $y = f(x)$.

Example 1. Find the domain and the range for the following functions.
(a) $f(x) = x^2$ (b) $g(x) = \dfrac{1}{x}$ (c) $h(x) = \sqrt{x}$.

Solution

(a) Since it is possible to compute the square of every real number, the domain of f consists of all real numbers. In symbols, $D = \mathbb{R}$. The square of any non-zero number is positive, and the square of zero is zero. Thus, the range of f consists of all y satisfying $y \geq 0$. In symbols, $R = [0, \infty)$.

(b) Division by zero is not allowed; thus, the domain of g consists of all real numbers x such that $x \neq 0$. Using set notation, we write $D = \{x \mid x \neq 0\}$.

We claim that the range is the set of all y such that $y \neq 0$. Since $1/x = 0$ has no solutions, it follows that no real number x can be mapped by g into $y = 0$; thus, $y = 0$ does not belong to the range of g. Pick any $y \neq 0$. From $1/x = y$ we get $x = 1/y$; thus, $g(1/y) = 1/(1/y) = y$, and so y is in the range of g (since it is obtained as the image of $1/y$ under g). So $R = \{y \mid y \neq 0\}$.

(c) The square root of a negative number is not defined; thus, $D = [0, \infty)$. By convention, $\sqrt{x} \geq 0$, and, consequently, $R = [0, \infty)$. ∎

Exercise 1. Find the domain and the range for the following functions.
(a) $f(x) = x^3$ (b) $g(x) = \dfrac{1}{x^2}$ (c) $h(x) = |x|$.

♣

Example 2. Find the domain for each of the following functions.
(a) $f(x) = \dfrac{1}{x+4}$ (b) $g(x) = \dfrac{x+2}{x^2-2x}$ (c) $h(x) = \sqrt{1-x^2}$.

Solution

(a) Since the division by zero is not allowed, it follows that all x in the domain of f must satisfy $x+4 \neq 0$. Thus, $D = \{x \mid x \neq -4\}$.

(b) As in (a), we require that $x^2 - 2x \neq 0$. Since $x^2 - 2x = x(x-2) = 0$ implies that $x = 0$ or $x = 2$, we conclude that $D = \{x \mid x \neq 0 \text{ and } x \neq 2\}$.

(c) The square root function is not defined for negative numbers; thus, all x in the domain of h must satisfy $1 - x^2 \geq 0$, or $x^2 \leq 1$. Computing the square root of both sides, we get $\sqrt{x^2} \leq 1$, i.e., $|x| \leq 1$ (recall that $\sqrt{x^2} = |x|$). Thus, $-1 \leq x \leq 1$. Alternatively, we write $D = \{x \mid -1 \leq x \leq 1\}$.

We can avoid formalities in solving $x^2 \leq 1$ and proceed as follows. The square of a number between -1 and 1 is smaller than 1. Also, $1^1 = 1 \leq 1$ and $(-1)^2 = 1 \leq 1$. Thus, the solution of $x^2 \leq 1$ is the interval $[-1, 1]$. ∎

Exercise 2. Find the domain of the following functions.
(a) $f(x) = \dfrac{1}{\sqrt{x-2}}$ (b) $g(x) = \dfrac{x}{x^3-x}$ (c) $h(x) = \sqrt{1-\dfrac{1}{x}}$. ♣

Graph of a Function. The graph of a function $y = f(x)$ is the set of points $(x, f(x))$ for all values of x in the domain of f. In other words, the graph of f consists of points, and each point carries two pieces of data: the value of the independent variable x and the corresponding value $f(x)$ of the function f at x.

In general, the graphs of functions that we study in calculus are curves. An elementary way of sketching the graph is by plotting points.

The fact that a function assigns a unique value to each x in its domain means that its graph cannot contain two or more points that have the same x-coordinate. In other words, every vertical line that crosses the graph must cross it exactly once (this is known as the vertical line test).

We have already seen a few graphs of functions. For example, the graph of $y = mx + b$ is a line of slope m and y-intercept b; the graph of the function $y = ax^2 + bx + c$ is a parabola. An ellipse is not the graph of a function (the vertical line test does not work! — however, an ellipse is the graph of an equation).

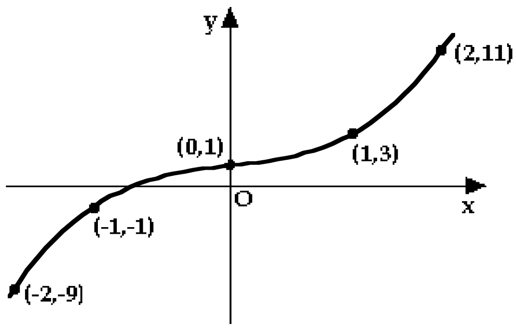
Example 3. Sketch the graph of the function $y = x^3 + x + 1$.

Solution

We choose (convenient) values for x and compute corresponding values for y. For example, when $x = 0$, we get that $y = 1$; thus, the point $(0, 1)$ belongs to the graph. In the same way, we can compute as many points as we wish. The table below shows some of them.

x	-2	-1	$-1/2$	0	$1/2$	1	2
$y = x^3 + x + 1$	-9	-1	$3/8$	1	$13/8$	3	11
point on the graph	$(-2,-9)$	$(-1,-1)$	$(-1/2,3/8)$	$(0,1)$	$(1/2,13/8)$	$(1,3)$	$(2,11)$

Now we use these points (the more points we have the better the picture will look like) to produce the following graph:



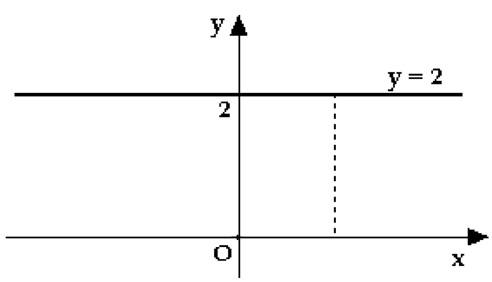
∎

Exercise 3. Sketch the graph of the function $y = \sqrt{x^3+1}$ by plotting points.

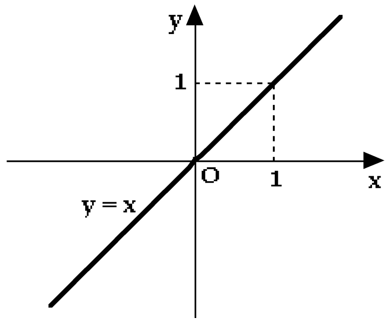
♣

Next, we present a list of important functions; for each function, we state its domain and range and show its graph.

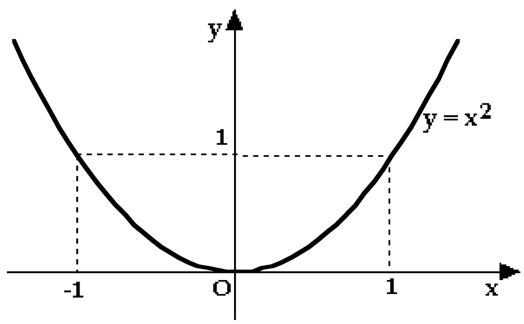
(1) $f(x) = c$ (c is a real number) is a constant function. It assigns the same number c to all real values x. Its domain consists of all real numbers. The range consists of a single point $\{c\}$. The graph below shows the constant function $f(x) = 2$.



(2) $f(x) = x$; $D = \mathbb{R}$, $R = \mathbb{R}$; the graph is a line of slope 1 through the origin.

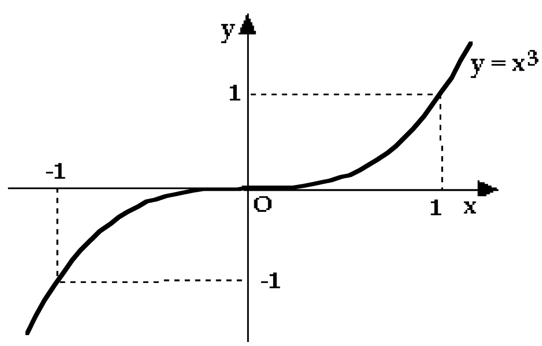


(3) $f(x) = x^2$; $D = \mathbb{R}$, $R = [0, \infty)$; the graph is a parabola with the vertex located at the origin, symmetric with respect to the y-axis.



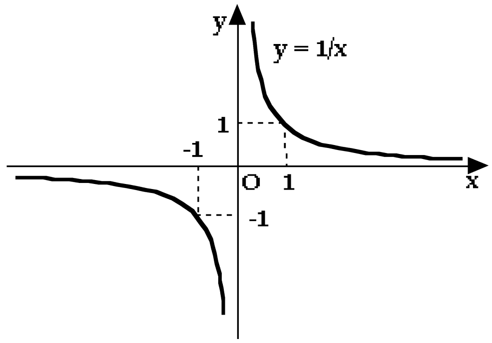
The graphs of x^n, where $n = 4, 6, 8, \ldots$, look similar to the graph of $f(x) = x^2$.

(4) $f(x) = x^3$; $D = \mathbb{R}$, $R = \mathbb{R}$; the graph is a cubic parabola; it is symmetric with respect to the origin.

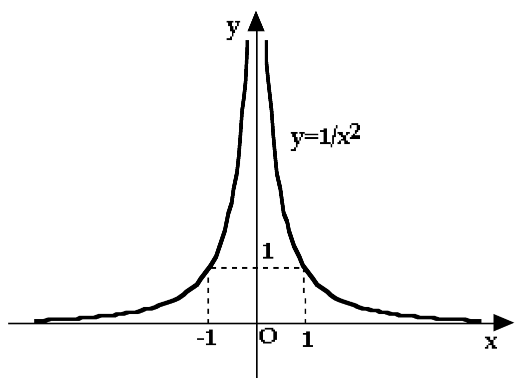


The graphs of x^n, where $n = 5, 7, 9, \ldots$, look similar to the graph of $f(x) = x^3$.

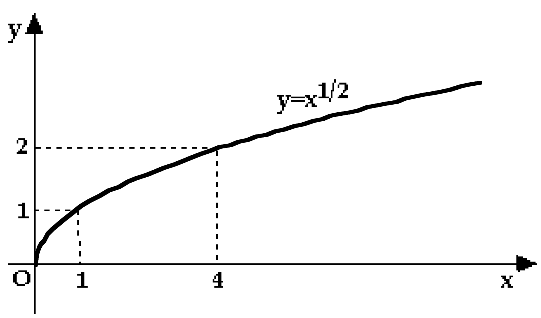
(5) $f(x) = 1/x$; $D = \{x \mid x \neq 0\}$, $R = \{y \mid y \neq 0\}$; the graph is a hyperbola, whose asymptotes are the x-axis and the y-axis. It is symmetric with respect to the origin.



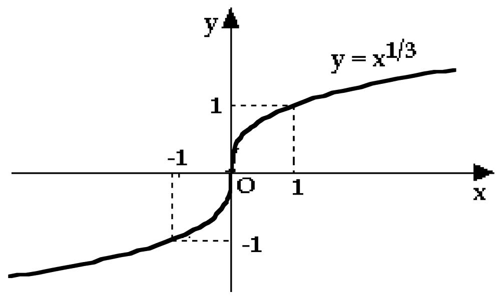
(6) $f(x) = 1/x^2$; $D = \{x \mid x \neq 0\}$, $R = \{y \mid y > 0\}$; the graph is a hyperbola, whose asymptotes are the x-axis and the y-axis. It is symmetric with respect to the y-axis.



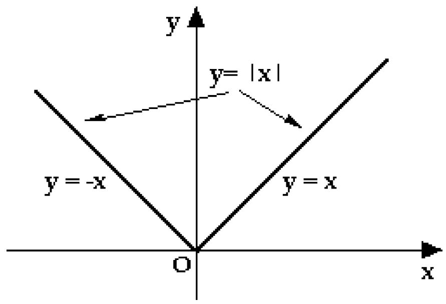
(7) $f(x) = \sqrt{x}$; $D = [0, \infty)$, $R = [0, \infty)$; the graph has a shape of a parabola.



(8) $f(x) = \sqrt[3]{x}$; $D = \mathbb{R}$, $R = \mathbb{R}$; the graph has a shape of a cubic parabola.



(9) $f(x) = |x|$; recall that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$; $D = \mathbb{R}$, $R = [0, \infty)$.

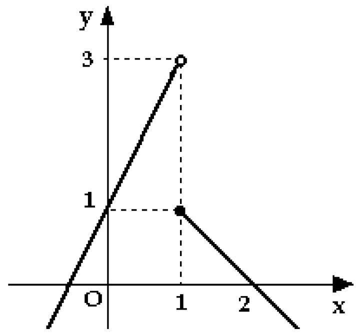


Example 4. Sketch the graph of the function

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ -x + 2 & \text{if } x \geq 1. \end{cases}$$

Solution

If $x < 1$, then $y = 2x + 1$; it is a line of slope 2 and the y-intercept 1. (Alternatively, we could compute two points that lie on it (say, $(0, 1)$ and $(1, 3)$) and join them with a straight line.) For $x \geq 1$, we get the line $y = -x + 2$ (of slope -1 and the y-intercept 2). The graph of f is given in the figure below. The filled dot in the graph indicates the fact that the value of f at $x = 1$ is $y = -x + 2 = -1 + 2 = 1$. The empty dot indicates that a point does not belong to the graph. (Note that two filled dots, one above the other, would violate the vertical line test!).



∎

Exercise 4. Sketch the graphs of the following functions.
(a) $f(x) = -4$ (b) $f(x) = -x^2$ (c) $f(x) = |x| + x$.

♣

New Functions From Old (Part I). Two functions can be added, subtracted, multiplied and divided (as long as the one in the denominator is not zero) to form new functions. For example, combining constant functions and the positive integer powers of x using addition, subtraction and multiplication, we obtain polynomials. In general, a polynomial of degree n is of the form $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ are real numbers. A quotient $p(x)/q(x)$ of two polynomials $p(x)$ and $q(x)$ (provided that $q(x) \neq 0$) is called a rational function. A function that can be obtained from polynomials using elementary algebraic operations and by taking roots is called an algebraic function. For example,

$$\sqrt{x}, \qquad \frac{x^4 + \sqrt[3]{x-1}}{x^2 + \sqrt{x}}, \qquad \frac{x^2 - 1}{x^6 + x - 1}, \qquad \sqrt[5]{\frac{1 - \sqrt{x}}{x}}$$

are algebraic functions. The functions that are not algebraic are called transcendental. Trigonometric, logarithmic and exponential functions are examples of transcendental functions. We will study such functions later.

Another common way of obtaining new functions from old is by composing functions.

Composition of functions f and g is the function $g \circ f$, defined by $$(g \circ f)(x) = g(f(x))$$

In words, we take an x and apply f to it, thus getting $f(x)$. Then we apply g to $f(x)$, to get the value $g(f(x))$ of the composition $g \circ f$ at x.

It is important to pay attention to the order in which the functions are composed. In general, $g \circ f \neq f \circ g$, as the following example shows.

Example 5. Let $f(x) = 1/x$ and $g(x) = x + 4$. Compute $g \circ f$ and $f \circ g$.

Solution

By definition,

$$(f \circ g)(x) = f(g(x)) = f(x+4) = \frac{1}{x+4}.$$

On the other hand,

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 4.$$

Clearly, $g \circ f \neq f \circ g$. ∎

The composition of three (or more) functions is computed analogously.

Example 6. Let $f(x) = \sqrt{x}$, $g(x) = 1 - \sqrt{x}$ and $h(x) = x^2 + 1$. Compute
(a) $f \circ g$ (b) $g \circ f \circ h$ (c) $f \circ h \circ f$.

Solution

(a) By definition,

$$(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}) = \sqrt{1 - \sqrt{x}}.$$

(b) As in the case of a composition of two functions,

$$(g \circ f \circ h)(x) = g(f(h(x))) = g(f(x^2+1)) = g(\sqrt{x^2+1}) = 1 - \sqrt{\sqrt{x^2+1}} = 1 - \sqrt[4]{x^2+1}.$$

(c) Similarly,

$$(f \circ h \circ f)(x) = f(h(f(x))) = f(h(\sqrt{x})) = f((\sqrt{x})^2 + 1) = f(x+1) = \sqrt{x+1}.$$ ∎

Exercise 5. Let $f(x) = 3x + 4$ and $g(x) = x/2$. Compute
(a) $f \circ g$ (b) $f \circ f$ (c) $g \circ f$.

♣

Exercise 6. Let $f(x) = x^5$ and $g(x) = x^2 + x + 1$. Compute
(a) $f \circ g$ (b) $f \circ f$ (c) $g \circ f$.

♣

New Functions From Old (Part II). By applying certain operations to the graph of a function $y = f(x)$ we obtain new functions.

Assume that $c > 0$ is a constant. The function $y = f(x) + c$ is obtained by adding c to every value of $f(x)$; geometrically, it means that the points on the graph of $y = f(x) + c$ are obtained by moving points on the graph of $y = f(x)$ up for c units. Similarly, the graph of $y = f(x) - c$ is obtained by moving the graph of $f(x)$ c units down.

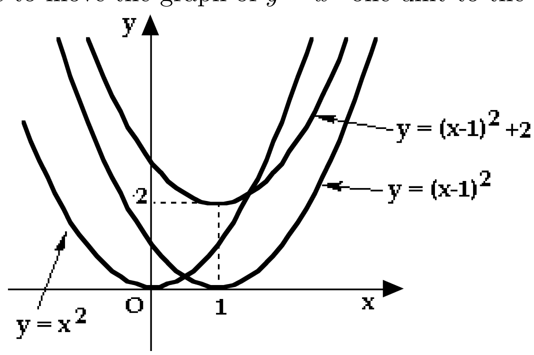
If $F(x) = f(x+c)$, then the value of F at x is the same as the value of f at $x+c$ (which is c units to the right of x); Thus, the graph of $F(x) = f(x+c)$ is obtained from the graph of $f(x)$ by moving it to the left for c units. The case of $F(x) = f(x-c)$ is argued analogously.

Let $c > 0$; to obtain the graph of		
$y = f(x) + c$,	move the graph of $y = f(x)$	c units up
$y = f(x) - c$,	move the graph of $y = f(x)$	c units down
$y = f(x+c)$,	move the graph of $y = f(x)$	c units to the left
$y = f(x-c)$,	move the graph of $y = f(x)$	c units to the right

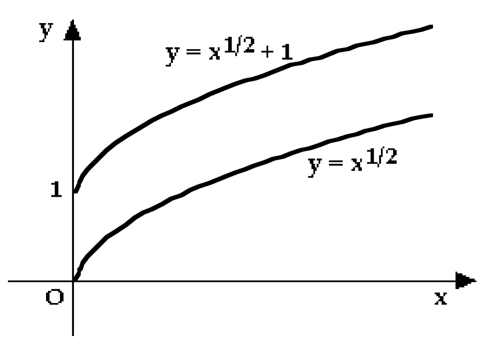
Example 7. Sketch the graphs of the following functions.
(a) $y = (x-1)^2 + 2$ (b) $y = \sqrt{x} + 1$ (c) $y = \dfrac{1}{x-4}$.

Solution

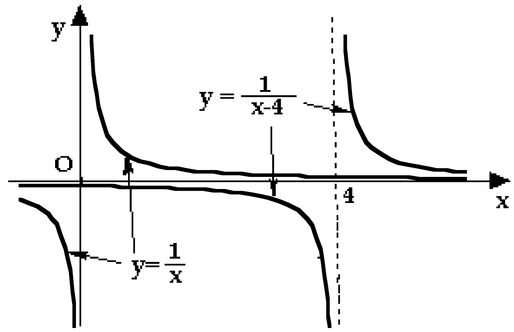
(a) The given graph is obtained from $y = x^2$ by replacing x by $x - 1$ and then by adding 2 to it; thus, we have to move the graph of $y = x^2$ one unit to the right and 2 units up.



(b) The graph of $y = \sqrt{x} + 1$ is obtained from the graph of $y = \sqrt{x}$ by moving it one unit up.



(c) The given graph is obtained from $y = 1/x$ by replacing x by $x - 4$; thus, all we need to do is to move the graph of $y = 1/x$ four units to the right.



∎

Exercise 7. Sketch the graphs of the following functions.
(a) $y = (x+2)^2 + 3$ (b) $y = \sqrt{x-4} - 2$ (c) $y = \dfrac{1}{(x-1)^2} + 2$.

♣

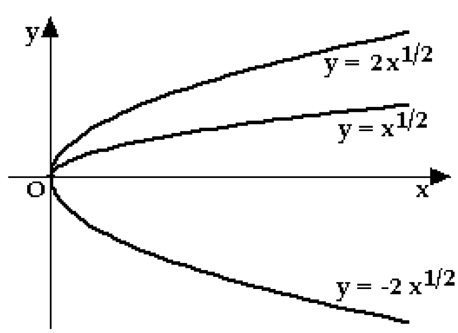
A few more ways of constructing new graphs from old are given in the following table.

To obtain the graph of
$y = cf(x)$ $(c > 1)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = cf(x)$ $(0 < c < 1)$, compress the graph of $y = f(x)$ vertically by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ with respect to the x-axis

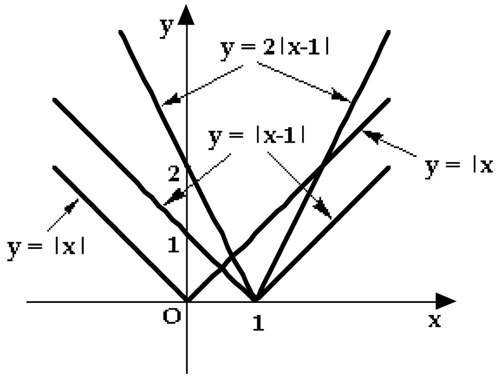
Example 8. Sketch the graphs of the following functions.
(a) $y = -2\sqrt{x}$ (b) $y = 2|x-1|$.

Solution

(a) Take the graph of $y = \sqrt{x}$, expand it by a factor of 2 and then reflect with respect to the x-axis; see figure below.



(b) Start with the graph of $y = |x|$, move it 1 unit to the right and then expand it by a factor of 2.



Exercise 8. Sketch the graphs of the following functions.

(a) $y = -x^3/2$ (b) $y = 3\sqrt{x+3} + 1$.

In the sections to follow, we will study other commonly used functions, such as trigonimetric, exponential and logarithmic functions.

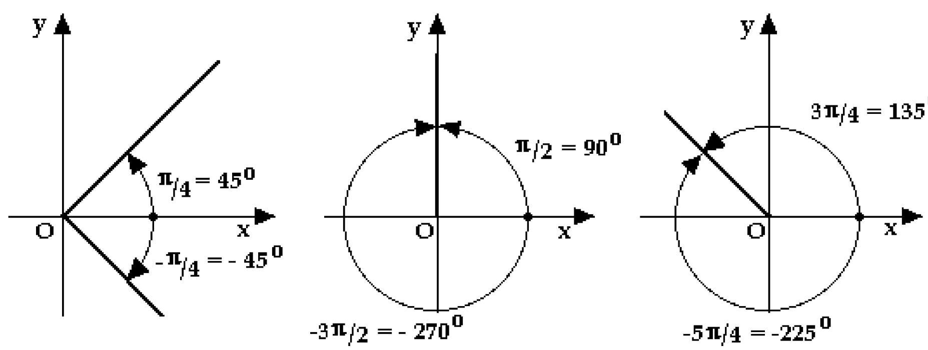
6. Trigonometry

This section contains review material on:

- Trigonometric ratios and trigonometric functions
- Trigonometric identities and trigonometric equations

Angles. Recall that a positive angle is measured counterclockwise from the direction of the positive x-axis. If it is measured clockwise, it is negative; see the figure below. The units commonly used are degrees (^{o}) and radians (rad). By convention, we use radians (unless stated otherwise). For example, $\sin 1$ denotes the value of the trigonometric function sine for 1 radian (using a calculator, we get $\sin 1 \approx 0.841471$).

A full revolution equals 360 degrees $= 2\pi$ radians. Thus, 1 degree equals $2\pi/360 = \pi/180$ radians (that is, to convert from degrees to radians, we multiply by $\pi/180$). Conversely, 1 radian equals $360/(2\pi) = 180/\pi$ degrees (and in order to convert radians into degrees, we multiply by $180/\pi$). For example, 90 degrees equals $90\frac{\pi}{180} = \frac{\pi}{2}$ radians. Similarly, $\frac{5\pi}{4}$ radians equals $\frac{5\pi}{4}\frac{180}{\pi} = 225$ degrees.

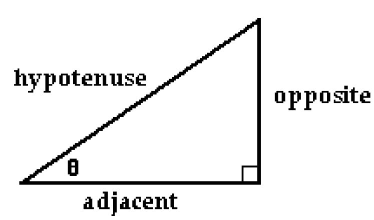


Exercise 1.
(a) Express 225 degrees in radians (b) Express $\frac{7\pi}{6}$ radians in degrees.

♣

Trigonometric Ratios. For an acute angle, the trigonometric ratios are defined as ratios of lengths of sides in a right triangle.

Basic trigonometric ratios
sine: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
cosine: $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
tangent: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\text{opposite}}{\text{adjacent}}$



The remaining three ratios are usually defined as the reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

Trigonometric ratios

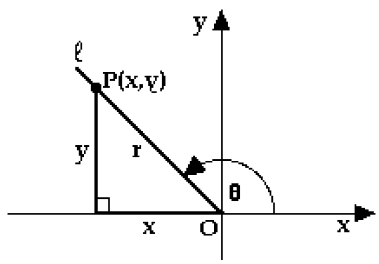
cosecant: $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

secant: $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

cotanent: $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\text{adjacent}}{\text{opposite}}$

For general angles (such as obtuse or negative angles) the above definition does not apply, and we proceed as follows.

Let θ be an angle defined by the x-axis and a line ℓ, see the figure below.



Choose a point P anywhere on the line ℓ (as long as it is not the origin), and denote by (x, y) its coordinates. Let r be the distance between P and the origin (recall that $r = \sqrt{x^2 + y^2} > 0$). We define:

Trigonometric ratios for general angles

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$

Note that $\sin\theta$ and $\cos\theta$ are always defined. However, that is not true for the remaining four ratios. The ratios $\tan\theta$ and $\sec\theta$ are not defined when $x = 0$, and $\cot\theta$ and $\csc\theta$ are not defined when $y = 0$.

For an acute angle, the two definitions agree.

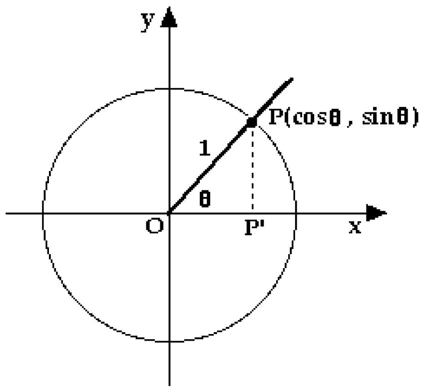
Note that

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = 1,$$

since $x^2 + y^2 = r^2$. Thus, we have obtained the basic trigonometric identity:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

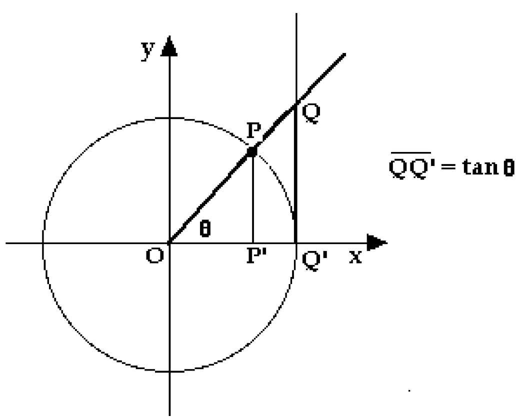
It is also possible to use a unit circle to define trigonometric ratios. Let P be the point of intersection of a circle of radius 1 and the line whose angle (positive or negative) with respect to the x-axis is θ; see the figure below.



By definition, the coordinates of P are $(\cos\theta, \sin\theta)$. In other words, $\overline{OP'} = \cos\theta$ and $\overline{PP'} = \sin\theta$. Now, draw the vertical line that intersects the x-axis at $(1, 0)$ and label its intersection with the line OP by Q, see the figure below. The triangles OPP' and OQQ' are similar. Thus,

$$\frac{\overline{QQ'}}{\overline{OQ'}} = \frac{\overline{PP'}}{\overline{OP'}} \qquad \text{implies} \qquad \frac{\overline{QQ'}}{1} = \frac{\sin\theta}{\cos\theta}.$$

Thus, $\overline{QQ'} = \tan\theta$.

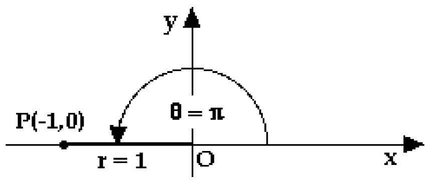


Values of Trigonometric Ratios for Special Angles.

(1) $\theta = 0$ (radians). Looking at the unit circle we see that, when $\theta = 0$, $\overline{OP'} = 1$ and $\overline{PP'} = 0$; in other words, $\cos 0 = 1$ and $\sin 0 = 0$. Consequently, $\tan 0 = \sin 0/\cos 0 = 0$, and $\sec 0 = 1/\cos 0 = 1$. The ratios $\cot 0$ and $\csc 0$ are not defined.

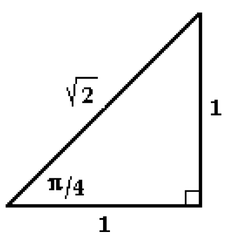
(2) $\theta = \pi/2$ (radians). In this case (use the unit circle definition again), $\overline{OP'} = 0$ and $\overline{PP'} = 1$ (in other words, the coordinates of P are $(0,1)$). Thus, $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$. It follows that $\tan\frac{\pi}{2}$ and $\sec\frac{\pi}{2}$ are not defined. Finally, $\cot\frac{\pi}{2} = \cos\frac{\pi}{2}/\sin\frac{\pi}{2} = 0/1 = 0$ and $\csc\frac{\pi}{2} = 1/\sin\frac{\pi}{2} = 1$.

(3) $\theta = \pi$ (radians). For a change, we use the definition for general angles: pick a point $P(x = -1, y = 0)$; then $r = \sqrt{(-1)^2 + 0^2} = 1$.



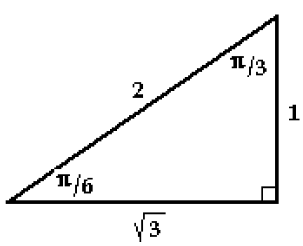
It follows that $\cos\pi = x/r = -1/1 = -1$ and $\sin\pi = y/r = 0/1 = 0$. Consequently, $\tan\pi = 0$ and $\sec\pi = -1$. The ratios $\cot\pi$ and $\csc\pi$ are not defined.

(4) $\theta = \pi/4$ (radians). The values of trigonometric ratios can be read from the triangle below.



From the definition for acute angles, we get $\sin\frac{\pi}{4} = \text{ opposite / hypotenuse } = 1/\sqrt{2}$, $\cos\frac{\pi}{4} = \text{ adjacent / hypotenuse } = 1/\sqrt{2}$ and $\tan\frac{\pi}{4} = \text{ opposite / adjacent } = 1$.

(5) $\theta = \pi/6$ and $\theta = \pi/3$ (radians). The values of trigonometric ratios can be read from the triangle below.



From the definition for acute angles, we get

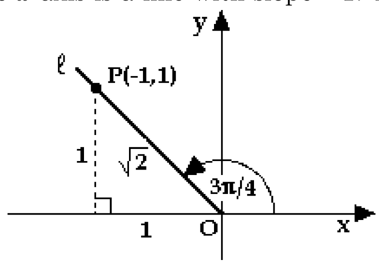
$$\sin\frac{\pi}{6} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan\frac{\pi}{6} = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}}.$$

Similarly, $\sin\frac{\pi}{3} = \sqrt{3}/2$, $\cos\frac{\pi}{3} = 1/2$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

Example 1. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for
(a) $\theta = 3\pi/4$ (b) $\theta = 2\pi/3$.

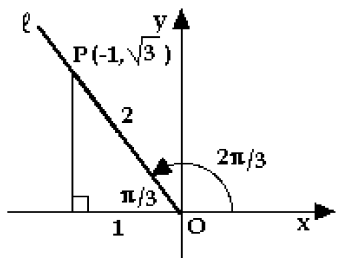
Solution

(a) We use the definition for general angles. The line that makes the angle of $\theta = \frac{3\pi}{4}$ radians with respect to the x-axis is a line with slope -1. Thus, we can choose the point $(x = -1, y = 1)$ as P.



In that case, $r = \sqrt{1+1} = \sqrt{2}$, and it follows that $\sin\frac{3\pi}{4} = y/r = 1/\sqrt{2}$, $\cos\frac{3\pi}{4} = x/r = -1/\sqrt{2}$ and $\tan\frac{3\pi}{4} = y/x = -1$.

(b) We use the definition for general angles. Placing the triangle that we used to compute the ratios for $\pi/6$ and $\pi/3$ (see (5) above), as shown in the figure, we see that we can use the point $(x = -1, y = \sqrt{3})$ as P.



It follows that $r = \sqrt{x^2+y^2} = \sqrt{1+3} = 2$, and thus $\sin\frac{2\pi}{3} = y/r = \sqrt{3}/2$, $\cos\frac{2\pi}{3} = x/r = -1/2$ and $\tan\frac{2\pi}{3} = \sqrt{3}$. ∎

Exercise 2. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for
(a) $\theta = 5\pi/6$ (b) $\theta = -\pi/6$.

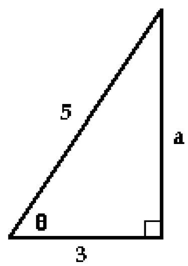
♣

Example 2. If $0 < \theta < \pi/2$, and $\cos\theta = 3/5$, find the values of $\sin\theta$, $\tan\theta$ and $\sec\theta$.

Solution

By definition, $\sec\theta = 1/\cos\theta = 5/3$.

Using the fact that $\cos\theta$ is the ratio of the adjacent side to the hypotenuse in an acute triangle (and that is given, since $0 < \theta < \pi/2$!), we label the triangle as follows:



By Pythagorean theorem, the opposite side is equal to $a = \sqrt{5^2 - 3^2} = 4$. Thus, $\sin\theta = a/5 = 4/5$ and $\tan\theta = a/3 = 4/3$. ∎

Exercise 3. If $0 < \theta < \pi/2$, and $\csc\theta = 3$, find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

♣

Exercise 4. Find the values of $\sin\theta$, $\cos\theta$, $\tan\theta$ and $\sec\theta$ for $\theta = -3\pi/2$.

♣

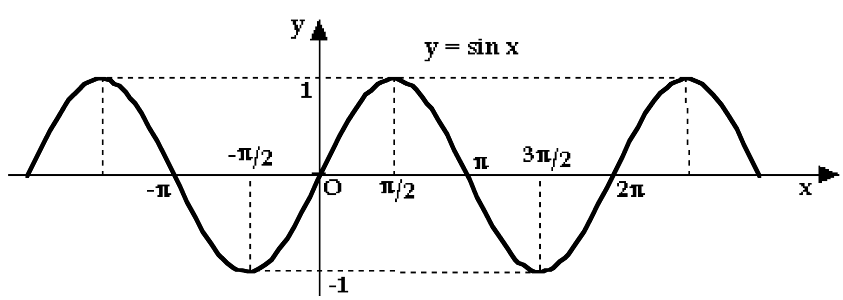
Trigonometric Functions. Let x denote an angle (in radians). Using the general method of defining trigonometric ratios, we can compute the values of the functions $y = \sin x$ and $y = \cos x$ for all real numbers x (keep in mind that x denotes an angle in radians).

Since the angles x and $x + 2\pi$ are the same (think of an angle and what it looks like one full revolution later), it follows that

Periodicity of $\sin x$ and $\cos x$
$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$

These formulas state that the values of sin and cos repeat after 2π radians. In other words, $\sin x$ and $\cos x$ are periodic with period equal to 2π.

By plotting points we obtain the graphs of the two functions. Given below is the graph of $y = \sin x$.

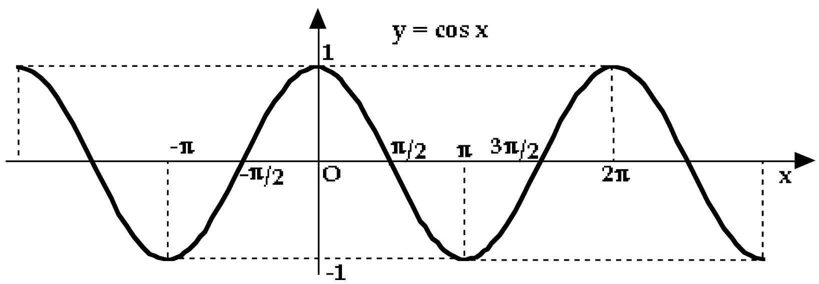


The part of the graph of $\sin x$ over the interval $[0, 2\pi]$ is called the main period. That part is repeated in both directions to produce the whole graph.

Note that $\sin x = 0$ when $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. In words, $\sin x = 0$ when x is an integer multiple of π, i.e., when $x = k\pi$ (k is an integer). We have to remember this fact.

$$\boxed{\sin x = 0 \quad \text{if and only if} \quad x = k\pi \ (k = \text{integer})}$$

Given below is the graph of $y = \cos x$.



Note that $\cos x = 0$ when $x = \ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$. In words, $\cos x = 0$ at $\pi/2$ and all points that are a multiple of π away from it. Thus,

$$\boxed{\cos x = 0 \quad \text{if and only if} \quad x = \frac{\pi}{2} + k\pi \ (k = \text{integer})}$$

The part of the graph of $\cos x$ over the interval $[0, 2\pi]$ is called the main period. That part is repeated in both directions to produce the whole graph.

Note that

$$\boxed{-1 \le \sin x \le 1 \quad \text{and} \quad -1 \le \cos x \le 1}$$

Recall the basic trigonometric identity

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

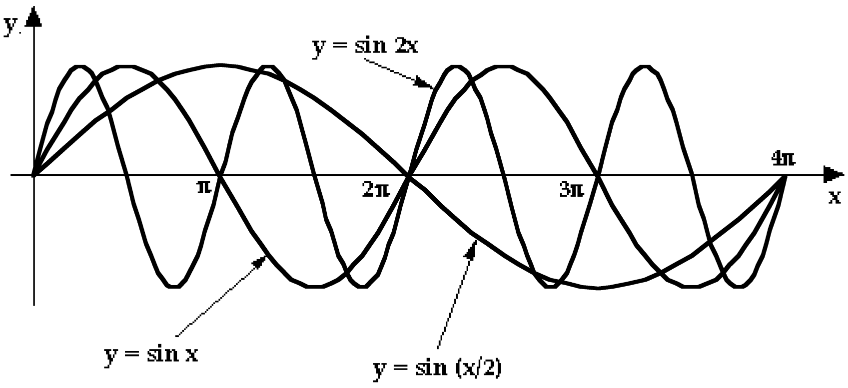
Example 3. Sketch the graphs of $y = \sin x$, $y = \sin 2x$ and $y = \sin(x/2)$ in the same coordinate system.

Solution

(a) Recall that the main period of $\sin x$ is defined to be the interval from $x = 0$ to $x = 2\pi$. Replacing x by $2x$, we get that the main period of $\sin 2x$ is the interval from $2x = 0$ (i.e., $x = 0$) to $2x = 2\pi$ (i.e., $x = \pi$). In other words, the period of $\sin 2x$ is π.

Rephrasing the above argument, we can show that the period of $\sin(ax)$ is $2\pi/a$.

Thus, the graph of $\sin 2x$ is obtained from the graph of $\sin x$ by compressing it along the x-axis by the factor of 2. The period of $\sin(x/2)$ is $2\pi/(1/2) = 4\pi$. Thus, its graph is obtained by stretching $\sin x$ along the x-axis by a factor if 2. See the figure below.



∎

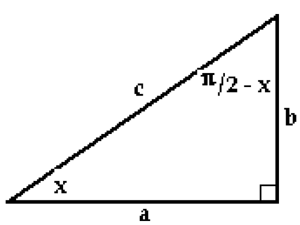
Exercise 5. What is the period of $\cos(ax)$?

♣

Exercise 6. Sketch the graphs of $y = \cos x$, $y = \cos 3x$ and $y = \cos 0.5x$ in the same coordinate system.

♣

Consider the following triangle.



Using the definition of sin and cos, we get $\cos x = a/c$, $\sin x = b/c$, $\cos\left(\frac{\pi}{2} - x\right) = b/c$ $\sin\left(\frac{\pi}{2} - x\right) = a/c$. We have thus obtained the following formulas.

$$\boxed{\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad \text{and} \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x}$$

Next, we list useful formulas involving $\sin x$ and $\cos x$.

Relation between x and $-x$

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

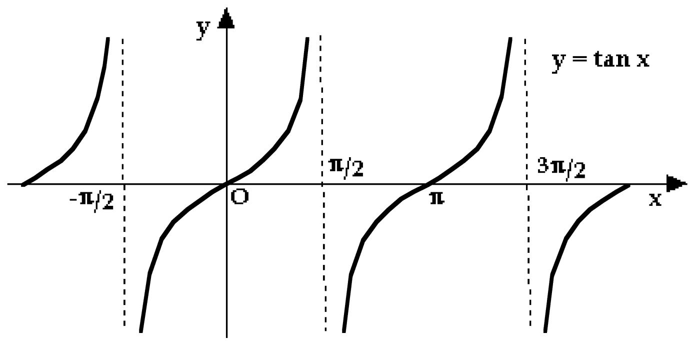
Addition and subtraction formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Double angle formulas

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

The function $y = \tan x = \frac{\sin x}{\cos x}$ is not defined when $\cos x = 0$; i.e., it is not defined when $x = \frac{\pi}{2} + k\pi$. The graph of $y = \tan x$ is given below.

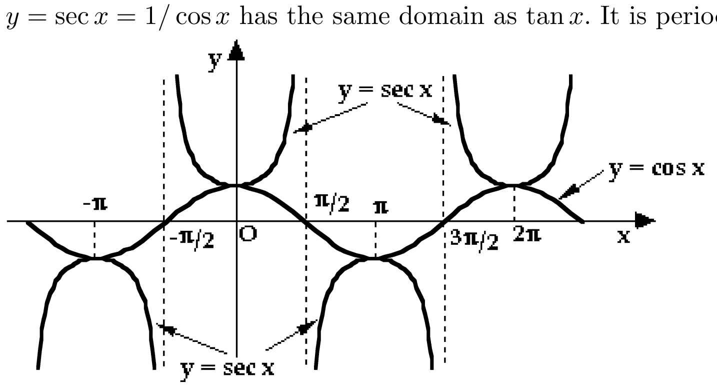


$y = \tan x$ is periodic with period π. The part of the graph over the interval $(-\pi/2, \pi/2)$ is its main period. $y = \tan x = 0$ whenever $\sin x = 0$.

Exercise 7. What is the period of $\tan(ax)$?

♣

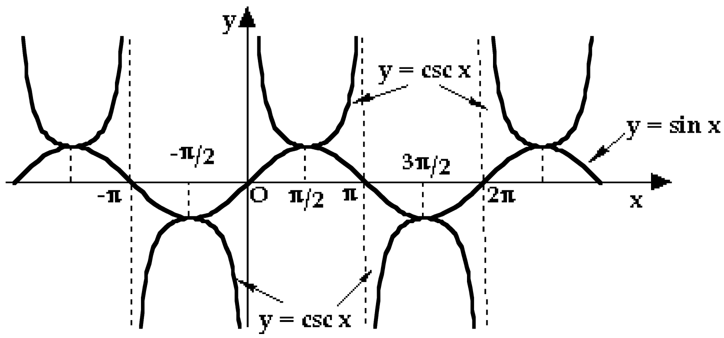
The function $y = \sec x = 1/\cos x$ has the same domain as $\tan x$. It is periodic with period 2π.



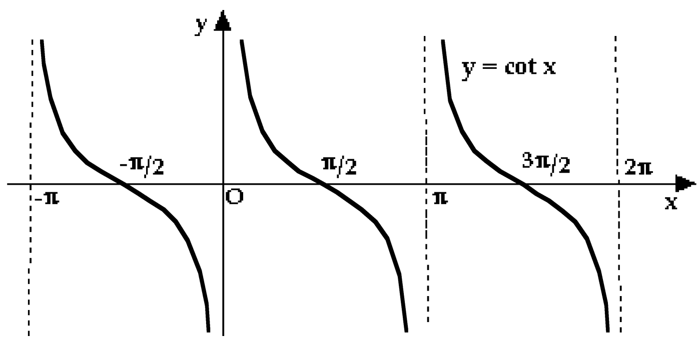
There is a useful relationship between the tangent and the secant, given by

$$\boxed{\tan^2 x + 1 = \sec^2 x}$$

The graph of $\csc x = 1/\sin x$ is given below.



The graph of $\cot x = 1/\tan x = \cos x/\sin x$ is given below. The domain of both $\csc x$ and $\cot x$ consists of all x such that $x \neq k\pi$ (k=integer).



Example 4. Prove the following formulas.

(a) $\sin(\pi - x) = \sin x$

(b) $(\sin x + \cos x)^2 = 1 + \sin 2x$

(c) $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = \dfrac{2}{\cos^2 x}$

Solution

(a) Using the subtraction formula for sin, we get

$$\sin(\pi - x) = \sin\pi\cos x - \cos\pi\sin x = 0\cdot\cos x - (-1)\sin x = \sin x.$$

(b) Squaring the left side,

$$(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x,$$

using $\sin^2 x + \cos^2 x = 1$ and the double angle formula for sin, we get

$$= 1 + 2\sin x\cos x = 1 + \sin 2x.$$

(c) Computing the common denominator on the left side, we get

$$\frac{1}{1-\sin x}+\frac{1}{1+\sin x}=\frac{1+\sin x}{1-\sin^2 x}+\frac{1-\sin x}{1-\sin^2 x}=\frac{2}{1-\sin^2 x}=\frac{2}{\cos^2 x}.$$

We used the identity $\sin^2 x+\cos^2 x=1$. ∎

Exercise 8. Prove the following formulas.

(a) $\sin(\pi/2+x)=\cos x$ (b) $\tan^2 x+1=\sec^2 x$

(c) $\sin^2 x-\tan^2 x+\sin^2 x\tan^2 x=0$.

♣

Example 5. Using addition formulas, prove the following identities.

(a) $\sin 2x=2\sin x\cos x$ (b) $\cos 2x=1-2\sin^2 x$.

Solution

(a) $\sin 2x=\sin(x+x)=\sin x\cos x+\cos x\sin x=2\sin x\cos x$.

(b) As in (a),

$$\cos 2x=\cos(x+x)=\cos x\cos x-\sin x\sin x=\cos^2 x-\sin^2 x$$

now use the identity $\sin^2 x+\cos^2 x=1$ to eliminate $\cos^2 x$

$$=(1-\sin^2 x)-\sin^2 x=1-2\sin^2 x.$$ ∎

Example 6. Show that $\cos 3x=4\cos^3 x-3\cos x$.

Solution

Write $3x=2x+x$, and start with the addition formula for cos :

$$\begin{aligned}\cos 3x&=\cos(2x+x)\\&=\cos 2x\cos x-\sin 2x\sin x\end{aligned}$$

use the double angle formulas

$$\begin{aligned}&=(2\cos^2 x-1)\cos x-(2\sin x\cos x)\sin x\\&=2\cos^3 x-\cos x-2\cos x\sin^2 x\end{aligned}$$

replace $\sin^2 x$ using the basic trigonometric identity $\sin^2 x=1-\cos^2 x$

$$\begin{aligned}&=2\cos^3 x-\cos x-2\cos x(1-\cos^2 x)\\&=2\cos^3 x-\cos x-2\cos x+2\cos^3 x\\&=4\cos^3 x-3\cos x.\end{aligned}$$ ∎

Exercise 9. Using the idea of the previous example, derive a formula that expresses $\sin 3x$ in terms of $\sin x$.

♣

Exercise 10. Show that $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$.

♣

In what follows, the symbol k denotes an integer.

Trigonometric Equations. To find a solution to a trigonometric equation, we find all solutions in the main period first, and then add the multiple of the period.

Example 7. Solve the following equations.
(a) $\sin x = 1$ (b) $\tan x = 1$.

Solution

(a) Looking at the graph of $\sin x$, we see that $x = \frac{\pi}{2}$ is the only solution of $\sin x = 1$ in the main period of $\sin x$. Thus, all solutions are given by $x = \frac{\pi}{2} + 2k\pi$.

(b) There is only one solution to $\tan x = 1$ in the main period of tangent (which is the interval from $-\pi/2$ to $\pi/2$): $x = \frac{\pi}{4}$. Since the period of the tangent is π, all solutions are given by $x = \frac{\pi}{4} + k\pi$. ∎

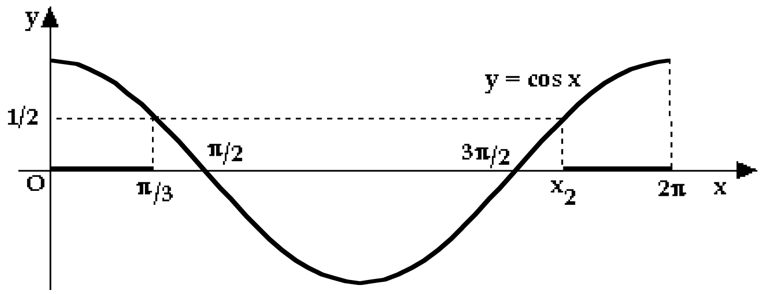
Exercise 11. Solve the following equations.
(a) $\cos x = -1$ (b) $\tan x = -1$.

♣

Example 8. Solve the following equations.
(a) $\cos x = 1/2$ (b) $\sin x = -1/2$.

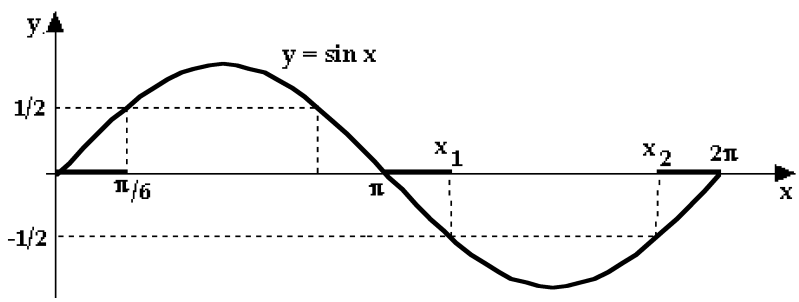
Solution

(a) From the graph below we see that there are two solutions of the given equation in the main period.



One of them is $x_1 = \frac{\pi}{3}$. Due to symmetry, the other solution is $\pi/3$ units to the left of 2π; thus, $x_2 = 2\pi - \pi/3 = 5\pi/3$. It follows that all solutions are given by $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$ (k is an integer).

(b) From the graph we see that there are two solutions of $\sin x = -1/2$ in the main period.



We know that $\sin \frac{\pi}{6} = \frac{1}{2}$. By symmetry, x_1 is $\pi/6$ units to the right of π, so $x_1 = \pi + \pi/6 = 7\pi/6$. Again, by symmetry, x_2 is $\pi/6$ units to the left of 2π; thus, $x_2 = 2\pi - \pi/6 = 11\pi/6$. Thus, the solutions are $x = \frac{7\pi}{6} + 2k\pi$ and $x = \frac{11\pi}{6} + 2k\pi$. ∎

Exercise 12. Solve the following equations.
(a) $\cos x = -\sqrt{3}/2$ (b) $\tan x = \sqrt{3}$ (c) $\sin x = \sqrt{2}/2$

♣

Example 9. Solve the equation $\sin 2x = \cos x$.

Solution

Using the double angle formula, we get

$$\begin{aligned} \sin 2x &= \cos x \\ 2\sin x \cos x &= \cos x \\ \cos x(2\sin x - 1) &= 0. \end{aligned}$$

Thus, $\cos x = 0$ or $2\sin x - 1 = 0$. If $\cos x = 0$, then $x = \frac{\pi}{2} + k\pi$ (this equation has been solved earlier). If $\sin x = 1/2$, then $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{5\pi}{6} + 2k\pi$ (look at the graph of Example 8(b)). Thus, the solution is $x = \frac{\pi}{2} + k\pi$, $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{5\pi}{6} + 2k\pi$. ∎

Exercise 13. Solve the equation $\sin 2x = \sin x$.

♣

Example 10. Solve the equation $2 + \cos 2x = 3\cos x$.

Solution

Using the double angle formula for $\cos x$, we rewrite $2+\cos 2x = 3\cos x$ as $2+2\cos^2 x - 1 = 3\cos x$. Thus $2\cos^2 x - 3\cos x + 1 = 0$, and $(2\cos x - 1)(\cos x - 1) = 0$.

It follows that $2\cos x - 1 = 0$, and $\cos x = 1/2$ (in which case $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$, see Example 8(a)) and $\cos x - 1 = 0$, and $\cos x = 1$ (in which case $x = 2k\pi$). Thus, the solution is $x = 2k\pi$, $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$. ∎

The latter two examples show that, if an equation contains different arguments of trig functions (such as sin and/or cos of x and $2x$), it is a good idea to reduce the expressions to a single argument (which is usually x).

Example 11. Solve the equation $4\sin 2x\cos 2x = 1$.

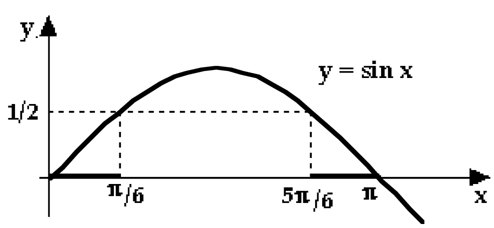
Solution

Using the double angle formula for $\sin x$, we get

$$4\sin 2x\cos 2x = 2(2\sin 2x\cos 2x) = 2\sin 4x = 1,$$

and thus $\sin 4x = 1/2$.

Now, $\sin A = 1/2$ implies $A = \pi/6$ or $A = 5\pi/6$, see the figure below.



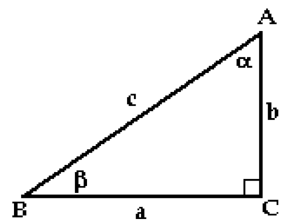
Thus (replacing A by $4x$), $4x = \frac{\pi}{6} + 2k\pi$ and $4x = \frac{5\pi}{6} + 2k\pi$, and the solutions are $x = \frac{\pi}{24} + \frac{k\pi}{2}$ and $x = \frac{5\pi}{24} + \frac{k\pi}{2}$. ∎

Exercise 14. Solve the equation $2\cos 2x - 1 = 0$.

♣

Additional exercises.

Exercise 15. Let ABC be a right triangle, where $\angle C = 90^0$; see the figure below.



(a) Given that $a = 21/5$ and $b = 4$, find $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$.

(b) Given that $\cos\beta = 12/13$ and $c = 13$, find a, b, $\sin\beta$, and $\tan\beta$.

(c) Given that $c = 1$ and $a = 0.6$, find all six trigonometric ratios for angle β.

Exercise 16. What quadrant do the following angles belong to? (Recall the convention that states that if no units for angles are explicitly stated, then the units are radians.)

(a) $36\pi/7$ (b) 999^0 (c) 989^0 (d) $44\pi/5$.

Exercise 17. Without a calculator, determine the sign of the following expressions.

(a) $\tan(13\pi/3)$ (b) $\sin(500^0)$ (c) $\cos(37\pi)$ (d) $\sin(\pi/12) + \cos(\pi/7)$.

Exercise 18. Without a calculator, determine which of the following is larger.

(a) $\sin 1^0$ or $\sin 1$ (b) $\cos 2^0$ or $\cos 2$ (c) $\tan 1^0$ or $\tan 1$.

Exercise 19. Simplify the following expressions.

(a) $\sec^2 x - \sin^2 x - \cos^2 x$

(b) $\dfrac{\cos x}{1 + \sin x} + \tan x$

(c) $\dfrac{\sin x}{1 + \cos x} + \dfrac{\sin x}{1 - \cos x}$

(d) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$.

Exercise 20. Solve the following equations.

(a) $\tan x = -\sqrt{3}/3$

(b) $\cot x = -1$

(c) $\cos x = \sqrt{3}/2$.

Exercise 21. Solve the following equations.

(a) $\sin x = \sqrt{2}/2$

(b) $\tan x + \cot x = 0.5$

(c) $\cos^2 x - \cos x - 2 = 0$.

Exercise 22. Sketch the graphs of the following functions.

(a) $\cos(x + \pi/4)$ (b) $\sin(x - \pi)$ (c) $\tan(x + 1)$.

Exercise 23. Prove the following identities.

(a) $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

(b) $\dfrac{1}{1 + \tan^2 x} + \dfrac{1}{1 + \cot^2 x} = 1$.

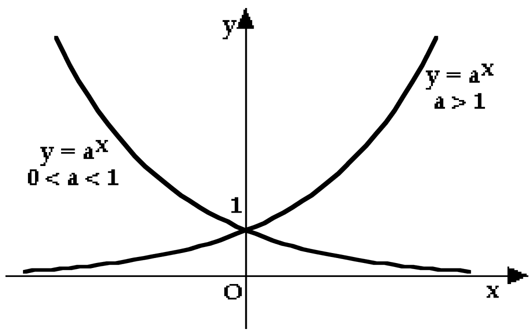
7. Exponential and Logarithmic Functions

This section contains review material on:

- Exponential functions and the natural exponential function
- Logarithmic functions and the natural logarithmic function

Exponential Functions. An exponential function is a function of the form $y = a^x$, where $a > 0$ and x is any real number. Although we can sometimes compute a power of a negative number, such as $(-4)^3$, the exponential function is defined only for positive bases. The domain of $y = a^x$ consists of all real numbers. Since $a^x > 0$ for all x (remember that $a > 0$!), it follows that the range of the exponential function $y = a^x$ consists of positive numbers only.

By plotting points, we obtain the graph of $y = a^x$.

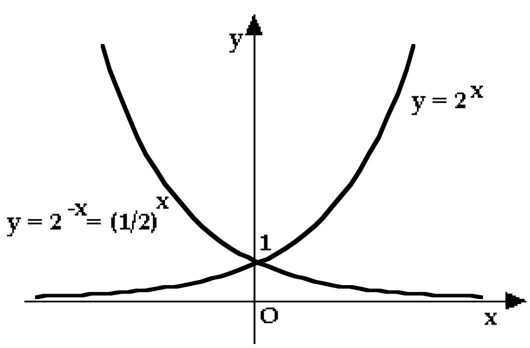


Since $a^0 = 1$, the graph of $y = a^x$ goes through the point $(0, 1)$ on the y-axis. If $a > 1$, the graph of $y = a^x$ is increasing. For $0 < a < 1$, it is decreasing. In either case, the x-axis is its horizontal asymptote.

Example 1. Sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ in the same coordinate system.

Solution

Since $2^{-x} = (2^{-1})^x = (1/2)^x$, we are asked to plot the functions 2^x and $(1/2)^x$. By plotting points, we obtain the following picture.

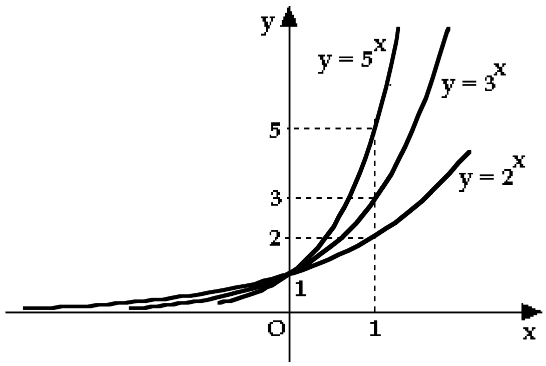


∎

Example 2. Sketch the graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$ in the same coordinate system.

Solution

By computing the values of the given functions for different x, we see that as the basis a in $y = a^x$ increases, the graph increases faster and faster.



∎

Exercise 1. Sketch the graphs of the following functions.

(a) $y = 2^x + 4$ (b) $y = 2^{x-4}$ (c) $y = -2^x$ (d) $y = -2^{-x}$

♣

Exercise 2. Sketch the graphs of $y = 2^{-x}$, $y = 3^{-x}$ and $y = 4^{-x}$ in the same coordinate system.

♣

Although algebraic rules for working with exponential functions have been given already, we repeat them here for convenience.

$$\boxed{\begin{array}{c} a^0 = 1 \qquad a^1 = a \qquad a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \\ \dfrac{a^x}{a^y} = a^{x-y} \qquad \dfrac{1}{a^x} = a^{-x} \end{array}}$$

Example 3. Simplify the following expressions (i.e., reduce to a single exponential function).

(a) $4^{x+6} \cdot 8^{2-x}$ (b) $\dfrac{27^{2x-3}}{9^{x-4}}$ (c) $(2^x)^3 \cdot (4^{2-x})^4$.

Solution

(a) Using the above formulas, we get

$$\begin{aligned} 4^{x+6} \cdot 8^{2-x} &= (2^2)^{x+6} \cdot (2^3)^{2-x} = 2^{2(x+6)} \cdot 2^{3(2-x)} \\ &= 2^{2x+12} \cdot 2^{6-3x} = 2^{(2x+12)+(6-3x)} = 2^{-x+18}. \end{aligned}$$

(b) Similarly,

$$\frac{27^{2x-3}}{9^{x-4}} = \frac{(3^3)^{2x-3}}{(3^2)^{x-4}} = \frac{3^{6x-9}}{3^{2x-8}} = 3^{(6x-9)-(2x-8)} = 3^{4x-1}.$$

(c) Start by exponentiating the exponents:

$$(2^x)^3 \cdot (4^{2-x})^4 = 2^{3x} \cdot 4^{8-4x} = 2^{3x} \cdot (2^2)^{8-4x} = 2^{3x} \cdot 2^{16-8x} = 2^{-5x+16}.$$

∎

Exercise 3. Simplify the following expressions (i.e., reduce to a single exponential function).

(a) $5^{x-2} \cdot 25^{3-x}$ (b) $3^{x-1} \cdot 9^{x-2} \cdot 27^{x-3}$ (c) $\dfrac{8^{x+4}}{16^{x-2}}$.

♣

Example 4. Solve each of the following equations for x.

(a) $4^x = 16^{2x-2}$ (b) $2^{x^3} = 0.25$ (c) $3^{2x} - 6 \cdot 3^x - 27 = 0$.

Solution

(a) Simplify so that both sides have the same basis:

$$\begin{aligned} 4^x &= 16^{2x-2} \\ 4^x &= (4^2)^{2x-2} \\ 4^x &= 4^{4x-4} \end{aligned}$$

It follows that $4x - 4 = x$ and $x = 4/3$.

(b) Use the technique from (a):

$$2^{x^3} = 0.25 = \frac{1}{4} = \frac{1}{2^2} = 2^{-2}.$$

Thus, $x^3 = -2$ and so $x = \sqrt[3]{-2}$.

(c) The idea lies in the fact that $3^{2x} = (3^x)^2$; this implies that the given equation is a quadratic equation in 3^x. Let $y = 3^x$; then $3^{2x} - 6 \cdot 3^x - 27 = 0$ reads $y^2 - 6y - 27 = 0$. From

$$y^2 - 6y - 27 = (y+3)(y-9) = 0,$$

we conclude that $3^x = y = -3$ or $3^x = y = 9$.

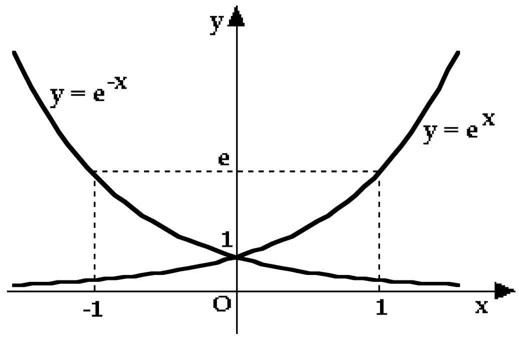
Since $3^x > 0$, the equation $3^x = -3$ has no solutions. From $3^x = 9$, we get $x = 2$. Thus, the only solution is $x = 2$. ∎

Exercise 4. Solve each of the following equations for x.

(a) $0.5^{x^2} = 0.125$ (b) $3^x(3^x - 3) = 0$ (c) $2^{2x} - 5 \cdot 2^x + 4 = 0$.

♣

In the case when $a = e \approx 2.71828$, we obtain the so-called special exponential function $y = e^x$. This function is used in a number of applications, from population problems to compound interest and radioactive decay. The graphs of $y = e^x$ and $y = e^{-x} = 1/e^x$ are shown below.



Let us recall that (as any exponential function) the natural exponential function satisfies $e^0 = 1$ and $e^x > 0$ for all x.

Logarithms. The statement $a^m = n$ can also be written as $\log_a n = m$, where $\log_a$ is the logarithm to the base a. For example, $10^2 = 100$ is the same as $\log_{10} 100 = 2$. Similarly, $5^4 = 625$ can be restated as $\log_5 625 = 4$. The statement $\log_2 32 = 5$ is just another way of saying that $2^5 = 32$.

Substituting $m = \log_a n$ into $a^m = n$, we get $a^{\log_a n} = n$. In words, if we take a number (call in n), apply $\log_a$ to it and then exponentiate it (with the base a) we get our number back. Similarly, substituting $n = a^m$ into $m = \log_a n$, gives $\log_a a^m = m$. Thus, taking a number m, exponentiating it (with the base a) and then taking $\log_a$ does not change it.

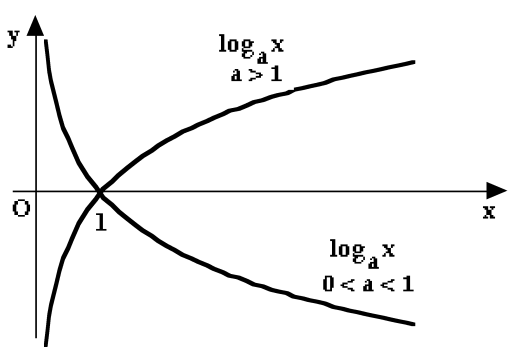
In other words, we say that exponentiating with the base a and applying logarithm to the base a are inverse of each other.

Note that from $n = a^m$ it follows that $n > 0$. Thus, $\log_a n$ is defined only for positive numbers n.

Logarithmic Functions. The logarithmic function $y = \log_a x$ is defined as the inverse function of the exponential function $y = a^x$. Consequently, when we apply the composition of the two functions (in any order) to a number x, we get it back:

$$a^{\log_a x} = x \qquad \text{and} \qquad \log_a a^x = x.$$

The domain of $\log_a x$ consists of positive numbers only. Its range is all of $\mathbb{R}$; see the graph below.



The graphs are the symmetric images of the graphs of $y = a^x$ with respect to the line $y = x$.

Since $a^0 = 1$, it follows that $\log_a 1 = 0$ (i.e., the value of $\log_a$ at 1 is 0). Thus, $\log_a x$ goes through the point $(1, 0)$ on the x-axis. If $a > 1$, $\log_a x$ is an increasing function; otherwise (if $0 < a < 1$), it is a decreasing function. In either case, the y-axis is its vertical asymptote.

Rules for logarithms

$$a^{\log_a x} = x \qquad \log_a a^x = x \qquad \log_a 1 = 0 \qquad \log_a a = 1$$

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a(x^n) = n \log_a x$$

$$\log_a(x/y) = \log_a x - \log_a y$$

Sometimes it might be useful to convert logarithms from one base to the other. The conversion formula is

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

The inverse function of the natural exponential function $y = e^x$ is called the natural logarithmic function, and is denoted by $\ln x$ (instead of $\log_e x$). Although we have already stated the properties of $y = \ln x$ when we talked about a general logarithmic function, we repeat it here.

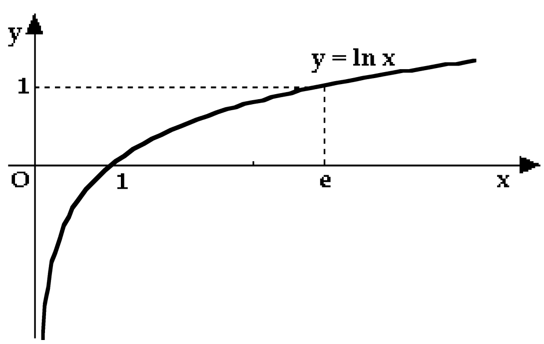
The domain of $\ln x$ is $(0, \infty)$. Its range consists of all real numbers.

By definition, e^x and $\ln x$ are inverse functions — thus, $e^{\ln x} = x$ (for all $x > 0$) and $\ln e^x = x$ (for all x). Moreover, $\ln 1 = 0$ and $\ln e = 1$ (the latter is true since $\ln e = \ln e^1 = 1$).

Natural logarithm can be used to simplify products, quotients and powers:

$$\ln(xy) = \ln x + \ln y \qquad \ln(x/y) = \ln x - \ln y \qquad \ln(x^n) = n \ln x.$$

The graph of $\ln x$ is given below (it is the symmetric image of $y = e^x$ with respect to the line $y = x$).



If needed, we can use conversion formulas

$$\ln x = \frac{\log_a x}{\log_a e} \qquad \text{and} \qquad \log_a x = \frac{\ln x}{\ln a}.$$

Example 5. Solve each of the following equations for x.

(a) $\log_2 x = 7$ (b) $\log_x 8 = 3$ (c) $\log_{16} 8 = x$ (d) $\log_2(\log_5 x) = 2$.

Solution

(a) Rewriting $\log_2 x = 7$ in the exponential form, we get $2^7 = x$; thus, $x = 128$. Alternatively, we could start with the equation $\log_2 x = 7$ and apply the exponential function 2^x to it, thus getting $2^{\log_2 x} = 2^7$; since $2^{\log_2 x} = x$, we get that $x = 2^7 = 128$.

(b) Rewriting $\log_x 8 = 3$ in the exponential form, we get $x^3 = 8$; thus, $x = 2$.

(c) Proceeding as in (a) or (b), we get $16^x = 8$. Thus $(2^4)^x = 2^3$, and $2^{4x} = 2^3$; it follows that $4x = 3$ and $x = 3/4$.

(d) Keep in mind the general principle: $\log_a B = C$ is equivalent to $B = a^C$. Applying this principle with $a = 2$, $B = \log_5 x$ and $C = 2$, we get $\log_5 x = 2^2 = 4$. Applying it once again, we get that $x = 5^4 = 625$. ∎

Exercise 5. Solve each of the following equations for x.

(a) $\log_x 4 = 1/2$ (b) $\log_3 x = 5$ (c) $\log_2 x^3 = \log_2(4x)$ (d) $16^{\log_4 x} = 4$.

♣

Example 6.

(a) Evaluate $e^{3\ln 2} \cdot e^{2\ln 3}$

(b) Express $2\ln 4 - \ln 8 - \ln 5$ as a single logarithm

(c) Solve $\ln(4x - 3) = 7$ for x

(d) Solve $\ln(\ln x) = 1$ for x

(e) Solve $\ln x + \ln(x + 7) = \ln 4 + \ln 2$ for x.

Solution

(a) We simplify exponents first and then use $e^{\ln x} = x$:

$$e^{3\ln 2} \cdot e^{2\ln 3} = e^{\ln 2^3} \cdot e^{\ln 3^2} = e^{\ln 8} \cdot e^{\ln 9} = 8 \cdot 9 = 72.$$

(b) Using $n \ln x = \ln x^n$ and $\ln x - \ln y = \ln(x/y)$, we get

$$\begin{aligned} 2\ln 4 - \ln 8 - \ln 5 &= \ln 4^2 - \ln 8 - \ln 5 = (\ln 16 - \ln 8) - \ln 5 \\ &= \ln(16/8) - \ln 5 = \ln 2 - \ln 5 = \ln(2/5). \end{aligned}$$

(c) Applying the inverse function e^x to both sides, we get

$$\begin{aligned} \ln(4x - 3) &= 7 \\ e^{\ln(4x-3)} &= e^7 \\ 4x - 3 &= e^7 \\ x &= \frac{e^7 + 3}{4}. \end{aligned}$$

(d) We repeat twice what we did in (c):

$$\begin{aligned}\ln(\ln x) &= 1\\ e^{\ln(\ln x)} &= e^1\\ \ln x &= e\\ e^{\ln x} &= e^e\\ x &= e^e.\end{aligned}$$

(e) Combining the terms on both sides we get

$$\begin{aligned}\ln x + \ln(x+7) &= \ln 4 + \ln 2\\ \ln x(x+7) &= \ln 8\\ x(x+7) &= 8\\ x^2 + 7x - 8 &= 0\\ (x+8)(x-1) &= 0.\end{aligned}$$

Thus, $x = -8$ and $x = 1$. The value $x = 1$ is a solution, since both terms on the right side of the given equation are defined. That is not true for $x = -8$, and so $x = 1$ is the only solution. ∎

Exercise 6.

(a) Evaluate $e^{\ln 4 + \ln 5}$

(b) Express $4\ln 2 + \ln 3 + 2$ as a single logarithm

(c) Solve $e^{3x-2} = 4$ for x

(d) Solve $\ln(x^2 + x - 1) = 0$ for x.

♣

Additional exercises.

Exercise 7.

(a) Simplify $10 \cdot 100^2 \cdot 1000^4$ by reducing to a single exponential function

(b) Reduce $3^7 + 6 \cdot 3^6$ to a single term

(c) Reduce $9 \cdot 27^3 + 2 \cdot 3^{11}$ to a single term

(d) Simplify $\dfrac{36^{n+3}}{6^{2n+5}}$ by reducing to a single exponential function.

Exercise 8. Without a calculator, evaluate the following expressions.

(a) $\dfrac{(0.5 \cdot 10)^{-3}}{16 \cdot 0.1^4}$

(b) $0.2^{-4} \cdot 16$

(c) $-32 \cdot \left(\frac{1}{2}\right)^4$.

Exercise 9. Without a calculator, find numeric values of the following expressions.

(a) $e^{(1/2)\ln 8}$ (b) $10^{\log_{10} 5}$ (c) $\log_{10} 100000$.

Exercise 10. Without a calculator, find numeric values of the following expressions.

(a) $\log_3(1/9)$ (b) $\ln\left(e^{\ln(e^2)}\right)$ (c) $e^{-\ln 23}$.

Exercise 11. Solve the following equations.

(a) $0.1^x = 100$ (b) $(1/4)^x = 2$ (c) $0.25^x = 16$.

Exercise 12. Solve the following equations.

(a) $0.1^{x+2} = 100^{1/3}$ (b) $e^{2x} + 2e^x - 8 = 0$.

8. Calculus: Limits and Derivatives

This section contains review material on:

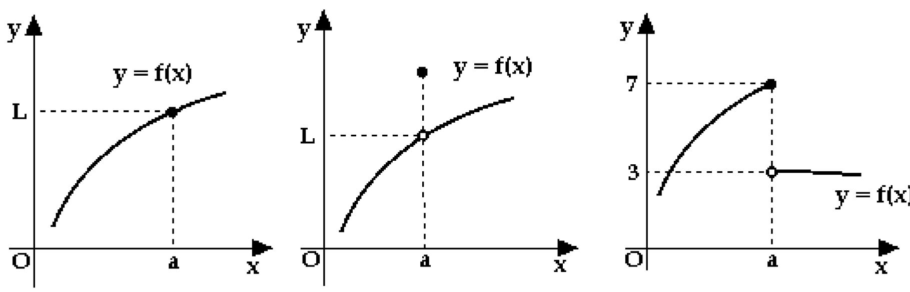
- Limits
- Derivatives

Limits. We do not intend to go into theoretical considerations about limits and other concepts of calculus, but rather concentrate on a few basic (mostly technical) issues.

We say that the limit of $f(x)$, as x approaches a, is L, and write $\lim_{x\to a} f(x) = L$, if we can make the values $f(x)$ as close to L as needed by choosing the values for x inside a small enough interval around a (for various reasons we require that $x \neq a$).

This statement is far from a precise definition, but is a good one to start with; it enables us to develop intuitive understanding of limits for functions of one variable.

Consider the following graphs.



(a) (b) (c)

For functions in (a) and (b), $\lim_{x\to a} f(x) = L$. According to our definition, the behaviour of f at a is irrelevant for its limit as x approaches a (remember, in the definition we required that $x \neq a$). Thus, the function in (b) would have had a limit equal to L even if it were not defined at a.

Consider the case (c). Can the limit of $f(x)$ as x approaches a be 7?

The answer is no — for the following reason: no matter how small interval around a we take, there will always be values of x (in this case, to the right of a, inside the interval) for which the function is approximately equal to 3 — and that is not close to 7.

Using a similar argument, we could rule out any other real number as a value of the limit of $f(x)$ as x approaches a. In such cases, we say that the limit does not exist.

Algebraically, we compute limits using limit laws

Limit laws
Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist; then
$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$

There are many more laws, most of which boil down to the following. Recall that an algebraic function is a function that is built from polynomials by using elementary algebraic operations and by taking roots. Then

$$\boxed{\text{If } f(x) \text{ is an algebraic function and } f(a) \text{ is defined, then } \lim_{x\to a} f(x) = f(a)}$$

So, in some cases it is possible to compute limits by substituting a for x.

Example 1. Compute $\lim\limits_{x\to 3} \dfrac{\sqrt{x}+3x}{x^2-4x+4}$.

Solution

Given function is an algebraic function; thus,

$$\lim_{x\to 3} \frac{\sqrt{x}+3x}{x^2-4x+4} = \frac{\sqrt{3}+3(3)}{(3)^2-4(3)+4} = \frac{\sqrt{3}+9}{1} = \sqrt{3}+9. \qquad \blacksquare$$

Exercise 1. Compute the following limits.

(a) $\lim\limits_{x\to -2} \dfrac{x^2-4x+2}{x-2}$ (b) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{x^2+x+1}-1}{\sqrt{x}+1}$.

♣

In some cases, we have to simplify an expression before taking limits. Let us consider a few examples.

Example 2. Compute the following limits.

(a) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$ (b) $\lim\limits_{x\to 0} \dfrac{x^3-x}{x}$ (c) $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x^2-9}$.

Solution

(a) Substituting $x = 1$, we get $\frac{x^2-1}{x-1} = \frac{0}{0}$, which is not defined (such expression is called an indeterminate form). Notice that it is possible to cancel the fraction:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2.$$

(b) As in (a), cancel the fraction:

$$\lim_{x\to 0} \frac{x^3-x}{x} = \lim_{x\to 0}(x^2-1) = -1.$$

(c) Factoring both the numerator and the denominator, we get

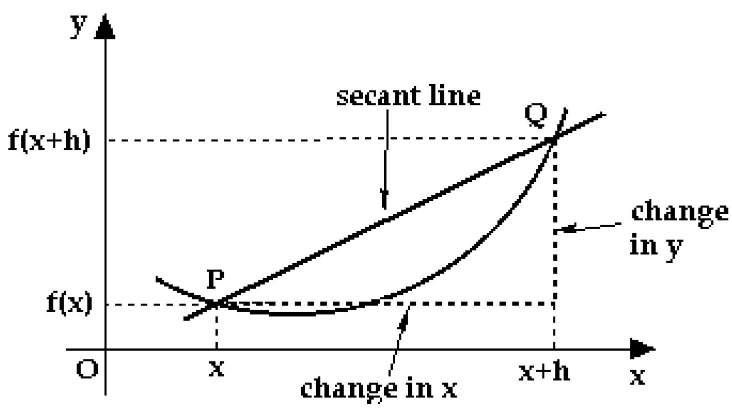
$$\lim_{x\to 3}\frac{x^2+x-12}{x^2-9} = \lim_{x\to 3}\frac{(x-3)(x+4)}{(x-3)(x+3)} = \lim_{x\to 3}\frac{x+4}{x+3} = \frac{7}{6.}$$ ∎

Exercise 2. Compute the following limits.

(a) $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$ (b) $\lim_{x\to 0}\frac{x^2+4x-21}{x^2-49}$ (c) $\lim_{x\to -7}\frac{x^2+4x-21}{x^2-49}$.

♣

Tangent and Derivative. Consider the graph of a function $y = f(x)$, and pick a point $P(x, f(x))$ on it.



Choose a nearby value of the variable, call it $x+h$ ($x+h$ is h units away from x; "nearby" means that h is small). The corresponding value of the function is $f(x+h)$. Now, we have two points on the curve: $P(x, f(x))$ and $Q(x+h, f(x+h))$. The slope of the line joining these two points (this line is called a secant line) is given by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}.$$

Now imagine that h gets closer and closer to zero, so that $x+h$ approaches x. In other words, imagine that the point Q slides along the curve towards the point P. The limiting position of the secant lines (joining P and Q) as Q approaches P is called the tangent line to the curve $y = f(x)$ at $(x, f(x))$. Its slope is given by

$$m = \text{slope of the tangent} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided that the limit in question exists.

This number is also called the derivative of $f(x)$ at x, and is denoted by $f'(x)$. Thus,

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

By computing $f'(x)$ at all x where that is possible, we obtain the derivative function. Thus, the derivative of a function is another function. The value of the derivative at a particular point is equal to the slope of the tangent line at that point.

Example 3. Find the equation of the line tangent to the graph of $y = x^2$ at the point $(1,1)$.

Solution

To get a line, we need a point (we have it) and a slope. The slope of a tangent is given by $m = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, where $f(x) = x^2$ and $x = 1$. Thus,

$$m = \lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2.$$

It follows that the equation of the desired tangent line is $y - 1 = 2(x-1)$; i.e., $y = 2x - 1$. ∎

Example 4. Find the equation of the line tangent to the graph of $y = 1/x$ at the point where $x = 2$.

Solution

The point of tangency has coordinates $x = 2$ and $y = 1/x = 1/2$. To get the slope, we substitute $f(x) = 1/x$ and $x = 2$ into the definition:

$$m = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} = \lim_{h\to 0} \frac{-h}{2(2+h)}\frac{1}{h} = \lim_{h\to 0} \frac{-1}{2(2+h)} = \frac{-1}{4}.$$

Thus, the equation of the tangent is $y - \frac{1}{2} = -\frac{1}{4}(x-2)$, or $y = -\frac{1}{4}x + 1$. ∎

Exercise 3. Find the equation of tangent to the graph of $y = 1/x^2$ at the point where $x = 1$.

♣

Example 5. Using the definition, compute the derivative of $f(x) = \sqrt{x}$.

Solution

The derivative of $f(x) = \sqrt{x}$ is given by

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

∎

Exercise 4. Using the definition, compute the derivatives of

(a) $y = \sqrt{x+1}$ (b) $y = 1/\sqrt{x}$

♣

Let $y = f(x)$. Besides $f'(x)$, commonly used notation for derivatives includes y', $\frac{dy}{dx}$ and $\frac{df}{dx}$.

Using the definition of the derivative, we could derive the following differentiation formulas (c and n denote constants).

Derivative of constant functions and of powers of x

$$\text{if } f(x) = c, \text{ then } f'(x) = 0; \qquad \text{in short, } c' = 0$$

$$\text{if } f(x) = x^n, \text{ then } f'(x) = nx^{n-1}; \qquad \text{in short, } (x^n)' = nx^{n-1}$$

Let $f(x)$ and $g(x)$ be two functions and denote by $f'(x)$ and $g'(x)$ their derivatives.

$$(f(x) \pm g(x))' = f'(x) \pm g'(x) \qquad \text{(sum and difference rules)}$$

$$(cf(x))' = cf'(x) \qquad \text{(constant times function rule)}$$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \qquad \text{(product rule)}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \qquad \text{(quotient rule)}$$

Example 6. Compute the derivatives of the following functions.

(a) $f(x) = 6x^2 + 7x + 4$ (b) $f(x) = x^3 + \dfrac{1}{x^3}$ (c) $f(x) = \sqrt{5}\,x + \sqrt{5x}$

(d) $y = x^{\sqrt{5}}$ (e) $y = \dfrac{\sqrt{3}}{x^{10}}$ (f) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$.

Solution

(a) Using the sum rule and the constant times function rule, we get

$$f'(x) = 6 \cdot 2x + 7 \cdot 1 + 0 = 12x + 7.$$

(b) Write $f(x) = x^3 + x^{-3}$; thus, $f'(x) = 3x^2 - 3x^{-4}$.

(c) Rewrite $f(x)$ as $f(x) = \sqrt{5}\,x + \sqrt{5}\sqrt{x} = \sqrt{5}\,x + \sqrt{5}x^{1/2}$. Thus,

$$f'(x) = \sqrt{5} \cdot 1 + \sqrt{5}\tfrac{1}{2}\,x^{-1/2} = \sqrt{5} + \frac{\sqrt{5}}{2\sqrt{x}}.$$

(d) Since $\sqrt{5}$ is a constant, we apply the x^n rule with $n = \sqrt{5}$; thus, $y' = \sqrt{5}x^{\sqrt{5}-1}$.

(e) Write $y = \sqrt{3}x^{-10}$; it follows that $y' = \sqrt{3}(-10)x^{-11}$.

(f) $f(x) = x^{3/2} + x^{2/3}$; thus, $f'(x) = \frac{3}{2}x^{1/2} + \frac{2}{3}x^{-1/3}$. ∎

Exercise 5. Compute the derivatives of the following functions.

(a) $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$ (b) $f(x) = \dfrac{6}{\sqrt[3]{x^4}}$ (c) $y = \dfrac{x^2+1}{\sqrt{x}}$

(d) $y = x^2 + \pi^2 + x^\pi$.

♣

Example 7. Find the equation of the line tangent to the curve $y = \dfrac{1}{x^4 + x^2 + x + 1}$ at the point where $x = 0$.

Solution

Substituting $x = 0$ into the formula for y, we get $y = 1$; so, the point of tangency is $(0, 1)$. The slope of the tangent line is given by $m = y'(0)$. Using the quotient rule,

$$y' = \frac{0(x^4 + x^2 + x + 1) - 1(4x^3 + 2x + 1)}{(x^4 + x^2 + x + 1)^2} = -\frac{4x^3 + 2x + 1}{(x^4 + x^2 + x + 1)^2}.$$

It follows that $y'(0) = -1$; the equation of the tangent is $y - 1 = -1(x - 0)$, i.e., $y = -x + 1$. ∎

Exercise 6. Find the equation of the line tangent to the curve $y = \dfrac{x+3}{x^2 + x + 3}$ at the point where $x = 0$.

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The derivatives of exponential, logarithmic and trigonometric functions are given below.

$$(e^x)' = e^x \qquad (a^x)' = a^x \ln a \qquad (\ln x)' = \frac{1}{x} \qquad (\log_a x)' = \frac{1}{x \ln a}$$
$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\tan x)' = \sec^2 x$$
$$(\csc x) = -\csc x \cot x \qquad (\sec x)' = \sec x \tan x \qquad (\cot x)' = -\csc^2 x$$

Example 8.

(a) Compute y' if $y = x^2 \sec x$.

(b) Compute y' if $y = \dfrac{\cos x - 1}{\sin x}$.

(c) Derive the formula for $(\tan x)'$.

Solution

(a) Using the product rule, we get $y' = 2x \sec x + x^2 \sec x \tan x$.

(b) By the quotient rule,

$$y' = \frac{-\sin x\ \sin x - (\cos x - 1)\cos x}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x + \cos x}{\sin^2 x} = \frac{\cos x - 1}{\sin^2 x}.$$

(c) Applying the quotient rule,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\ \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{1}{\cos^2 x} = \sec^2 x.$$ ∎

Exercise 7.

(a) Compute y' if $y = \dfrac{\tan x}{\sec x}$.

(b) Compute y' if $y = 3 \sin x \tan x + 3$.

(c) Derive the formula $(\sec x)' = \sec x \tan x$.

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Chain Rule. The derivative of the composition of two functions is computed as a product of their derivatives. To be precise: Let $f(x)$ and $g(x)$ be two functions and let $(g \circ f)(x) = g(f(x))$ be their composition.

Chain rule (version I)

If $h(x) = g(f(x))$, then $h'(x) = g'(f(x))\, f'(x)$

Note: in computing the composition $g(f(x))$, we apply f to x first, and then we apply g to $f(x)$. According to the chain rule, when doing the derivative, we proceed in the opposite order: g is done first, and then f. One more thing: the $f(x)$ part in $g'(f(x))$ states that, while doing the derivative of g, we do not change $f(x)$ (the $f(x)$ term is usually called the "inside").

Sometimes we think of the composition in the following way: $y = g(u)$ and $u = f(x)$ (i.e., y depends on u, and u depends on x). In that case, y depends on x and its derivative is

Chain rule (version II)

If $y = g(u)$ and $u = f(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

Example 9. Compute the derivatives of the following functions.

(a) $y = (x^2+1)^{14}$ (b) $y = \sqrt{\sin x + 1}$ (c) $y = \dfrac{1}{e^x+2}$

(d) $y = \sin(x^2+1)$ (e) $y = \cos(\sec x)$ (f) $y = (\sin x)^2 + \sin(x^2)$.

Solution

(a) We start by computing the derivative of the power of 14 :

$$y' = 14(x^2+1)^{13}(x^2+1)' = 14(x^2+1)^{13}\, 2x = 28x(x^2+1)^{13}.$$

(b) Writing $y = (\sin x + 1)^{1/2}$, we get

$$y' = \tfrac{1}{2}(\sin x+1)^{-1/2}(\sin x+1)' = \tfrac{1}{2}(\sin x+1)^{-1/2}\ \cos x = \tfrac{1}{2}\cos x(\sin x+1)^{-1/2}.$$

(c) $y = (e^x+2)^{-1}$; thus,

$$y' = (-1)(e^x+2)^{-2}(e^x+2)' = -e^x(e^x+2)^{-2} = -\frac{e^x}{(e^x+2)^2}.$$

(d) We start by computing the derivative of sin:

$$y' = \cos(x^2+1)\ (x^2+1)' = 2x\cos(x^2+1).$$

(e) $y' = -\sin(\sec x)\ (\sec x)' = -\sin(\sec x)\sec x\tan x$.

(f) We have to be careful about the order:

$$y' = 2(\sin x)^1(\sin x)' + \cos(x^2)\ (x^2)' = 2\sin x\cos x + 2x\cos(x^2) = \sin 2x + 2x\cos(x^2).$$

In simplifying, we used the formula $2\sin x\cos x = \sin 2x$. ∎

Exercise 8. Compute the derivatives of the following functions.
(a) $y = \dfrac{1}{x^3 + x - 2}$ (b) $y = (\sqrt{x} + 1)^2$ (c) $\tan(x^2) + \tan(x^2 + 1)$
(d) $y = \sec(e^x)$ (e) $y = \cos^2(x^2)$ (f) $y = x^2 \sin(1/x)$.

♣

Example 10. Compute the derivatives of the following functions.
(a) $y = e^{4x+2}$ (b) $y = \ln(\sin x + 2)$ (c) $y = 2^{3x}$
(d) $y = \ln(x^2 + 3x + e^x)$ (e) $y = e^{\sin x} + \sin(e^x)$ (f) $y = \sec\sqrt{x^2 + x}$.

Solution

(a) We start by doing the derivative of the exponential function:

$$y' = e^{4x+2}(4x + 2)' = 4e^{4x+2}.$$

(b) The derivative of $\ln x$ is $1/x$; thus,

$$y' = \frac{1}{\sin x + 2}(\sin x + 2)' = \frac{\cos x}{\sin x + 2}.$$

(c) The derivative of 2^x is $2^x \ln 2$; it follows that

$$y' = 2^{3x} \ln 2\ (3x)' = 3 \cdot 2^{3x} \ln 2.$$

(d) As in (b),

$$y' = \frac{1}{x^2 + 3x + e^x}(x^2 + 3x + e^x)' = \frac{2x + 3 + e^x}{x^2 + 3x + e^x}.$$

(e) $y' = e^{\sin x}(\sin x)' + \cos(e^x)(e^x)' = \cos x e^{\sin x} + e^x \cos(e^x)$.

(f) Write $y = \sec(x^2 + x)^{1/2}$ and recall that $(\sec x)' = \sec x \tan x$. Then

$$\begin{aligned} y' &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2}((x^2 + x)^{1/2})' \\ &= \sec(x^2 + x)^{1/2} \tan(x^2 + x)^{1/2} \tfrac{1}{2}(x^2 + x)^{-1/2}(2x + 1). \end{aligned}$$

∎

Exercise 9. Compute the derivatives of the following functions.
(a) $y = \ln\sqrt{x} + \sqrt{\ln x}$ (b) $y = \dfrac{1 - \ln x}{1 + \ln x}$ (c) $f(x) = 2^{e^x}$
(d) $y = e^x + x^e + e^e$ (e) $y = \sec^2(\ln x)$

♣

Example 11. Find dy/dx for the following functions.
(a) $y = 4u^2 - 3u + 2$, $u = e^x + 2e^{2x}$ (b) $y = \ln u$, $u = \sin x + \cos x$

Solution

(a) By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = (8u - 3)(e^x + 4e^{2x}) = 8((e^x + 2e^{2x}) - 3)(e^x + 4e^{2x}).$$

(b) As in (a),

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{u}(\cos x - \sin x) = \frac{\cos x - \sin x}{\sin x + \cos x}. \qquad \blacksquare$$

Exercise 10. Compute the derivatives of the following functions.

(a) $y = \sqrt{u^2 + 2}$, $u = \cot x$ (b) $y = \log_2 u$, $u = e^x + 4$

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9. SOLUTIONS TO EXERCISES

Basic Algebra

1. (a) π, $-1/2$, 0.33 (b) 1.

2. (a) 81 (b) 1/64 (c) not defined (d) 8.

3. (a) $3x^3 - 7x^2 + 7x - 4$ (b) $4x^2y^2 - 4xy^3 + y^4$ (c) $x^4 - 0.4x^2 + 0.04$
(d) $x^7 - x$ (e) $8x^3 - 12x^2 + 6x - 1$.

4. (a) $(x-2)(x+5)$ (b) $(2x-1)(x+4)$ (c) $2(x-3)(x+4)$
(d) $(x-1)(x^2+x+1)(x+1)(x^2-x+1)$.

5. (a) $(2x^2+y)(y-2)$ (b) $x^2(x-1)(x+1)$ (c) $3(3a-5b)(3a+5b)$
(d) $x(x-1)(x+1)(x^2+1)$.

6. (a) $-5/4$ (b) $1/\sqrt{2}$ (c) $5+2\sqrt{6}$ (d) -1 (e) $27/8$.

7. (a) $\dfrac{x^2+3x+9}{x+3}$ (b) $\dfrac{x^2(x-1)}{2x+1}$.

8. $\dfrac{x}{x-3}$.

9. $\dfrac{3(\sqrt{x}+\sqrt{2})}{x-2}$.

10. (a) $\dfrac{x+8-6\sqrt{x}}{4-x}$ (b) $\dfrac{x}{x-2}$ (c) $\dfrac{12x^2-x-3}{x^3-x}$.

Equations and Inequalities

1. $x=-1$.

2. $x=-47$.

3. (a) $x=5/3$ (b) $x=-9/5$.

4. (a) $x=9$ (b) $x=-1$ (c) $x=5/2$ (d) $x=-3$, $x=2$.

5. (a) $(2,\infty)$ (b) $[2,3]$ (c) $[5/3,2]$.

6. (a) $(-\infty,-7)$ or $(3,\infty)$ (b) $(-\infty,5)$ or $[8,\infty)$ (c) $(-\infty,-1]$ or $[0,1]$.

7. $x=-2$ and $x=5$.

8. (a) $1/2 < x < 5/2$ (b) $(-\infty,-8/3]$ or $[0,\infty)$.

9. (a) $x=3$, $y=-7$ (b) $x=1$, $y=5$ and $x=-2$, $y=2$.

Elements of Analytic Geometry

1. $\sqrt{85}$.

2. $y=-7x+10$.

3. Line, goes through the origin, of slope $-1/2$.

4. (a) No. Their slopes are $-1/2$ and 3 (b) $y=x/2-5/2$ (c) Yes. Their slopes are $3/2$ and $-2/3$.

5. (a) Ellipse with semi-axes 3 and $\sqrt{3}$; x-intercepts are ± 3, y-intercepts are $\pm\sqrt{3}$ (b) circle, centered at $(1,3)$, radius $\sqrt{10}$.

6. (a) x-intercepts at $x=-3\pm\sqrt{6}$; y-intercept at $y=3$; vertex at $(-3,-6)$; points upward

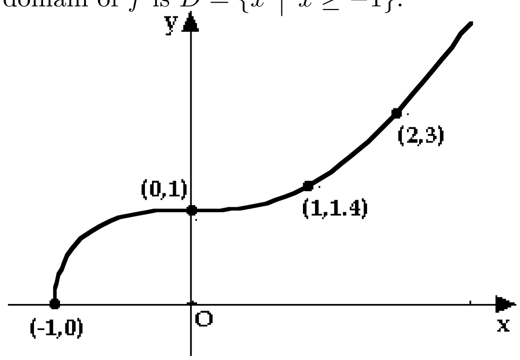
(b) x-intercept at $x=2$; y-intercept at $y=4$; vertex at $(2,0)$; points upward (c) no x-intercepts; y-intercept at $y=-1$; vertex at $(0,-1)$; points downward.

Functions

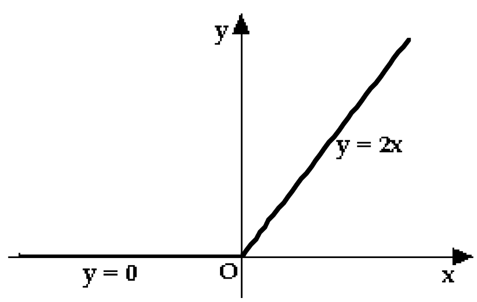
1. (a) $D=\mathbb{R}$, $R=\mathbb{R}$ (b) $D=\{x \mid x\neq 0\}$, $R=\{y \mid y>0\}$ (c) $D=\mathbb{R}$, $R=\{y \mid y\geq 0\}$.

2. (a) $D=\{x \mid x>2\}$ (b) $D=\{x \mid x\neq 0,1,-1\}$ (c) $D=\{x \mid x<0 \text{ or } x\geq 1\}=(-\infty,0)$ or $[1,\infty)$.

3. Note: the domain of f is $D=\{x \mid x\geq -1\}$.



4. (a) Horizontal line, crosses the y-axis at $(0,-4)$ (b) Mirror image of $y=x^2$ with respect to the x-axis (c) $f(x)=0$, if $x<0$ and $f(x)=2x$ if $x\geq 0$; see the graph below.



5. (a) $3x/2+4$ (b) $9x+16$ (c) $(3x+4)/2$.

6. (a) $(x^2+x+1)^5$ (b) x^{25} (c) $x^{10}+x^5+1$.

7. (a) Move the parabola $y=x^2$ 2 units to the left and 3 units up (b) Move the graph of $y=\sqrt{x}$ 4 units to the right and 2 units down (c) Move the graph of $y=1/x^2$ 1 unit to the right and then 2 units up.

8. (a) Take the graph of $y = x^3$, compress it by a factor of 2 and then mirror it with respect to the x-axis (b) Start with the graph of $y = \sqrt{x}$, move it 3 units to the left, then stretch by the factor of 3, and then move one unit up.

Trigonometry

1. (a) $5\pi/4$ radians (b) 210 degrees.

2. (a) $\sin\frac{5\pi}{6} = 1/2$, $\cos\frac{5\pi}{6} = -\sqrt{3}/2$ and $\tan\frac{5\pi}{6} = -1/\sqrt{3}$ (b) $\sin\left(-\frac{\pi}{6}\right) = -1/2$, $\cos\left(-\frac{\pi}{6}\right) = \sqrt{3}/2$ and $\tan\left(-\frac{\pi}{6}\right) = -1/\sqrt{3}$.

3. $\sin\theta = 1/3$, $\cos\theta = \sqrt{8}/3$ and $\tan\theta = 1/\sqrt{8}$.

4. $\sin(-3\pi/2) = -1$, $\cos(-3\pi/2) = 0$; $\tan(-3\pi/2)$ and $\sec(-3\pi/2)$ are not defined.

5. $2\pi/a$.

6. To obtain $\cos 3x$, compress the graph of $\cos x$ along the x-axis by a factor if 3; to obtain $\cos 0.5x$, stretch the graph of $\cos x$ along the x-axis by a factor of 2.

7. π/a.

8. (a) Use the addition formula for $\sin x$ (b) Write $\tan x = \sin x/\cos x$ and compute the common denominator (c) Combine the first and the third terms and use (b).

9. $\sin 3x = 3\sin x - 4\sin^3 x$.

10. Use the addition and the subtraction formulas for $\sin x$ and multiply out the terms that you get. Then combine terms and simplify using the basic trig identity.

11. (a) $x = \pi + 2k\pi$ (b) $x = -\frac{\pi}{4} + k\pi$.

12. (a) $x = \frac{5\pi}{6} + 2k\pi$ and $x = \frac{7\pi}{6} + 2k\pi$ (b) $x = \frac{\pi}{3} + k\pi$ (c) $x = \frac{\pi}{4} + 2k\pi$ and $x = \frac{3\pi}{4} + 2k\pi$.

13. $x = k\pi$, $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$.

14. $x = \frac{\pi}{6} + k\pi$ and $x = \frac{5\pi}{6} + k\pi$.

15. (a) $\sin\alpha = \cos\beta = 21/29$, $\cos\alpha = \sin\beta = 20/29$ (b) $a = 12$, $b = 5$, $\sin\beta = 5/13$, $\tan\beta = 5/12$ (c) $\sin\beta = 0.8 = 4/5$, $\cos\beta = 0.6 = 3/5$, $\tan\beta = 4/3$, $\csc\beta = 5/4$, $\sec\beta = 5/3$, and $\cot\beta = 3/4$.

16. (a) third (b) fourth (c) third (d) second.

17. (a) positive (b) positive (c) negative (d) positive.

18. (a) $\sin 1$ (b) $\cos 2^0$ (c) $\tan 1$.

19. (a) $\tan^2 x$ (b) $1/\cos x$ (c) $2/\sin x$ (d) 2.

20. (a) $x = -\frac{\pi}{6} + k\pi$ (b) $x = \frac{3\pi}{4} + k\pi$ (c) $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{11\pi}{6} + 2k\pi$.

21. (a) $x = \frac{\pi}{4} + 2k\pi$ and $x = \frac{3\pi}{4} + 2k\pi$ (b) no solution (c) $(2k+1)\pi$.

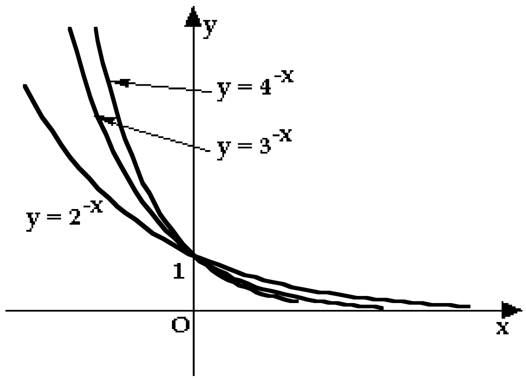
22. (a) graph of $\cos x$, moved $\pi/4$ units to the left (b) graph of $\sin x$, moved π units to the right (c) graph of $\tan x$, moved 1 unit to the left.

23. (a) cross-multiply (b) replace $\tan x$ by $\sin x/\cos x$ and $\cot x$ by $\cos x/\sin x$, and simplify fractions.

Exponential and Logarithmic Functions

1. (a) Move the graph of $y = 2^x$ 4 units up (b) Move the graph of $y = 2^x$ 4 units to the right (c) Reflect the graph of $y = 2^x$ with respect to the x-axis (d) $-2^{-x} = -(1/2)^x$; reflect the graph of $y = (1/2)^x$ with respect to the x-axis.

2.



3. (a) 5^{-x+4} (b) 3^{6x-14} (c) 2^{-x+20}.

4. (a) $x = \pm\sqrt{3}$ (b) $x = 1$ (c) $x = 0$ and $x = 2$.

5. (a) $x = 16$ (b) $x = 243$ (c) $x = 2$ only ($x = 0$ and $x = -2$ are not solutions since they are not in the domain of $\log_2 x$) (d) $x = 2$.

6. (a) 20 (b) $\ln(48e^2)$ (c) $x = (\ln 4 + 2)/3$ (d) $x = -2$ and $x = 1$.

7. (a) 10^{17} (b) 3^8 (c) 3^{12} (d) 6.

8. (a) 5 (b) 10^4 (c) -2.

9. (a) $\sqrt{8}$ (b) 5 (c) 5.

10. (a) -2 (b) 2 (c) 1/23.

11. (a) -2 (b) $-1/2$ (c) -2.

12. (a) $-8/3$ (b) $\ln 2$.

Calculus: Limits and Derivatives

1. (a) $-7/2$ (b) 0.

2. (a) 3/2 (b) 3/7 (c) 5/7.

3. $y = -2x + 3$.

4. (a) $y' = 1/(2\sqrt{x+1})$ (b) $y' = -1/(2x^{3/2})$.

5. (a) $f'(x) = 1/(2\sqrt{x}) + 1/(2x^{3/2})$ (b) $f'(x) = -8x^{-7/3}$ (c) $y' = (3x^2 - 1)/(2x^{3/2})$ (d) $y' = 2x + \pi x^{\pi-1}$.

6. $y = 1$.

7. (a) $y' = \cos x$ (b) $y' = 3\sin x + 3\sin x \sec^2 x$ (c) Apply the quotient rule to $\sec x = 1/\cos x$.

8. (a) $y' = -(3x^2+1)/(x^3+x-2)^2$ (b) $y' = (\sqrt{x}+1)/\sqrt{x}$ (c) $y' = 2x\sec^2(x^2) + 2x\sec^2(x^2+1)$ (d) $y' = e^x \sec(e^x)\tan(e^x)$ (e) $y' = -4x\sin(x^2)\cos(x^2)$ (f) $y' = 2x\sin(1/x) - \cos(1/x)$.

9. (a) $y' = \dfrac{1}{2x} + \dfrac{1}{2x\sqrt{\ln x}}$ (b) $y' = \dfrac{-2}{x(1+\ln x)^2}$ (c) $f'(x) = 2^{e^x} e^x \ln 2$

(d) $y' = e^x + ex^{e-1}$ (e) $2\sec^2(\ln x)\tan(\ln x)/x$.

10. (a) $-\dfrac{\cot x \csc^2 x}{\sqrt{\cot^2 x + 2}}$ (b) $\dfrac{e^x}{(e^x+4)\ln 2}$.