Affine Markov chain model of multifirm credit migration

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Abstract

This paper introduces and explores variations on a natural extension of the intensity based doubly stochastic framework for credit default. The essential addition proposed here is to introduce a Markov chain for the "credit rating" of each firm, which are independent conditioned on a stochastic time change, or equivalently a stochastic intensity. The stochastic time change is then combined with other stochastic factors, here the interest rate and the recovery rate, into a multidimensional affine process. The resulting general framework has the computational effectiveness of the intensity based models. This paper aims to illustrate the potential of the general framework by exploring a minimal implementation which is still capable of combining stochastic interest rates, stochastic recovery rates and the multifirm default process. Already within this minimal version we see very good reproduction of essential features of credit spread curves, default correlations and multifirm default distributions. Increased flexibility can also be achieved with a number of mathematical extensions of the basic framework. In a companion paper, [Hurd and Kuznetsov (2006)] we show how the same framework extends to large scale basket credit derivatives, particularly CDOs (collateralized debt obligations).

Key words: Credit risk, stochastic intensity, credit migration, stochastic recovery, default correlation, credit spread

1 Introduction

The intensity based approach to credit risk initiated by [Jarrow and Turnbull (1995)] and developed by many subsequent authors takes as its main ingredients the default time t^* of the firm, the interest rate r_t, the survival indicator $Y_t = 1\{t^* > t\}$, and the default intensity λ_t. In terms of these processes the instantaneous probability of default is

$$P(t < t^* < t + dt | t^* > t) := E[1\{Y_{t+dt} = 0\}|Y_t = 1] = \lambda_t dt. \tag{1}$$

Then one can derive the probability of survival to time t:

$$P(t^* > t) = E\left[e^{-\int_0^t \lambda_s ds}\right],$$

and the price of a zero-recovery, zero-coupon defaultable bond:

$$E\left[e^{-\int_0^T r_s ds} I\{t^* > T\}\right] = E\left[e^{-\int_0^T (r_s + \lambda_s) ds}\right] \tag{2}$$

In many applications one needs to have a finer structure for the creditworthiness of a firm than simply default or no default, and one is then led to the credit migration approach of [Jarrow et al. (1997)], [Lando (1998)], [Arvanitis et al. (1999)]. Here the idea is to generalize the $0, 1$ process Y_t to a finite state Markov chain on $\{0, 1, 2, \ldots, K\}$ where the states $\{0, 1, 2, \ldots, K\}$ represent credit rating or distance to default of the firm. 0 represents the default state, and the closer the rating of the company is to 0 the higher is its probability of default. For example, if $K = 7$ one can identify states with Moody's or Standard and Poor's ratings classes:

$$\{0, 1, 2, \ldots, 7\} \leftrightarrow \{\text{"default", CCC, B, BB, BBB, A, AA, AAA}\}.$$

One can also take K large and think of $k \leq K$ as analogous to "distance to default": in this case rating class models take on the characteristics of structural models such as the Black-Cox model [Black and Cox (1976)]. In general credit migration models, one specifies a stochastic intensity matrix $L_t = (\lambda_{kl,t})_{k,l \in \{0,1,2,\ldots,K\}}$ such that

$$P(Y_{t+dt} = l | Y_t = k) := E[1\{Y_{t+dt} = l\}|Y_t = k] = \lambda_{kl,t} dt, \quad \text{for } k \neq l.$$

It is clear that to be useful, such a dramatic generalization of the intensity framework must be constrained in some way. For example, to achieve tractable formulas for credit derivative prices, [Lando (1998)] and [Arvanitis et al. (1999)] assume that the matrix L_t is diagonalizable in a fixed time independent basis, and then specify the joint law of the eigenvalue processes.

The primary purpose of the present paper is to extend the credit migration framework developed in the above-mentioned works, with its flexibility and computational efficiency, from single firms to the multifirm setting. Our modelling of the credit states $Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(M)}$ of multiple firms, as expressed in Assumptions 1-5 of the paper, is guided by certain considerations:

1. For multiple firms, a natural simplification is to suppose that stochastic migration intensities, interest rate, and stochastic recovery are market factors that apply identically to all firms. Thus, when conditioned on market information, the credit migration processes $Y_t^{(1)}, Y_t^{(2)}, \dots, Y_t^{(M)}$ can be taken to be identically distributed independent Markov chains. This reduces the complexity of the framework, and also has the consequence that while market factors can influence defaults, defaults cannot influence the market as they do in contagion models such as [Davis and Lo (2001)].

2. The conditional independence of $Y_t^{(1)}, Y_t^{(2)}, \dots, Y_t^{(M)}$ implies that default correlation arises purely from correlated migration intensities. It is well known from intensity based approaches that the possible default correlations which arise from correlating continuous intensity processes are generally too weak to be realistic. For this reason, we allow stochastic intensities to be jump diffusions, and thereby obtain the possibility of strong default correlations.

3. The special case of a single market wide migration intensity can be usefully interpreted as a stochastic time change.

4. Mathematical consistency suggests that intensities, interest rates and recovery rates must be positive processes. Modelling experience then suggests that the affine class of positive jump-diffusion processes [Duffie et al. (2003)] is both rich and computationally efficient.

5. It is important that the multifirm structure should allow efficient computation of high dimensional basket credit portfolio derivatives such as CDOs by enabling a reduction to explicit low dimensional integrals. The computation of CDOs in the AMC framework is explored in a companion paper [Hurd and Kuznetsov (2006)].

It is worth explaining in more detail the "credit migration with stochastic time change" picture that holds in the simplest possible version of our framework, where there is a single time change factor for the entire credit market. Viewed in stochastic time, all firms undergo their credit migration independently with identical transition intensities, eventually jumping to the absorbing default state. Viewed in real time however, the stochastic clock variously speeds up or slows down each firm's migration process, thereby raising or lowering its true default intensity. The result is a complex structure of credit spreads and positive default correlations. The concept of a stochastic market time has been used by numerous authors to explain asset price dynamics: for example, stochastic volatility models can be viewed this way.

While the above considerations form the conceptual basis for the AMC multifirm framework, they are too restrictive to apply perfectly, or even adequately, to real credit markets. For this reason, we investigate in the later sections of the paper a number of different extensions that lead to greater flexibility while retaining the basic picture and its structural and computational advantages. Thus we demonstrate how to include multidimensional time changes that drive different modes of credit migration, enabling greater flexibility in modelling credit spread dynamics. These extra time changes can be introduced one at a time, providing a more systematic method than the general method proposed in [Lando (1998)]. We also show how to correctly fit the credit spread curves of individual firms, while preserving the market average credit spreads by rating class. We provide a third extension that gives a flexible default correlation structure while fixing all marginal default probabilities, somewhat analogous to the copula models of default.

In addition to the basic development of the multifirm framework, plus the above extensions, our paper provides detailed derivations of pricing formulas for bonds and credit default swaps, under a number of different stochastic recovery mechanisms. We show how to compute joint default distributions, enabling the study of the default correlation function. We also provide a brief discussion of one consistent method for a full-scale calibration of the model.

The organization of this paper is as follows. The main ingredients of the framework are the processes which govern credit migration, interest rate, time change and recovery. These are defined in Section 2 for both the reference probability measure P and a risk neutral measure Q. Assumptions 1-3 are the main structural hypotheses underlying the entire paper. Section 3 focusses on the computation of pure rating transition and default probabilities. Section 4 gives general formulas for defaultable zero coupon bonds and credit default swaps. Assumption 4 is introduced simply to have closed expressions for several key results. Section 5 is devoted to stating and proving the bond price formula under the recovery of market value mechanism. In Section 6, Assumption 5 extends the AMC framework to the multifirm setting, and the remainder of the section gives derivations of formulas for the joint default distribution and default correlation in a basic two firm context. We provide a number of simple extensions of the basic framework in Section 7. These variations should be useful for modelling some of the finer detail observed in real credit markets. Section 8 provides a numerical illustration of two simple specifications of the modelling framework where the market factors are modelled as affine jump diffusions. The two models are chosen to have identical marginal default probabilities, but different default correlations. In Section 9, we outline a coherent calibration scheme which in principle provides filtered estimates of unobservable market factors and maximum likelihood estimates of modelling parameters, all based on a panel of observed bond pricing data. Finally, the appendix gives detailed computations which underlie some of the basic results.

2 The one–firm setup

In this section we will consider two probabilistic settings. We begin with our modelling ingredients in the physical (or historical or natural) probability measure P. Then we develop a richer dynamical framework which we interpret as the risk-neutral framework, that is a probability measure Q under which all discounted traded asset price processes are martingales. The P measure is interpreted as a reference measure, chosen compatible with long term historical credit migration rates, under which the dynamics is as simple as possible. While it can be used as the physical measure by a risk manager, typically, a sophisticated risk manager who feels there are significant market drivers of "true" migration probabilities will choose yet a different P' which has dynamics as rich as Q dynamics.

Assumption 1. The P probability setting supports stochastic processes and random variables with the following properties:

(a) A continuous time, time-homogeneous Markov chain Y_t on the space of rating classes $\{0, 1, 2, \ldots, K\}$. Since a firm, once defaulted, stays in default, we take 0 to be an absorbing state. The Markov generator of the chain, $\mathcal{L}_Y$, is then a $K+1 \times K+1$ matrix with rows summing to zero and the first row consisting of zeros:

$$\mathcal{L}_Y = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ l_{10} & -l_{11} & l_{12} & \ldots & l_{1,K} \\ l_{20} & l_{21} & -l_{22} & \ldots & l_{2K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{K0} & l_{K1} & l_{K2} & \ldots & -l_{KK} \end{pmatrix} \tag{3}$$

The off-diagonal entries of $\mathcal{L}_Y$ record the instantaneous intensity for all possible migrations, since over any infinitesimal time interval

$$P(Y_{t+dt} = j | Y_t = k) := E^P[1\{Y_{t+dt} = j\} | Y_t = k] = l_{kj} dt, \quad \text{for } k \neq j. \tag{4}$$

We assume there is always a non-zero intensity for the firm to migrate into any other rating class, which means

$$l_{ij} > 0, \text{ if } i > 0 \text{ and } j \neq i. \tag{5}$$

Finally, we denote the initial rating class of the firm by $Y_0 = y$.

(b) The default time t^*, the mathematical idealization of the moment the firm enters bankruptcy, is defined as the first time the process Y_t hits the absorbing state 0.

(c) A vector-valued process $\mathbf{X}_t = [r_t, l_t, \lambda_t]'$ where r_t is the spot interest rate process, the recovery rate process is $R_t = e^{-l_t}$, and λ_t is the stochastic migration intensity process. We will shortly specify more properties of $\mathbf{X}_t$: we think of these as *market* or *macroeconomic* factors which will be used to drive the default dynamics of all firms under the risk neutral measure Q. (Note that in this article vectors will be written in bold font as $\mathbf{x} = \{x^1, x^2, ..., x^n\}$.)

Under the above assumptions, one can prove that the matrix $\mathcal{L}_Y$ is diagonalizable $\mathcal{L}_Y = VDV^{-1}$ and that the diagonal eigenvalue matrix $D = -\text{diag}\{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_K\}$ has $\alpha_0 = 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_K$ all with positive real parts. The eigenvector matrix

$V = (v_{ij})_{i,j=0\dots K}$ is a matrix whose columns are the corresponding eigenvectors of $\mathcal{L}_Y$. The elements of V^{-1} will be denoted as $V^{-1} = (\tilde{v}_{ij})_{i,j=0\dots K}$. The *transition semigroup* $\mathcal{P}_Y(t)$ for the process Y is a $K+1 \times K+1$ matrix valued function such that for any $t \geq 0$

$$P(Y_t = j | Y_0 = y) := E^P_{0,y}[I\{Y_t = j\}] = (\mathcal{P}_Y)_{yj}(t), \quad y, j \in \{0, \dots, K\} \tag{6}$$

It satisfies the Kolmogorov equation

$$\frac{d}{dt}\mathcal{P}_Y(t) = \mathcal{L}_Y \mathcal{P}_Y(t), \qquad \mathcal{P}_Y(0) = I,$$

and as can be seen from the diagonal form of $\mathcal{L}_Y$, can be computed as

$$(\mathcal{P}_Y)_{yj}(t) = (e^{t\mathcal{L}_Y})_{yj} = \sum_{i=0}^{K} v_{yi}\tilde{v}_{ij}e^{-\alpha_i t}. \tag{7}$$

The next assumption completes the specification of the model under the historical probability measure P.

Assumption 2. Under P,

(a) Y_t is independent of the process $\mathbf{X}_t$;

(b) The process $\mathbf{X}_t$ is a given by a linear transformation $\mathbf{X}_t = M\mathbf{Z}_t$ of an underlying N-dimensional "solvable" time-homogeneous Markov state process $\mathbf{Z}_t$. Here M is a $3 \times N$ dimensional matrix. By "solvable", we mean that

$$G^P(t, \mathbf{Z}_0; \mathbf{u}, \mathbf{v}) := E^P_{0,\mathbf{Z}_0}[e^{-\int_0^t \mathbf{u}\cdot\mathbf{Z}_s ds - \mathbf{v}\cdot\mathbf{Z}_t}] \tag{8}$$

is given by an explicit expression which is analytic in all its arguments. At the end of this section, we give a list of pertinent examples of such formulas.

We now turn to the model under the risk-neutral measure Q. Without going into technical details, the relationship between P and Q is captured by a version of the Girsanov theorem applicable to measure changes of jump-diffusions (see [Elliott (1982)]). Under this theorem, it is possible to define new measures equivalent to P under which each non-zero off-diagonal entry of the migration intensity matrix becomes a positive stochastic process. As well, the allowable Girsanov transformations can change the drifts and jump intensities of the market process $\mathbf{Z}_t$, but not its diffusive volatilities. By use of this theorem, and by assuming the absence of arbitrage, we find we may assume:

Assumption 3. (a) There exists a risk neutral measure Q, equivalent to P, such that all discounted traded assets are Q martingales.

(b) Under Q, the Markov chain Y_t has stochastic transition intensities $\lambda_t(\mathcal{L}_Y)_{ij}$: thus for infinitesimal time intervals dt

$$Q(Y_{t+dt} = j | Y_t = k) := E^Q[1\{Y_{t+dt} = j\} | Y_t = k] = \lambda_t l_{kj} dt, \quad \text{for } k \neq j. \tag{9}$$

(our use here of the Girsanov theorem imposes the further assumption that $\lambda_t > 0$ almost surely).

(c) The process $\mathbf{Z}_t$ is Markov and solvable under Q, and the function

$$G^Q(t, \mathbf{Z}_0; \mathbf{u}, \mathbf{v}) := E^Q_{0,\mathbf{Z}_0}[e^{-\int_0^t \mathbf{u}\cdot\mathbf{Z}_s ds - \mathbf{v}\cdot\mathbf{Z}_t}] \tag{10}$$

has the same functional form as the function G^P. In other words, the stochastic differential equation (SDE) governing $\mathbf{Z}_t$ under Q has the same form as its SDE under P.

Remark 1. Formula (9) has a very important and natural interpretation, namely the Q dynamics of Y_t is equal in distribution to the P-dynamics of the *time-changed* process $\tilde{Y}_t := Y_{\tau_t}$ defined by

$$\tau_t = \int_0^t \lambda_s ds, \quad t > 0.$$

To see this, note that from (4), for infinitesimal time intervals dt

$$E^P[1\{Y_{\tau_{t+dt}} = j\} | Y_{\tau_t} = k] = E^P[1\{Y_{\tau_t + \lambda_t dt} = j\} | Y_{\tau_t} = k] = \lambda_t l_{kj} dt, \quad \text{for } k \neq j. \tag{11}$$

Examples of SDEs for market factors: Explicit formulas for both G^P, G^Q exist in the following two classes of examples. From now on, we confine our discussion to the processes under the risk neutral measure Q: thus each W_t is a Q Brownian motion, and each jump process J_t is taken to have a Lévy measure ν supported on $(0, \infty)$ under Q.

1. (OU processes) The simplest formulation generalizes the Vasicek interest rate model and assumes that $\mathbf{Z}_t$ is a three-dimensional Ornstein-Uhlenbeck process of the following general form:

$$d\mathbf{Z}_t = A(\bar{\mathbf{Z}} - \mathbf{Z}_t) + d\mathbf{W}_t \tag{12}$$

for A a 3×3 matrix and $\bar{\mathbf{Z}} \in \mathbb{R}^3_+$. Despite its inconsistency with fundamental constraints $r_t > 0, l_t > 0, \lambda_t > 0$, for reasonable choices of $A, M, \bar{\mathbf{Z}}$ the constraints hold with high probability. This model may prove to be rather tractable despite this deficiency, and provide a good fit with observations.

2. (Positive affine processes) We suppose there are N_1 independent factors $Z_t^{(1)}, \dots, Z_t^{(N_1)}$ each of which follows a Cox-Ingersoll-Ross (CIR) mean-reverting diffusion process. The CIR process with parameters (a, b, c, z_0) is the solution of the SDE
$$dZ_t = (a - bZ_t)dt + \sqrt{2cZ_t}dW_t, \quad Z_0 = z_0 \tag{13}$$
As well, we may take N_2 independent factors $Z_t^{(N_1+1)}, \dots, Z_t^{(N_1+N_2)}$ each of which is a positive mean-reverting pure jump process governed by an SDE of the form
$$dZ_t = -aZ_t dt + dJ_t, Z_0 = z_0. \tag{14}$$
Here J_t is a non-decreasing pure jump process with identical independent increments, characterized by its jump measure ν which is supported on $(0, \infty)$. A natural choice is the two parameter exponential family
$$\nu(dy) = cde^{-cy}dy \tag{15}$$
with $c, d > 0$. By taking $r_t, -\log(R_t)$ and λ_t each to be a linear combination of the Zs with nonnegative coefficients one can produce a flexible, solvable family of positive mean-reverting processes. That is, one takes $\mathbf{X}_t = M\mathbf{Z}_t$, for a $3 \times (N_1 + N_2)$ matrix M with non-negative entries, and $\mathbf{Z}_t = [Z_t^{(1)}, \dots, Z_t^{(N_1+N_2)}]$. In the Appendix, we provide explicit formulas for a realization of this structure with $N_1 = N_2 = 1$.

3 Rating transition probabilities

The (risk neutral) transition probabilities of the credit rating migration process Y_t, in particular the distribution of the time of default, are explicit in terms of the G^Q function. In what follows, let the three row vectors of the matrix M be denoted $\mathbf{M}_r, \mathbf{M}_l, \mathbf{M}_\lambda$ so that
$$[r_t, l_t, \lambda_t] = [\mathbf{M}_r \cdot \mathbf{Z}_t, \mathbf{M}_l \cdot \mathbf{Z}_t, \mathbf{M}_\lambda \cdot \mathbf{Z}_t].$$
For the remainder of the paper, we concentrate on risk neutral probabilities: computations of physical probabilities are by and large straightforward, and left to the reader.

Lemma 1. *The rating transition probabilities for the process Y_t are given by*
$$Q_{0,\mathbf{Z}_0,y}(Y_t = j) = \sum_{i=0}^{K} v_{yi}\tilde{v}_{ij} G^Q(t, \mathbf{Z}_0; \alpha_i \mathbf{M}_\lambda, \mathbf{0}). \tag{16}$$

Proof: We use the time change formula (11), the law of iterated expectations, and (7):

$$
\begin{aligned}
Q_{0,\mathbf{Z}_0,y}(Y_t = j) &= E^Q_{0,\mathbf{Z}_0,y}[E^Q[1\{Y_t = j\}|\{\mathbf{Z}_s\}_{s\le t}]] \\
&= E^Q_{0,\mathbf{Z}_0,y}[\exp(\tau_t \mathcal{L}_Y)_{yj}] \\
&= E^Q_{0,\mathbf{Z}_0,y}[\sum_{i=0}^{K} v_{yi}\tilde{v}_{ij}\exp(-\tau_t\alpha_i)] \\
&= \sum_{i=0}^{K} v_{yi}\tilde{v}_{ij} G^Q(t,\mathbf{Z}_0;\alpha_i\mathbf{M}_\lambda,\mathbf{0}).
\end{aligned}
$$

□

Default time is the first time Y_t hits the absorbing state 0 which means $\{t^* \le t\} = \{Y_t = 0\}$. Therefore default probability is computed from (16) with $j = 0$:

Corollary 2. *The probability of default at or before time $t > 0$ is given by*

$$
Q_{0,\mathbf{Z}_0,y}(t^* \le t) = \sum_{i=0}^{K} v_{yi}\tilde{v}_{i0} G^Q(t,\mathbf{Z}_0;\alpha_i\mathbf{M}_\lambda,\mathbf{0}). \tag{17}
$$

4 Pricing defaultable securities

In this section we derive pricing formulas for the basic default risky securities written on a firm with credit rating y at time t, namely zero–coupon bonds and credit default swaps. We first note that $B_t(T)$, the price at time t of a *riskless* zero–coupon bond with maturity T, is given by

$$
B_t(T) = E^Q_{t,\mathbf{Z}_t}\left[e^{-\int_t^T r_s ds}\right] = G^Q(T-t,\mathbf{Z}_t;\mathbf{M}_r,\mathbf{0}). \tag{18}
$$

The price at time t of a *defaultable* zero–coupon bond with maturity T with a specified recovery mechanism $R = 1 - L$ will be denoted $B_t(T, L)$. Then, using this notation, $B_t(T, 1)$ is the price of a defaultable bond with full loss (zero recovery) while $B_t(T, 0) := B_t(T)$ is the price of a default free bond.

Lemma 3. *The price of a y rated defaultable bond with zero recovery is given by*

$$
\begin{aligned}
B_t(T,1) &= E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} I\{t^* > T\}\right] \\
&= B_t(T) - \sum_{i=0}^{K} v_{yi}\tilde{v}_{i0} G^Q(T-t,\mathbf{Z}_t;\mathbf{M}_r+\alpha_i\mathbf{M}_\lambda,\mathbf{0}).
\end{aligned}
$$

Proof. We find

$$\begin{aligned} E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} I\{t^* > T\}\right] &= E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds}(1 - I\{Y_T = 0\})\right] \\ &= B_t(T) - E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} I\{Y_T = 0\}\right] \end{aligned}$$

where the second expectation can be computed as

$$\begin{aligned} E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} I\{Y_T = 0\}\right] &= E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} E^Q_{t,y}\left[I\{Y_T = 0\} | \{Z_s\}_{s \le T}\right]\right] \\ &= E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} \sum_{i=0}^{K} \upsilon_{yi} \tilde{\upsilon}_{i0} e^{-\alpha_i(\tau_T - \tau_t)}\right] \\ &= \sum_{i=0}^{K} \upsilon_{yi} \tilde{\upsilon}_{i0} G^Q(T - t, \mathbf{Z}_t; \mathbf{M}_r + \alpha_i \mathbf{M}_\lambda, \mathbf{0}) \end{aligned}$$

□

To show that our framework is capable of dealing with recovery which is correlated to other market factors, we now consider a nontrivial recovery mechanism called *recovery of treasury* (RT). Other recovery specifications often considered include *recovery of par* and *recovery of market value*, [Schönbucher (2003)]. Up to this point, all formulas have involved simple evaluation of the function G^Q. The more complicated structures we next consider involve more than one time interval. A final structural assumption on our framework enables us to find explicit formulas even for products which involve multiple time intervals.

Assumption 4. The process $\mathbf{Z}_t$ is *affine* under both P and Q [Duffie et al. (2003)]. That is, there are a scalar function $\phi(t, \mathbf{u}, \mathbf{v})$ and vector function $\psi(t, \mathbf{u}, \mathbf{v})$ such that the G functions have the exponential affine form

$$G^Q(t, \mathbf{Z}; \mathbf{u}, \mathbf{v}) = e^{-\phi(t,\mathbf{u},\mathbf{v}) - \psi(t,\mathbf{u},\mathbf{v}) \cdot \mathbf{Z}} \tag{19}$$

In the next lemma we compute the price of a credit default swap (CDS) under recovery of treasury (RT). One party, the insured, pays a constant rate CDS, called the *CDS spread*, up to $t^* \wedge T$, the minimum of the default time t^* and the maturity date. If default happens at or before maturity T, it is assumed that on that date the insured exercises their right to exchange the now defaulted note for a riskless bond of the same maturity T and par value. The defaulted note is assumed to have a recovery value of $R_{t^*} B_{t^*}(T, 0)$, so the net value to the insured of the insurance exchange is $(1 - R_{t^*}) B_{t^*}(T, 0)$.

The fair CDS spread is chosen to solve the balance equation CDS $\times V_0 = W_0$ at time 0, where the time t price of the *premium leg* which pays at rate 1 while $t < t^* \wedge T$ is

$$V_t = E^Q_{t,\mathbf{Z}_t,y}\left[\int_t^T e^{-\int_t^s r_u du} I\{t^* > s\} ds\right] \tag{20}$$

and price of the *insurance leg* at time $t < t^* \wedge T$ is

$$W_t = E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^{t^*} r_s ds}(1-R_{t^*})B_{t^*}(T,0)I\{t^* \le T\}\right]. \tag{21}$$

Lemma 4. *Under Assumption 4, the price of the premium leg is given by*

$$V_t = \int_t^T B_t(s,0)ds, \tag{22}$$

and the price of the insurance leg is

$$\begin{aligned} W_t &= \sum_{i=0}^K v_{yi}\tilde{v}_{i0}\int_t^T e^{-\phi(T-s,\mathbf{M}_r,0)}\alpha_i(\mathbf{M}_\lambda\cdot\mathbf{D_v})\Big[G^Q\left(s-t,\mathbf{Z}_t,\mathbf{M}_r+\alpha_i\mathbf{M}_\lambda,\psi(T-s,\mathbf{M}_r,0)\right)- \\ &\quad -G^Q\left(s-t,\mathbf{Z}_t,\mathbf{M}_r+\alpha_i\mathbf{M}_\lambda,\mathbf{M}_l+\psi(T-s,\mathbf{M}_r,0)\right)\Big]ds \end{aligned}$$

Here ψ and ϕ are given by (19) *and $\mathbf{D_v}G^Q$ denotes the derivative of G^Q in the $\mathbf{v}$ variable.*

Proof. The premium leg is straightforward, so we address only the more complicated insurance leg. We treat the increasing process $I_s := I\{t^* \le s\}$ as an integrator and use the law of iterated expectations to write

$$\begin{aligned} W_t &= E^Q_{t,\mathbf{Z}_t,y}\left[\int_t^T e^{-\int_t^s r_{s'}ds'}(1-R_s)B_s(T,0)\,dI_s\right] \\ &= E^Q_{t,\mathbf{Z}_t}\left[\int_t^T e^{-\int_t^s r_{s'}ds'}(1-R_s)B_s(T,0)\,dE^Q_{t,y}\left[I\{Y_s=0\}|\{\mathbf{Z}_u\}_{u\le s}\right]\right] \end{aligned}$$

Since $E^Q_{t,y}\left[I\{Y_s=0\}|\{\mathbf{Z}_u\}_{u\le s}\right] = \sum_{i=0}^K v_{yi}\tilde{v}_{i0}e^{-\alpha_i(\tau_s-\tau_t)}$, $r_s = \mathbf{M}_r\cdot\mathbf{Z}_s, R_s = e^{-\mathbf{M}_l\cdot\mathbf{Z}_s}$ and the default-free bond price is given by (18) we find

$$\begin{aligned} W_t &= \sum_{i=0}^K v_{yi}\tilde{v}_{i0}E^Q_{t,\mathbf{Z}_t}\Big[\int_t^T e^{-\int_t^s \mathbf{M}_r\cdot\mathbf{Z}_{s'}ds'-\phi(T-s,\mathbf{M}_r,0)-\psi(T-s,\mathbf{M}_r,0)\cdot\mathbf{Z}_s} \\ &\qquad \times(1-e^{-\mathbf{M}_l\cdot\mathbf{Z}_s})(-\alpha_i\mathbf{M}_\lambda\cdot\mathbf{Z}_s)e^{-\alpha_i\int_t^s \mathbf{M}_\lambda\cdot\mathbf{Z}_{s'}ds'}ds\Big]. \end{aligned}$$

In order to finish the proof, one just needs to use the formula

$$E^Q_{0,\mathbf{Z}_0}[(-\mathbf{w}\cdot\mathbf{Z}_t)e^{-\int_0^t \mathbf{u}\cdot\mathbf{Z}_s ds-\mathbf{v}\cdot\mathbf{Z}_t}] = (\mathbf{w}\cdot D_\mathbf{v})G^Q(t,\mathbf{Z}_0;\mathbf{u},\mathbf{v}).$$

□

Finally, we note that a riskless bond is equivalent to a defaultable bond with recovery RT plus the insurance leg of a CDS. Thus

$$B_t(T,RT) = B_t(T,0) - W_t. \tag{23}$$

The *credit yield spread* on bonds with recovery RT is defined to be the function

$$h(t,T,X) = \frac{\log B_t(T,0) - \log B_t(T,RT)}{T-t}. \tag{24}$$

5 Pricing under recovery of market value

Bond pricing with "recovery of market value" makes the assumption that at the default time t^*, the defaulted bond pays the holder a single cash amount of R_{t^*} times the market value of the bond at the instant prior to default. The random variable $R_{t^*} = [R_{t^*,1}, \ldots, R_{t^*,K}]'$ may depend on the credit rating Y_{t^*-}, as well as the market processes $\mathbf{Z}_s, s \leq t$. The following useful formula generalizes the well-known result of Duffie and Singleton [Duffie and Singleton (1999)], and has a formally similar proof:

Theorem 5. *The RMV value of a y rated defaultable zero coupon bond is*

$$B_t(T, RMV) = \begin{cases} \sum_{k=1}^K E^Q_{t,\mathbf{Z}_t}\left[e^{-\int_t^T r_s ds}\left(e^{\int_t^T \tilde{\mathcal{L}}_s \lambda_s ds}\right)_{yk}\right] & y \neq 0 \\ 0 & y = 0 \end{cases} \tag{25}$$

Here $\tilde{\mathcal{L}}_s = (\tilde{l}_{s,jk})_{j,k=1,\ldots,K}$ is the $K \times K$ recovery adjusted migration matrix process with components $\tilde{l}_{s,jk} = l_{jk} + \delta_{jk} R_{s,j} l_{j0}$.

Proof. Let the vector $\mathbf{V}_t = (V_{t,1}, \ldots, V_{t,K})'$ denote the values of $B_t(T, RMV)$ in each nondefault rating, and let U_t denote the matrix valued process $\exp[-\int_0^t (r_s - \mathcal{L}\lambda_s) ds]$. One can write

$$\mathbf{V}_t = E^Q_{t,\mathbf{Z}_t}\left[U_t^{-1} U_T \mathbf{e} + U_t^{-1} \int_t^T U_s(\tilde{\mathcal{L}}_s - \mathcal{L})\mathbf{V}_s \lambda_s ds\right] \tag{26}$$

where $\mathbf{e} = (1, \ldots, 1)'$. This equals $U_t^{-1}\left(\mathbf{N_t} - \int_0^t U_s(\tilde{\mathcal{L}}_s - \mathcal{L})\mathbf{V}_s\lambda_s ds\right)$ where $\mathbf{N}_t = E^Q_{t,\mathbf{Z}_t}\left[U_T\mathbf{e} + \int_0^T U_s(\tilde{\mathcal{L}}_s - \mathcal{L})\mathbf{V}_s\lambda_s ds\right]$ is a martingale. It follows that

$$d\mathbf{V}_t = (r_t - \mathcal{L}\lambda_t)\mathbf{V}_t dt - (\tilde{\mathcal{L}}_t - \mathcal{L})\lambda_t \mathbf{V}_t dt + U_t^{-1} d\mathbf{N}_t$$

Since the last term is a martingale, the expectation of the integral of this SDE yields the result.

□

If $R_{s,k} = R_k$ are constants, we can then compute this formula by diagonalizing $\tilde{\mathcal{L}}$ and proceeding as before. In this case, we can say that the RMV bond price equals the price of a zero recovery bond computed with the recovery adjusted migration matrix. We note for future use the following formula for the yield of a zero coupon bond:

$$Y_t(T, RMV) = \frac{1}{T-t} \log \sum_{k,l=1}^K v_{yk}\tilde{v}_{kl} G^Q(T-t, \mathbf{Z}_t; \mathbf{M}_r + \alpha_k \mathbf{M}_\lambda, \mathbf{0}). \tag{27}$$

where $v, \tilde{v}$ and α arise from the diagonalization of the recovery adjusted matrix $\tilde{\mathcal{L}}$.

6 The Multiple Firm Framework

The simplest way to extend the credit framework to M firms is to assume:

Assumption 5. Firms are distinguishable in their creditworthiness at time t only by their rating class at that time. The spot interest process r_t, the recovery process $R_t = e^{-l_t}$, and the stochastic migration intensity λ_t, are defined as for the one firm model. The credit migration Markov chain processes $Y_t^{(1)}, \ldots, Y_t^{(M)}$ are identical, independent, and satisfy the instantaneous transition equations (4) under P. They are identical, conditionally independent, and satisfy (9) under Q. The default time t_i^* of firm i is defined as the first time the corresponding process $Y_t^{(i)}$ hits the absorbing state 0.

This strong assumption implies that different firms of the same rating will have identical credit yield curves, and thus it certainly fails for real rating systems such as Moody's and Standard and Poor's that are influenced by factors other than credit worthiness. Nonetheless, we argue that the assumption is a good approximation for an ideal system that rates firms solely on the basis of their instantaneous credit worthiness. In Section 7, we indicate two simple and natural extensions of the modelling framework which weaken Assumption 5: one extension breaks the indisinguishability assumption, the other breaks the independence assumption.

In the above multifirm framework, all credit derivatives on a single firm are computed as before. As a first multifirm computation, we find the joint default probability distribution of any two firms:

Lemma 6. *The joint probability $Q_{0,\mathbf{Z}_0,y_1,y_2}(t_1^* \le s, t_2^* \le t)$ for two firms with $Y_0^{(1)} = y_1, Y_0^{(2)} = y_2$ is given by*

$$Q_{0,\mathbf{Z}_0,y_1,y_2}(t_1^* \le s, t_2^* \le t) = E^Q_{0,\mathbf{Z}_0,y_1,y_2}\left[I\{Y_s^{(1)} = 0\}I\{Y_t^{(2)} = 0\}\right] \quad (28)$$
$$= \sum_{k,l=0}^{K} v_{y_1 k}\tilde{v}_{k0} v_{y_2 l}\tilde{v}_{l0} E^Q_{0,\mathbf{Z}_0}\left[e^{-\alpha_k \tau_s - \alpha_l \tau_t}\right],$$

Under Assumption 4, the expectation $E^Q_{0,\mathbf{Z}_0,y_1,y_2}\left[e^{-\alpha_k \tau_s - \alpha_l \tau_t}\right]$ is given explicitly in (A.10) *in Appendix A.*

For the remainder of this section, we focus on the dependence between default times of the two firms. An important measure of dependence is the *default correlation* between events $\{t_1^* \le t\}, \{t_2^* \le t\}$ defined as a function of t and the two rating classes y_1, y_2 to be

$$\rho_{12}(t) = \frac{E_{12}(t) - E_1(t)E_2(t)}{\sqrt{(E_1(t) - E_1(t)^2)(E_2(t) - E_2(t)^2)}}, \quad (29)$$

where

$$E_{12}(t) \;=\; E^Q_{0,\mathbf{Z}_0,y_1,y_2}\left[I\{t_1^* \le t\}I\{t_2^* \le t\}\right],$$

and

$$E_n(t) \quad = \quad E^Q_{0,\mathbf{Z}_0,y_n}\left[I\{t^*_n \le t\}\right], n = 1, 2$$

The following formulas are proved by mimicking the proofs done so far:

Lemma 7. *The above expectations E_1, E_2, E_{12} are given by*

$$E_{12}(t) \quad = \quad E^Q_{0,\mathbf{Z}_0,y_1,y_2}\left[I\{Y^{(1)}_t = 0\}I\{Y^{(2)}_t = 0\}\right] \tag{30}$$

$$= \quad \sum_{k,l=0}^{K} v_{y_1k}\tilde{v}_{k0}\ v_{y_2l}\tilde{v}_{l0}G^Q(t, \mathbf{Z}_0; (\alpha_k + \alpha_l)\mathbf{M}_\lambda, \mathbf{0}), \tag{31}$$

and

$$E_n(t) \quad = \quad E^Q_{0,\mathbf{Z}_0,y_n}\left[I\{Y^{(n)}_t = 0\}\right] \tag{32}$$

$$= \quad \sum_{ik=0}^{K} v_{y_nk}\tilde{v}_{k0}G^Q(t, \mathbf{Z}_0; \alpha_k\mathbf{M}_\lambda, \mathbf{0}), \quad n = 1, 2. \tag{33}$$

7 Extensions

To this point, we have considered the simplest possible multifirm structure, which although it admits many desired features, is far too restrictive for practical purposes. Let us now briefly discuss three extensions which add flexibility for fitting both credit spread curve dynamics and default correlations, while retaining the essential feature of computational feasibility.

7.1 Multidimensional Time Change

Models with S-dimensional time change bring greater dynamic flexibility, and can be formulated by replacing the single generator $\mathcal{L}$ by matrices $\mathcal{L}^{(1)}, \dots, \mathcal{L}^{(S)}$ which can be diagonalized by the same matrix $\mathcal{L}^{(i)} = VD^{(i)}V^{-1}$, and hence span a commuting subgroup:

$$\mathcal{L}^{(i)}\mathcal{L}^{(j)} = \mathcal{L}^{(j)}\mathcal{L}^{(i)}, \quad i, j \le S$$

We can assume Q dynamics of the form

$$Q(Y_{t+dt} = j|Y_t = k) = \sum_{s=1}^{S} \lambda^{(s)}_t l^{(s)}_{kj} dt, \qquad \text{for } k \ne j,$$

where now $\mathbf{X}_t = [r_t, l_t, \lambda^{(1)}_t, \dots, \lambda^{(S)}_t]$ is a $2+S$ dimensional vector-valued process.

S can be increased one step at a time, thereby introducing new modes of credit migration, and consequently new degrees of freedom in the shapes of credit spreads.

The most natural first step arises with the special stochastic matrix

$$\mathcal{L}^{(2)} := \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -1 \end{bmatrix}, \tag{34}$$

which commutes with any stochastic matrix. Its inclusion adds a rating independent default rate to each firm. One can see that its effect is to shift all credit spread curves up by a similar amount and it can be used to model the liquidity risk premium that is the usual explanation for the gap between the highest rated firms and the corresponding risk free curve.

7.2 Fitting Yield Curves

Multidimensional time change also leads us to a simple method to fit yield curves of each firm exactly on a single date $t = 0$: this is necessary in practise to break the symmetry between identically rated firms and to fit observed bond prices exactly. The basic idea is similar to the Hull-White extensions of short-rate models: we add a deterministic function $f(s)$ to the default intensity which can then be used to fit the bond data exactly. It is simplest to apply such a correction using the special generator $\mathcal{L}^{(2)}$ given by (34).

We suppose that at time 0, we have a multifirm model of the above form, with all market parameters underlying $\mathbf{X}, \mathcal{L}$ calibrated to credit spread dynamics averaged over rating classes (see Section 9 for a more detailed discussion). We also suppose that for the ith firm, we observe a family of zero coupon bond prices $\{B^{(i)}(T)\}$ of various maturities $T \in \mathcal{T}^{(i)}$.

For the ith firm, we introduce a deterministic function $f^{(i)}(t)$ which is piecewise constant over the tenor structure $\mathcal{T}^{(i)}$, and replace the Q Markov chain probabilities by

$$Q(Y^{(i)}_{t+dt} = j | Y^{(i)}_t = k) = (\lambda_t l_{kj} + f^{(i)}(t)\delta_{j0})dt, \quad \text{for } k \neq j. \tag{35}$$

One can now use the following lemma to recursively find a unique solution for the values of $f^{(i)}(t)$ in terms of the observed bond prices.

Lemma 8. *Under dynamics* (35)*, the zero-coupon bond price with recovery mechanism R, is given by:*

$$e^{-\int_t^T f(s)ds} B_t(T, 1-R).$$

where $B_t(T, 1-R)$ is computed using the dynamics with $f(t) = 0$.

Proof. Let $\tilde{Y}_t$ be the process defined by

$$Q(\tilde{Y}_{t+dt} = l | \tilde{Y}_t = k) = \lambda_t l_{kl} dt, \quad \text{for } k \neq l$$

and let ξ be the exponential random variable with intensity $f(t)$:

$$Q(\xi \leq t + dt | \xi > t) = f(t)dt$$

Define a new process $\hat{Y}_t$ as follows: $\hat{Y}_t = \tilde{Y}_t$ if $\xi \leq t$ and $\hat{Y}_t = 0$ if $\xi > t$. Then one can check that $\hat{Y}_t$ and $Y_t^{(i)}$ defined by (35) are identically distributed processes. Thus, for example, the zero recovery bond price can be computed as

$$\begin{aligned} E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} I\{Y_T^{(i)} \neq 0\}\right] &= E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} I\{\tilde{Y}_T \neq 0\} I\{\xi > T\}\right] \\ &= e^{-\int_t^T f(s)ds} E^Q_{t,\mathbf{Z}_t,y}\left[e^{-\int_t^T r_s ds} I\{\tilde{Y}_T \neq 0\}\right] = e^{-\int_t^T f(s)ds} B_t(T,1). \end{aligned}$$

The same argument applies for bond prices with general recovery R . □

7.3 Correlations

Recall that in our basic framework, the credit migration processes $Y^{(i)}$ are independent under the reference measure P, but become dependent under Q via the influence of the market factors $\mathbf{X}_t$. We now describe a natural extension of the dynamics which changes credit migration correlations (even under P), while keeping the marginal migration probabilities fixed. This construction, reminiscent of the copula construction, should be important in pricing correlation sensitive securities such as CDOs.

Observe that a homogeneous Markov chain remains a homogeneous Markov chain if subjected to a time change by any independent increasing Lévy process. For illustration purposes, we consider here time changes by the specific process η_t defined by its characteristic exponent (assumed to be the same under both P and Q):

$$\Psi(u) := -\log E[e^{-u\eta_1}] = \frac{u}{1 + \beta u} \tag{36}$$

where $\beta \geq 0$ is taken as a fixed parameter, but the same construction goes through for more general η. This particular choice of η is the pure jump Lévy process with an exponential jump measure $\nu(dx) = \beta^{-2} e^{-x/\beta} dx$, with its average rate of increase normalized to 1. β is a measure of stochasticity: as $\beta \to 0$, the process η_t becomes equal in distribution to the deterministic time t. Using the diagonal form $\mathcal{L} = VDV^{-1}$, and the inverse function $\Psi^{-1}(u) = \frac{u}{1-\beta u}$, we define the matrix $\Psi^{-1}(\mathcal{L}) := -V\Psi^{-1}(-D)V^{-1}$ where

$$\Psi^{-1}(-D) = \text{diag}\{0, \Psi^{-1}(\alpha_1), \ldots, \Psi^{-1}(\alpha_K)\}.$$

There is a maximal value $0 \leq \bar{\beta} < \infty$ such that $\Psi^{-1}(\mathcal{L})$ is a stochastic matrix for all $\beta \leq \bar{\beta}$. One can now compute that if $\beta \leq \bar{\beta}$ and $\widetilde{Y}_t$ is the time-homogeneous

Markov chain with transition generator $\Psi^{-1}(\mathcal{L})$, then the time changed process $\widetilde{Y}_{\eta_t}$ has the original generator $\mathcal{L}$. The multifirm process subject to the same time change $[\widetilde{Y}^{(1)}_{\eta_t}, \dots, \widetilde{Y}^{(M)}_{\eta_t}]$ will be correlated, but has marginals which are identical in distribution to Y_t. The parameter β plays a role analogous to the correlation parameter in the one factor normal copula: correlation goes to zero as $\beta \to 0$, and increases as $\beta \uparrow \bar{\beta}$.

The following lemma gives the essential formulas for the two firm distribution.

Lemma 9. *The joint default probability $Q_{0,\mathbf{Z}_0,y_1,y_2}(t_1^* \le s, t_2^* \le t)$ in this extended model is given for $s \le t$ by*

$$Q_{0,\mathbf{Z}_0,y_1,y_2}(t_1^* \le s, t_2^* \le t) = \sum_{k,l=0}^{K} v_{y_1 k}\tilde{v}_{k0} v_{y_2 l}\tilde{v}_{l0} E^Q_{0,\mathbf{Z}_0}\left[e^{-\tau_s \Psi(\Psi^{-1}(\alpha_k)+\Psi^{-1}(\alpha_l))-(\tau_t-\tau_s))\alpha_l}\right] \quad (37)$$

Under Assumption 4, the expectation $E_{0,\mathbf{Z}_0}\left[e^{\alpha\tau_s+\beta\tau_t}\right]$ can be computed explicitly (see (A.10) *in Appendix A).*

Default correlation in this extended model is given by (29) *with*

$$\begin{aligned} E_{12}(t) &= E^Q_{0,\mathbf{Z}_0,y_1,y_2}\left[I\{\tilde{Y}^{(1)}_{\eta_t}=0\}I\{\tilde{Y}^{(2)}_{\eta_t}=0\}\right] \\ &= \sum_{k,l=0}^{K} v_{y_1 k}\tilde{v}_{k0} v_{y_2 l}\tilde{v}_{l0} G(t,\mathbf{Z}_0;\Psi(\Psi^{-1}(\alpha_k)+\Psi^{-1}(\alpha_l))\mathbf{M}_\lambda,\mathbf{0}), \quad (38) \end{aligned}$$

and $E_n(t)$ given by (32)*.*

Proof. Again, using iterated expectations, the joint default probability can be written

$$\begin{aligned} & E^Q_{0,\mathbf{Z}_0,y_1,y_2}\left[I\{\widetilde{Y}^{(1)}_{\eta(s)}=0\}I\{\widetilde{Y}^{(2)}_{\eta(t)}=0\}\right] \\ &= E^Q_{0,\mathbf{Z}_0,y_1,y_2}\left[(e^{\eta(\tau_s)\Psi^{-1}(\mathcal{L})})_{y_1 0}(e^{\eta(\tau_t)\Psi^{-1}(\mathcal{L})})_{y_2 0}\right] \\ &= \sum_{k,l=0}^{K} v_{y_1 k}\tilde{v}_{k0} v_{y_2 l}\tilde{v}_{l0} E^Q_{0,\mathbf{Z}_0}\left[e^{-\eta(\tau_s)(\Psi^{-1}(\alpha_k)+\Psi^{-1}(\alpha_l))-(\eta(\tau_t)-\eta(\tau_s))\Psi^{-1}(\alpha_l)}\right] \\ &= \sum_{k,l=0}^{K} v_{y_1 k}\tilde{v}_{k0} v_{y_2 l}\tilde{v}_{l0} E^Q_{0,\mathbf{Z}_0}\left[e^{-\tau_s \Psi(\Psi^{-1}(\alpha_k)+\Psi^{-1}(\alpha_l))-(\tau_t-\tau_s)\alpha_l}\right] \quad (39) \end{aligned}$$

The proof of (38) is similar. □

Remark 2. The restriction $\beta \le \bar{\beta}$ implies a bound on the amount of correlation possible using this method. We will see in section 8 that in practice one can push for higher correlations by applying this method with $\beta > \bar{\beta}$, without a breakdown in the computations.

8 Illustrative Models

In this section we give two model specifications designed to give similar credit spread curve dynamics, but different correlation structures. Our main intention is to illustrate how the various pieces of the model can be put together, and how the resulting formulas can be computed.

We adopt the "credit rating" interpretation of the state space $\{0, \dots, K\}$. Then the first step is to determine the Markov generator $\mathcal{L}_Y$, a $K+1 \times K+1$ matrix, such that $e^{\mathcal{L}_Y}$ is a reasonable approximation of the one-year historical rating transition matrix as published by a rating agency such as Moody's or Standard and Poor's. This guarantees that the model dynamics under the reference measure P is compatible with historical default data. Here we simply take:

$$\mathcal{L}_Y = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2856 & -0.4318 & 0.0928 & 0.0250 & 0.0142 & 0.0142 & 0.0000 & 0.0000 \\ 0.0753 & 0.0479 & -0.1928 & 0.0568 & 0.0073 & 0.0034 & 0.0021 & 0.0000 \\ 0.0273 & 0.0144 & 0.1181 & -0.2530 & 0.0813 & 0.0089 & 0.0025 & 0.0005 \\ 0.0049 & 0.0020 & 0.0174 & 0.0701 & -0.1711 & 0.0713 & 0.0047 & 0.0007 \\ 0.0010 & 0.0000 & 0.0048 & 0.0107 & 0.0688 & -0.1172 & 0.0309 & 0.0010 \\ 0.0000 & 0.0000 & 0.0030 & 0.0030 & 0.0105 & 0.0787 & -0.1043 & 0.0091 \\ 0.0000 & 0.0000 & 0.0000 & 0.0031 & 0.0020 & 0.0083 & 0.1019 & -0.1153 \end{pmatrix} \tag{40}$$

which generates a typical one-year transition matrix taken from [Jarrow et al. (1997)]. A good discussion on how to find an approximate generator matrix from a given one-year transition matrix can be found in [Kreinin and Sidelnikova (2001)].

We assume we have one of two versions of a two-factor positive affine form of the processes $\mathbf{X}_t = [r_t, l_t, \lambda_t]' = M\mathbf{Z}_t$ with $N_1 = N_2 = 1$, each based on processes $Z_t^{(1)}, Z_t^{(2)}$. Z_1 is taken to be a CIR process with parameters $\alpha = 0.3790, \bar{Z} = 1, \sigma = 0.3486$ and initial value $Z_1(0) = 1$. Z_2 is taken to be a mean-reverting jump process, with generator

$$\mathcal{L}_Z^2 f(x) = -xf'(x) + \int_0^\infty [f(x+y) - f(x)]\nu(dy)$$

with an exponential jump measure

$$\nu(dy) = cde^{-cy}dy$$

and with parameters $d = 1/3, c = 1/3$ and initial value $Z_2(0) = 1$.

1. **Model A:** Under P we take identical independent Markov chains $Y_t^{(j)}$ with transition generator $\mathcal{L}^A := \mathcal{L}_Y$. Under Q, the dynamics follows (9) with

$$M = \begin{bmatrix} .0365 & 0 \\ 0.2 & 0.2 \\ 1.0 & 1.0 \end{bmatrix}$$

Thus the interest rate process r_t is CIR, with parameters selected to yield typical values for observed bond prices. The intensity process λ_t is a mixture of CIR and the Z_2 process and is positively correlated with r_t. Its parameters lead to a long term average "credit risk premium" value of 2. We also include a dynamic (log) recovery process l_t in the span of Z_1 and Z_2.

2. **Model B** is constructed to have identical marginal default distributions to Model A, but with higher default correlation. Thus, we follow method of Section 7.3 with the Lévy time change η_t^β, $\beta = 1$, and $\mathcal{L}^B := \Psi^{-1}(\mathcal{L}^A)$, and otherwise identical parameters. Note that in this case $\bar{\beta}$ is very small (because some off-diagonals of $\mathcal{L}_Y$ are close to zero) and $\beta > \bar{\beta}$. Thus the model is not strictly a Markov chain model (it generates small nonpositive "probabilities"), but we find that the resulting formulas nonetheless produce plausible values except for very short time intervals.

We computed the credit spread curves based on defaultable bonds, for each rating class, computed with recovery of treasury using the formula (24). The results are of course the same for both models, and are summarized in Figure 1. This graph illustrates that the AMC model is capable of reproducing the general features observed in real market data, such as hump-backed, decreasing and increasing curves for firms of (respectively) intermediate, low and high credit quality. Figures 2 and 3 illustrate the dependence of these curves on the initial values $Z_1(0), Z_2(0)$. Similar graphs showing dependence on all other underlying parameter values are easily generated, but space doesn't permit their inclusion here.

Default correlation is a very important issue in credit risk, so we next considered the joint properties of two firms. Figures 4 and 5 show the joint default distributions for two firms, one B and one BBB, under Models A and B. Observe their similarities and differences, in particular the existence of simultaneous jumps in Model B. Figures 6 and 7 give the default correlation function $\rho_{12}(t)$ between a BBB firm and all rating classes in both models. One can see from these results that as predicted, Model A shows relatively low correlation, while Model B shows generally higher correlation.

Taken together, these plots illustrate a range of features observed in real market data, and show that the modelling framework has the right qualitative features to be useful for credit risk.

9 Calibration Issues

A fully specified model in the AMC framework amounts to the specification of the parametric form of the $\mathbf{Z}$ Markov state process, the recovery mechanism R, the number S and and size $K+1$ of commuting Markov chain generators. As a guide to how to actually implement the AMC framework, we now consider the steps needed to calibrate a multifirm model with $S = 1$, $K = 7$, and $N = 3$ market factors, where we adopt the RMV bond valuation formula (27) with constant recovery fractions.

The aim is to capture the average pricing of the entire bond market for firms of all ratings, as driven by the Markov state process $\mathbf{Z}_t$. In this context it is natural to assume exchangeability of firms (Assumption 5), with the understanding that once the model is calibrated, the credit yield curve fitting technique described in the previous section can be used to capture the exact yield curves of each individual firm on one fixed date.

It is convenient to suppose that pure interest rate theory nests as a 2-factor submodel inside the full AMC model, leading to a splitting of parameters $\Theta = (\Theta_{IR}, \Theta_{CR})$ and the process components $\mathbf{Z}_t = (\mathbf{Z}_{t,IR}, Z_{t,CR})$. Then, as argued by Duffee (1999), Dai and Singleton (2003), and Bakshi et al (2006), an effective strategy begins by applying a Kalman Filter/Maximization Likelihood estimation (KFML estimation) procedure (see [Harvey (2001)]) to calibrate the interest rate submodel to a panel of observed default-free yield curves on dates $t = 1, 2, \ldots, T$. Suppose that this relatively standard exercise has been completed, giving estimates of the pure interest rate parameters $\hat{\Theta}_{IR}$ jointly with the time series of the unobserved interest rate components $\{\hat{\mathbf{Z}}_{t,IR}\}_{t=1,\ldots,T}$. We now investigate how in principle to do a KFML estimation for the one-dimensional time series $Z_{t,CR}$ and the remaining parameters Θ_{CR} (amongst which we include the recovery adjusted Markov chain generator $\tilde{\mathcal{L}}$), based on a panel of corporate bond data consisting of a time series of N observed zero coupon bond yields $\mathbf{Y}_t = \{Y_t^1, \ldots, Y_t^N\}$, on each of the dates $t = 1, \ldots, T$, each yield referring to a bond with known credit rating and maturity.

KFML relies on formulas for certain basic quantities, all of which are explicitly computable in the AMC model: the *Markov probability density* $\rho(Z_t|Z_{t-1}, \Theta)$, the *measurement equation*

$$\mathbf{Y}_t = \mathbf{F}(Z_t, \Theta) + \eta_t, \tag{41}$$

where $\mathbf{F}$ is the bond yield formula (27) and η_t are independent zero mean Gaussians, and the *prior density* $\rho(\Theta)$. It proceeds by iteration of two steps until adequate convergence is achieved.

1. Kalman filter step: from a single sample $\hat{\mathbf{Y}}$ of $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$, and an estimate $\hat{\Theta}$ of Θ, it produces for $t = 1, 2, \ldots, T$ a maximum likelihood estimate $\hat{Z}_t = \hat{Z}_t(\mathbf{Y}_1, \ldots, \mathbf{Y}_t)$ conditional on $\mathbf{Y}_1, \ldots, \mathbf{Y}_t$ by computing

$$\hat{Z}_t := \text{argmax}_Z \exp\left[-\frac{1}{2}\sigma_\eta^{-2}|\hat{\mathbf{Y}}_t - \mathbf{F}(Z, \hat{\Theta})|^2\right]\rho(Z|\hat{Z}_{t-1}, \hat{\Theta}) \tag{42}$$

sequentially starting with an initial value $\hat{Z}_0$.

2. Maximization likelihood step: from a single sample $\hat{\mathbf{Y}}$ of $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$, and estimates $\hat{Z}_1, \ldots, \hat{Z}_T$, it produces a maximum likelihood estimate

$$\hat{\Theta} = \text{argmax}_\Theta \sigma_\eta^{-NT} \exp\left[-\frac{1}{2}\sigma_\eta^{-2}|\hat{\mathbf{Y}}_t - \mathbf{F}(Z, \Theta)|^2\right]\rho(Z|\hat{Z}_{t-1}, \Theta)\rho(\Theta) \tag{43}$$

Practical implementation of this method meets a number of difficulties whose resolution goes beyond the scope of the present paper. We mention here how to avoid two major obstacles. First, to bypass difficulties associated with nonlinearities in the AMC model, the KF step can be reduced to linear algebra in a Gaussian approximation obtained by matching first and second moments, $Z_t|\hat{Z}_{t-1} \sim N(\mu_t(\hat{Z}_{t-1}), \text{var}_t(\hat{Z}_{t-1}))$, and making linear approximations of the function $\mathbf{F}$ about $\mu_t(\hat{Z}_{t-1})$. However, care must be taken to verify the accuracy of the linearized filter. Second, we note that the complexity of the ML step, a high dimensional nonlinear optimization, can be reduced by a careful specification of the prior density $\rho(\Theta)$. In particular, bond prices are strongly sensitive to default probabilities but not migration probabilities, so it follows that an effective, parsimonious calibration of the Markov chain generator $\tilde{\mathcal{L}}$ can arise if we take a prior density where $\tilde{\mathcal{L}} - \tilde{\mathcal{L}}_{\text{hist}}$ is diagonal, mean zero, and Gaussian, and $\tilde{\mathcal{L}}_{\text{hist}}$ is estimated from historical transition frequencies and losses given default.

10 Conclusion

This article has introduced a flexible yet computationally efficient multifirm model of credit risk. In the basic version, the variations in credit quality across firms are determined by the credit ratings alone. Each firm undergoes a credit migration process which is correlated with market conditions only through the stochastic time change τ_t. The speed of the single time change process provides a measure of the credit environment experienced by firms at each time: when the speed is high, firms migrate quickly and hence default quickly, and the opposite when the speed is low.

The dynamics built into the basic model reflects in a plausible way the true dynamics of the market. In focussing on dynamics, our model contrasts with static copula methods for credit risk, which are the current industrial standard models for multifirm credit products. A detailed comparison of the relative advantages of the two frameworks is needed. The simple model we present here produces a wide range of possible behavior, and the graphs we show pass visual inspection to be plausible representation of the real market.

We also explored a number of extensions which will be necessary in practice to fit the complexities of real credit markets. All versions we propose share a common structure, and lead to extremely efficient algorithms for pricing standard credit products.

For basket credit products written on a large number of names, such as CDOs and basket CDSs, it is natural to distinguish firms only by ratings class, a simplification which makes an AMC description acceptable. This case is investigated in a companion paper [Hurd and Kuznetsov (2006)] where it is found that the computational complexity for CDOs and certain basket credit derivatives can be reduced enormously, and are computable almost as quickly as basic one firm computations.

In summary, we have introduced a versatile family of credit risk models capable in principle of reproducing most of the important features of real markets. The framework appears to be deserving of future development.

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Appendix: Explicit computations in the positive affine setting

We consider here the setting of positive affine processes discussed in Section 2, in the special case of $N_1 = N_2 = 1$. The extension to larger values of N_1, N_2 follows in straightforward fashion by independence.

Proposition 10. *In this setting, the function G^Q defined by (10) is given explicitly by*

$$G^{CIR}(t, z_0^{(1)}; u^{(1)}, v^{(1)}) G^{MR}(t, z_0^{(2)}; u^{(2)}, v^{(2)}) \tag{A.1}$$

where

1. *G^{CIR} has the exponential affine form*

$$G^{CIR}(t, z; u, v) = e^{-\phi^{CIR}(t,u,v) - z\psi^{CIR}(t,u,v)}, \tag{A.2}$$

where the functions ϕ^{CIR} and ψ^{CIR} are explicit:

$$\begin{cases} \psi^{CIR}(t, u, v) = -\psi_2 + \left(1 + \frac{c}{\gamma}(v + \psi_1)\left(e^{\gamma t} - 1\right)\right)^{-1}(v + \psi_2), \\ \phi^{CIR}(t, u, v) = -a\psi_1 t + \frac{a}{c}\log\left(1 + \frac{c}{\gamma}(v + \psi_1)\left(e^{\gamma t} - 1\right)\right) \end{cases} \tag{A.3}$$

with constants ψ_1, ψ_2 and γ given by

$$\begin{cases} \gamma = \sqrt{b^2 + 4uc} \\ \psi_1 = \frac{b+\gamma}{2c} \\ \psi_2 = \frac{b-\gamma}{2c} \end{cases} \tag{A.4}$$

2. *G^{MR} has the exponential affine form*

$$G^{MR}(t, z; u, v) = e^{-\phi^{MR}(t,u,v) - z\psi^{MR}(t,u,v)}, \tag{A.5}$$

where the functions ϕ^{MR} and ψ^{MR} are explicit:

$$\begin{cases} \psi^{MR}(t, u, v) = \left(v - \frac{u}{b}\right)e^{-bt} + \frac{u}{b} \\ \phi^{MR}(t, u, v) = dt - \frac{cd}{cb+u}\log\left(\frac{(cb+u)e^{bt} - u + vb}{cb+vb}\right), \end{cases} \tag{A.6}$$

Proof. These results are derived from the Feynman-Kac formula which states that when Z_t is a continuous time Markov process with generator $\mathcal{L}$ then

$$G(t, z) = E_{0,z}[e^{-\int_0^t uZ_s ds} F(Z_t)] \tag{A.7}$$

is characterized as the unique solution of the equation

$$\begin{cases} -[\partial_t f](t, z) + [\mathcal{L}f](t, z) - uzf(t, z) = 0 \quad t > 0 \\ f(0, z) = F(z) \end{cases} \tag{A.8}$$

The generator for the CIR process is

$$[\mathcal{L}^{\mathrm{CIR}}f](z) = (a - bz)f'(z) + czf''(z),$$

and one can verify by inspection that the solution $G(t,z)$ with $G(0,z) = e^{-vz}$ is given by $G(t,z) = e^{-\phi^{CIR}(t,u,v) - z\psi^{CIR}(t,u,v)}$ with ϕ^{CIR}, ψ^{CIR} given by (A.3). The result now follows by taking $t = 0$.

Similarly, the generator of the MR process is

$$\mathcal{L}^{\mathrm{MR}}f(z) = -bzf'(z) + cd\int_0^\infty (f(z+y) - f(z))e^{-cy}dy.$$

Plugging in the affine form for G^{MR} leads to the following system of ordinary integral-differential equations

$$\begin{cases} \frac{d\psi}{dt} = -b\psi + u, & \psi(0,u,v) = v \\ \\ \frac{d\phi}{dt} = \frac{d\psi}{c+\psi}, & \phi(0,u,v) = 0. \end{cases} \tag{A.9}$$

which can be solved explicitly to yield (A.6). $\square$

Finally we derive an explicit formula for the expectations appearing in Lemmas 6 and 9.

Proposition 11. *If Assumptions 1-4 hold and* $\tau_t = \int_0^t \mathbf{u}\cdot\mathbf{Z}_s ds$ *then*

$$E_{0,\mathbf{Z}_0}\left[e^{-\alpha\tau_s - \beta\tau_t}\right] = e^{-\phi(t-s,\beta\mathbf{u},0)}G(s,x;\alpha+\beta,\psi(t-s,\beta\mathbf{u},0)) \tag{A.10}$$

when $s \le t$ *(the symmetric formula holds when* $t \le s$*).*

Proof. We can rewrite the expression as an iterated expectation

$$E_{0,\mathbf{Z}_0}\left[e^{-(\alpha+\beta)\tau_s}E_s\left[e^{-\beta(\tau_t-\tau_s)}\right]\right]. \tag{A.11}$$

and compute the inner expectation to obtain

$$E_{0,\mathbf{Z}_0}\left[e^{-(\alpha+\beta)\tau_s}e^{-\phi(t-s,\beta\mathbf{u},0)-\psi(t-s,\beta\mathbf{u},0)\cdot\mathbf{Z}_s}\right].$$

The final expectation is done using the definition (10), leading to the final result. $\square$

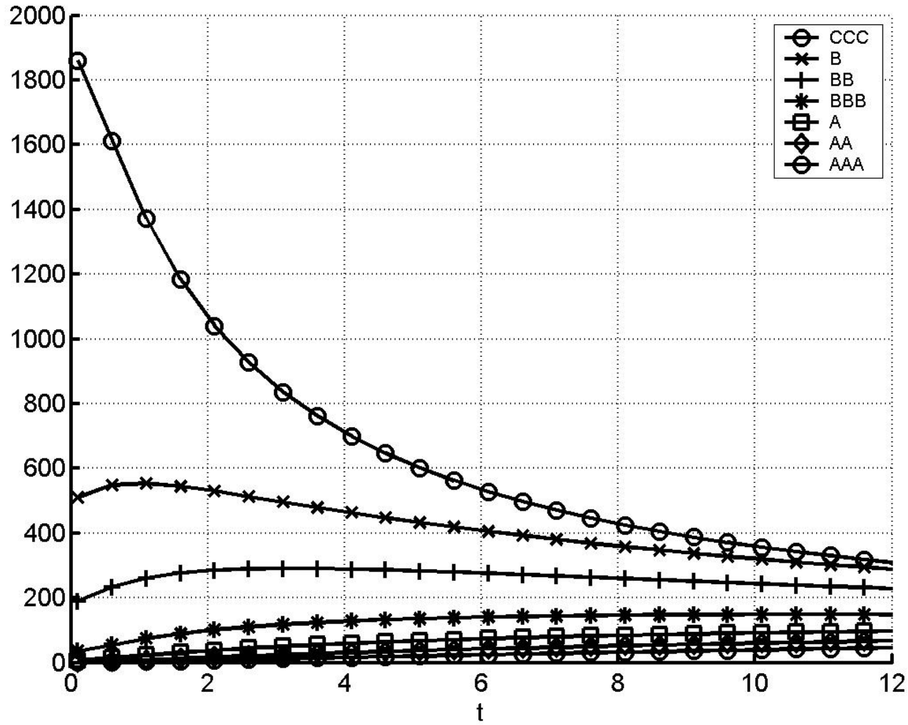

Figure 1: The credit spreads for all rating classes in Models A (and also Model B). These are computed with the “Recovery of treasury” mechanism.

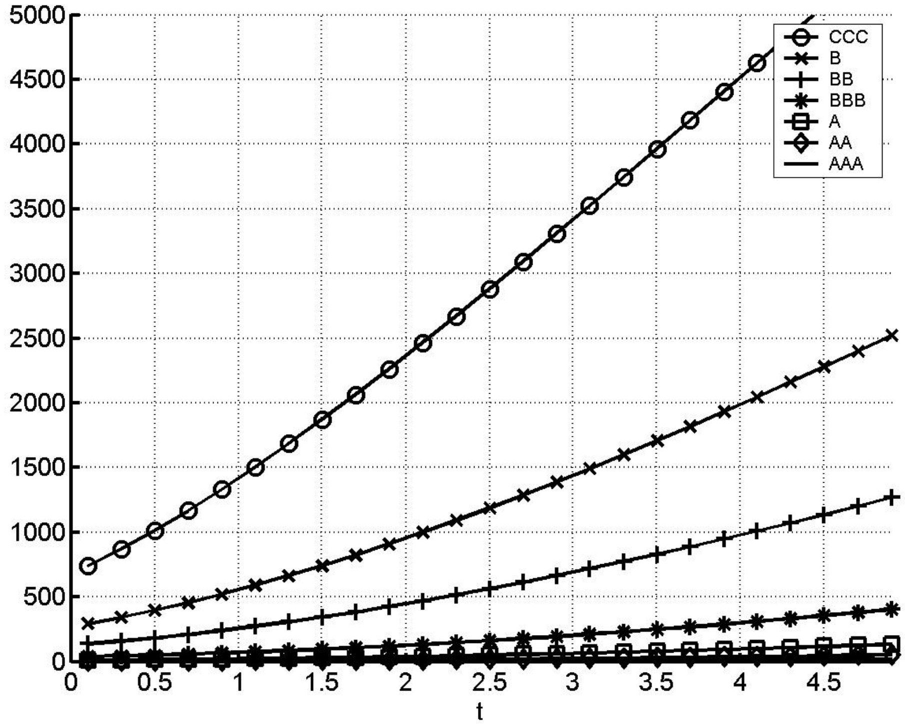

Figure 2: The one year credit spreads for all ratings, as a function of the initial value Z_0^1, computed in Model A.

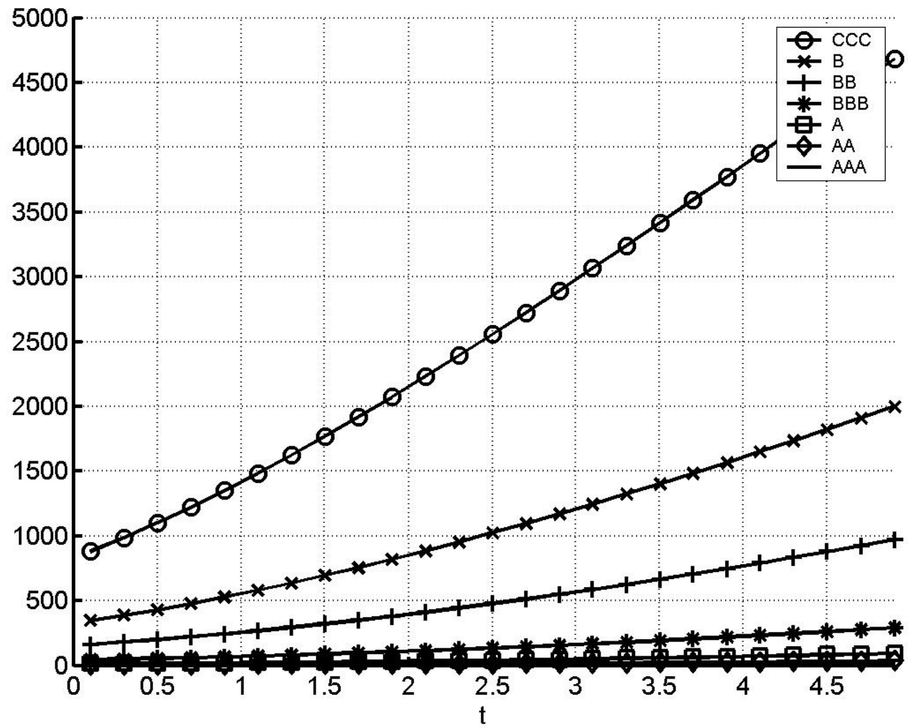

Figure 3: The one year credit spreads for all ratings, as a function of the initial value Z_0^2, computed in Model A.

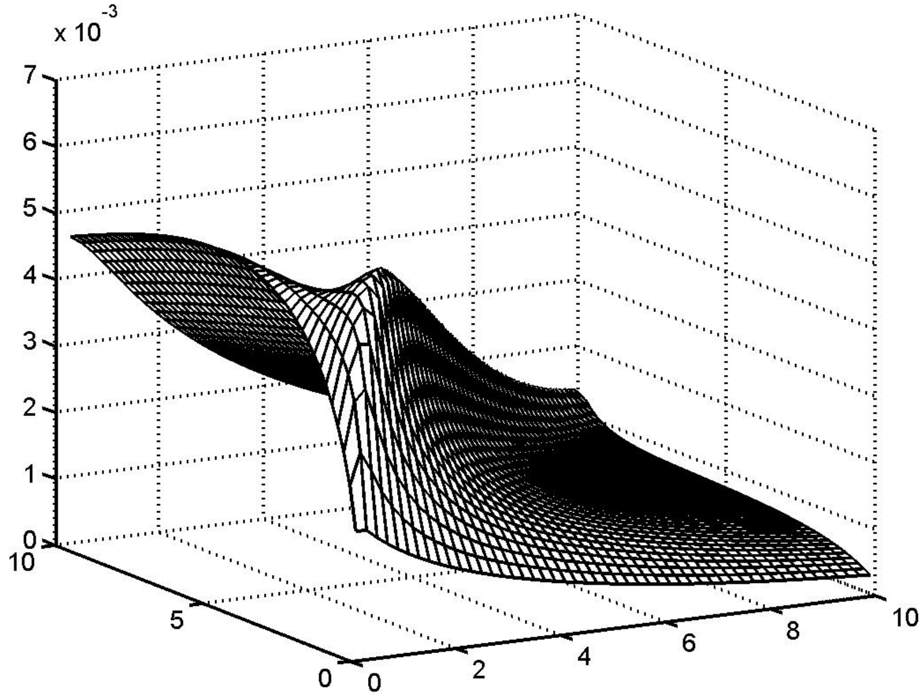

Figure 4: The joint default distribution for a B and a BBB rated firm in Model A.

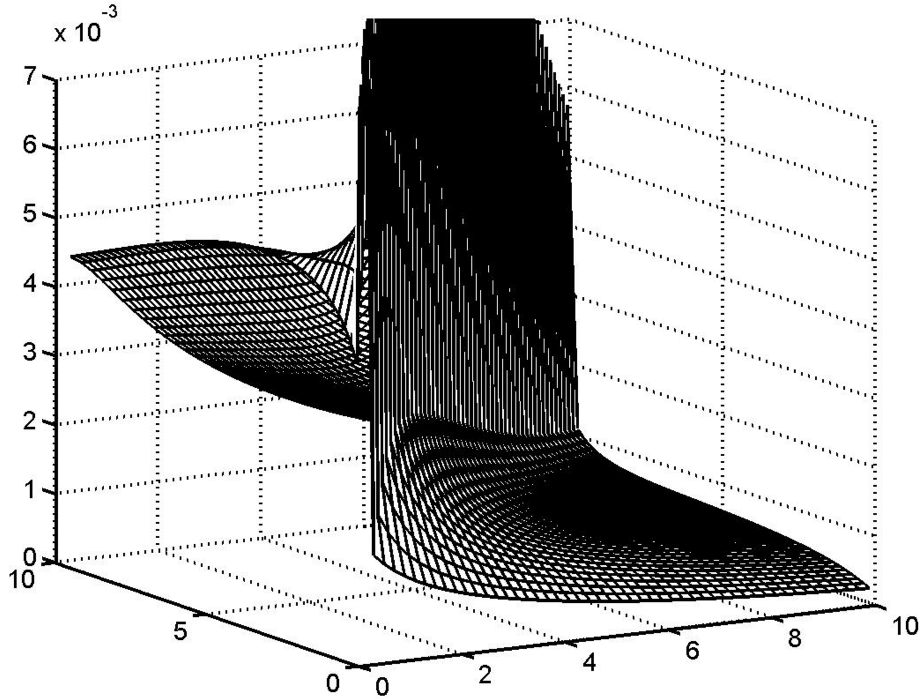

Figure 5: The joint default distribution for a B and a BBB rated firm in Model B. Note the ridge along the diagonal represents the singular support for simultaneous defaults.

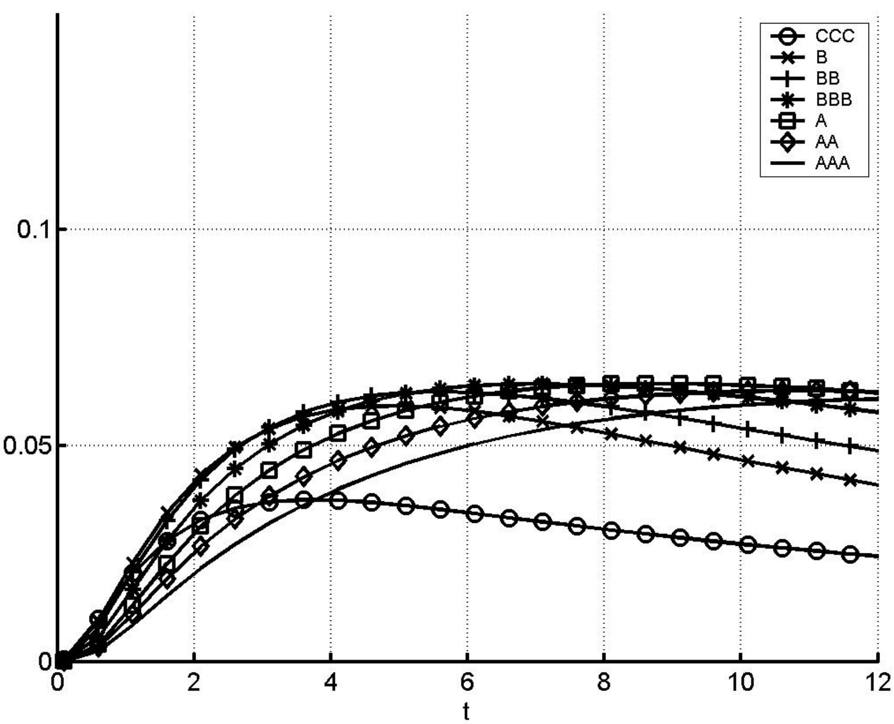

Figure 6: The default correlation as a function of time for a BBB firm versus firms of all ratings, in model A.

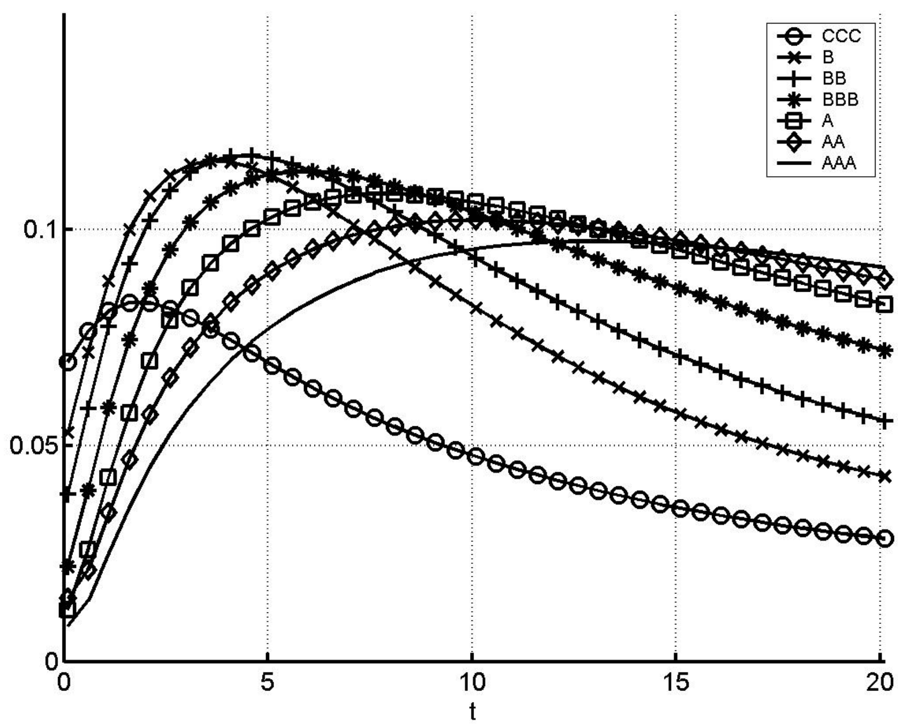

Figure 7: The default correlation as a function of time for a BBB firm versus firms of all ratings, in model B.