Saddlepoint approximations in portfolio credit risk

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August 28, 2008

1 Introduction

The classic method known variously as the saddlepoint approximation, the method of steepest descents, the method of stationary phase, or the Laplace method, applies to contour integrals that can be written in the form

$$I(s) = \int_{\mathcal{C}} e^{sf(\zeta)} d\zeta \tag{1}$$

where f, an analytic function, has a real part that goes to minus infinity at both ends of the contour $\mathcal{C}$. Such integrals arise frequently in applications of the Fourier transform. The fundamental idea is that the value of the integral when $s > 0$ is large should be dominated by contributions from the neighbourhoods of points where the real part of f has a saddlepoint. Early use was made of the method by Debye to produce asymptotics of Bessel functions, as reviewed in for example [9]. Daniels [4] wrote a definitive work on the saddlepoint approximation in statistics. Later, these ideas evolved into the theory of large deviations, initiated by Varadhan in [8], which seeks to determine rigorous asymptotics for the probability of rare events. A textbook description of the saddlepoint approximation can be found in [3].

If we write $\zeta = x + iy$, elementary complex analysis implies that the surface over the (x, y) plane with graph $\Re f$ has zero mean curvature, so any critical point ζ^* (a point where $f' = 0$) will be a saddlepoint of the modulus $|e^{sf(\zeta)}|$. The level curves of $\Re f$ and $\Im f$ form families of orthogonal trajectories: The curves of steepest descent of $\Re f$ are the level curves of $\Im f$, and vice versa. Thus the curve of steepest descent of the function $\Re f$ through ζ^* is also a curve on which $\Im f$ is constant. In other words, it is a curve of "stationary phase". On such a curve, the modulus of $e^{sf(\zeta)}$ will

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have a sharp maximum at ζ^*. If the contour $\mathcal{C}$ can be deformed to follow the curve of steepest descent through a unique critical point ζ^*, and the modulus of $e^{sf(\zeta)}$ is negligible elsewhere, the dominant contribution to the integral for s large can be computed by a local computation in the neighbourhood of ζ^*. In more complex applications, several critical points may need to be accounted for. To see how these ideas work in practice, we first review a statistical application due to Daniels.

2 Daniels' Application to Statistics

Daniels [4] presented an asymptotic expansion for the probability density function $f_n(x)$ of the mean $\bar{X}_n$ of n iid copies of a continuous random variable X with cumulative probability function $F(x)$ and probability density function $f(x) = F'(x)$. Assuming that the moment generating function

$$M(\tau) = e^{\Psi(\tau)} = \int_{-\infty}^{\infty} e^{\tau x} f(x) dx$$

is finite for τ in an open interval $(-c_1, c_2)$ containing the origin, the Fourier inversion theorem implies that

$$f_n(x) = \frac{n}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{n(\Psi(\tau) - \tau x)} d\tau \tag{2}$$

for any real $\alpha \in (-c_1, c_2)$.

For each x in the support of f, one can show that the saddlepoint condition

$$\Psi'(\tau) - x = 0$$

has a unique real solution $\tau^* = \tau^*(x)$. The contour for (2) with $\alpha = \tau^*$ is the tangent line to the steepest descents curve at τ^*. A Taylor expansion of $\Psi(\tau) - \tau x$ about τ^* and the substitution $w = -i\sqrt{n\Psi''(\tau^*)}(\tau - \tau^*)$ implies

$$\Psi(\tau) - \tau x = \Psi(\tau^*) - \tau^* x + \sum_{k \geq 2} \frac{1}{k!} \Psi^{(k)}(\tau^*) \left(\frac{iw}{(n\Psi^{(2)}(\tau^*))^{1/2}} \right)^k$$

and therefore one can write the integral in the form

$$\begin{aligned} f_n(x) \quad \sim \quad & \frac{\sqrt{n}}{2\pi\sqrt{\Psi''(\tau^*)}} \int_{-\infty}^{\infty} e^{n(\Psi(\tau^*) - \tau^* x) - w^2/2} \\ & \Big[1 - i n^{-1/2} (\Psi''(\tau^*))^{-3/2} \Psi^{(3)}(\tau^*) w^3/3! + \\ & n^{-1} \Big((\Psi''(\tau^*))^{-2} \Psi^{(4)}(\tau^*) w^4/4! - (\Psi''(\tau^*))^{-3} (\Psi^{(3)}(\tau^*)) w^6/(2!3!^2) \Big) + \ldots \Big] dw. \end{aligned}$$

Each term in this expansion is a Gaussian integral that can be evaluated in closed form. The odd terms all vanish, leaving an expansion in powers of n^{-1}:

$$f_n(x) \sim g_n(x) \left[1 + n^{-1} \left(\frac{\Psi^{(4)}(\tau^*)}{8(\Psi''(\tau^*))^2} - \frac{5(\Psi^{(3)}(\tau^*))^2}{24(\Psi''(\tau^*))^3} \right) + O(n^{-2}) \right] \tag{3}$$

where the leading term (called "the saddlepoint approximation") is given by

$$g_n(x) = \left(\frac{n}{2\pi\Psi''(\tau^*)}\right)^{1/2} e^{n(\Psi(\tau^*)-\tau^* x)}.$$

The function $I(x) = \sup_\tau \tau x - \Psi(\tau) = \tau^* x - \Psi(\tau^*)$ that appears in this expression is the Legendre transform of the cumulant generating function Ψ, and is known as the "rate function" or "Cramér function" of the random variable X. The *large deviation principle*

$$\lim_{n\to\infty} \frac{1}{n} \log P(\bar{X}_n > x) = -I(x), \quad \text{for } x > E[X]$$

holds for very general X. Another observation is that the Edgeworth expansion of statistics comes out in a similar way, but takes 0 instead of τ^* as the centre of the Taylor expansion.

One can show using a lemma due to Watson [9] that (3) is an *asymptotic expansion*, which means roughly that when truncated at any order of n^{-1}, the remainder is of the same magnitude as the first omitted term. A more precise statement of the magnitude of the remainder is difficult to establish: the lack of a general error analysis is an acknowledged deficiency of the saddlepoint method.

3 Applications to Portfolio Credit Risk

The problem of portfolio credit risk measures and the problem of evaluating arbitrage free pricing of *collateralized debt obligations* (CDOs)[1] both boil down to computation of the probability distribution of the portfolio loss at a set of times.

3.1 Credit Risk Measures

In risk management, the key quantities that determine the economic capital requirement for a risky portfolio are the *Value at Risk* (VaR) and *Conditional Value at Risk* (CVar)[2] for a fixed time horizon T and a fixed confidence level $\alpha < 1$. We consider the computation of these risk measures for a large portfolio of credit risky instruments such as corporate loans or credit default swaps. Let $(\Omega, \mathcal{F}, P)$ be a probability space that contains all of the random elements: here P is the physical measure (for an analogous treatment of CDO pricing it will be the risk–neutral probability measure). The portfolio is defined by the following basic quantities: M reference obligors with notional amounts $N_j, j = 1, 2, \ldots, M$; the random default time τ_j of the jth credit; the fractional recovery R_j after default of the jth obligor, assumed to be deterministic; and the loss $l_j = (1 - R_j)N_j/N$ caused by default of the jth obligor

[1] See article **.

[2] See Section ??.

as a fraction of the total notional $N = \sum_j N_j$. In terms of these quantities, the cumulative portfolio loss up to time t as a fraction of the total notional is given by:

$$L(t) = \sum_j l_j I(\tau_j \le t).$$

Then the two important credit risk measures are defined as follows.

$$\begin{aligned}
\mathrm{VaR}_{\alpha,T} &= N \inf\{x | P[L_T > x] > \alpha\} \\
\mathrm{CVaR}_{\alpha,T} &= \frac{E[(NL_T - \mathrm{VaR}_{\alpha,T})^+]}{1-\alpha}.
\end{aligned}$$

3.2 Saddlepoint Approximations for Credit Risk

As is commonly done, we now assume a *conditional independence* structure for default times. That is, we assume there exists a d–dimensional random variable Y, the "condition", such that the default times τ_j are mutually conditionally independent under the condition $Y = y$. The marginal distribution of Y, denoted by P_Y, and its probability density function $\rho_Y(y), y \in \mathbb{R}^d$, are taken to be known. This implies that conditioned on $Y = y$, the fractional loss $L(t)$ is a sum of independent (but not identical) Bernoulli random variables. For fixed values t, y of the time and condition Y, we note that $\hat{L} := L(t)|_{Y=y} \sim \sum_j l_j X_j$ where $X_j \sim \mathrm{Bern}(p_j(t,y)), p_j = \mathrm{Prob}(\tau_j \le t | Y = y)$. We suppose that the conditional default probabilities $p_j = p_j(t,y)$ are known.

The following functions are associated to the random variable $\hat{L}$:

1. The *probability distribution function* (PDF) $\rho(x) := F^{(-1)}(x)$ (in our simple example, it is a sum of delta functions supported on the interval $[0,1]$);
2. The *cumulative distribution function* (CDF) $F^{(0)}(x) = E[I(\hat{L} \le x)]$;
3. The *higher conditional moment functions* $F^{(m)}(x) = (m!)^{-1}E[((x-\hat{L})^+)^m], m = 1, 2, \ldots$;
4. The *cumulant generating function* (CGF)

$$\Psi(u) = \log(E[e^{u\hat{L}}]) = \sum_{j=1}^{M} [1 - p_j + p_j e^{ul_j}].$$

When we need to make explicit the dependence on t, y we write $F^{(m)}(x|t,y), \Psi(u|t,y)$. The unconditional versions of these functions are given by

$$\begin{aligned}
F^{(m)}(x|t) &= E[F^{(m)}(x|t,Y)] = \int_{\mathbb{R}^d} F^{(m)}(x|t,y)\rho_Y(dy), \ \ m = -1, 0, \ldots \\
\Psi(u|t) &= \log(E[e^{\Psi(u|t,Y)}] = \int_{\mathbb{R}^d} \prod_j \log([1 - p_j(t,y) + p_j(t,y)e^{ul_j}]\rho_Y(dy).
\end{aligned}$$

According to these definitions, for all $m = 0, 1, \ldots$ we have the integration formula

$$F^{(m)}(x) = \int_0^x F^{(m-1)}(z)dz. \tag{4}$$

We see that VaR and CVaR are obtained directly from $E[F^{(m)}(x|T)]$ with $m = 0, 1$: Anticipating that the Y integrals will be done numerically, the credit problem thus boils down to finding an efficient method to compute $E[F^{(m)}(x|T, y)]$ for a large but finite set of values (x, y). A number of different strategies can be used to compute the conditional distribution accurately when M is large:

1. In the fully homogeneous case when $p_j = p, l_j = l$, the distribution is binomial;
2. When $l_j = l$ but p_j are variable (the homogeneous notional case), the distribution can be computed highly efficiently by a recursive algorithm due to [1, 6].
3. When both l_j, p_j are variable, [7, 5, 2, 10] all noted that a saddlepoint treatment of the distribution functions offers superior performance over naive Edgeworth expansions.

We now consider the fully nonhomogeneous case and begin by using the Laplace inversion theorem to write

$$\rho(x) = F^{(-1)}(x) = \frac{1}{2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\Psi(\tau) - \tau x} d\tau$$

Since ρ is a sum of delta functions, this formula must be understood in the distributional sense, and holds for any real α. When $\alpha < 0$,

$$F^{(0)}(x) = \frac{1}{2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{\Psi(\tau)} \frac{1 - e^{-\tau x}}{\tau} d\tau = -\frac{1}{2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \tau^{-1} e^{\Psi(\tau) - \tau x} d\tau. \tag{5}$$

In the last step in this argument, one term is zero because $e^{\Psi(\tau)}$ is analytic and decays rapidly as $\Re\tau \to -\infty$. Similarly, for $m = 1, 2, \ldots$ one can show that

$$F^{(m)}(x) = (-1)^{m+1} \frac{1}{2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \tau^{-m-1} e^{\Psi(\tau) - \tau x} d\tau \tag{6}$$

provided $\alpha < 0$. It will also be useful to consider the functions

$$G^{(m)}(x) := (-1)^{m+1} \frac{1}{2\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \tau^{-m-1} e^{\Psi(\tau) - \tau x} d\tau \tag{7}$$

defined when $\alpha > 0$. One can show by evaluating the residue at $\tau = 0$ that

$$F^{(0)}(x) = G^{(0)}(x) - 1 \tag{8}$$
$$F^{(1)}(x) = G^{(1)}(x) - E[L] + x \tag{9}$$

with similar formulas relating $F^{(m)}$ and $G^{(m)}$ for $m = 2, 3, \dots$.

Since the conditional portfolio loss is a sum of similar, but not identical, independent random variables, we can follow the argument of Daniels to produce an expansion for the functions $F^{(m)}$. Some extra features are involved: the cumulant generating function is not N times something, but rather a sum of N (easily computed) terms; we must deal with the factor τ^{-m-1}; we must deal with the fact that critical points of the exponent in these integrals may be on the positive or negative real axis and there is a pole at $\tau = 0$. To treat the most general case, we move the factor τ^{-m-1} into the exponent and consider the saddlepoint condition

$$\Psi'(\tau) - (m+1)/\tau - x = 0. \tag{10}$$

Proposition 5.1 from [10] shows that a choice of two real saddlepoints solving this equation is typically available:

Proposition 3.1. *Suppose that $p_j,\ l_j > 0$ for all j. Then*

1. *There is a solution τ^*, unique if it exists, of $\Psi'(\tau) - x = 0$ if and only if $0 < x < \sum_j l_j$. If $E[\hat{L}] > x > 0$ then $\tau^* > 0$ and if $E[\hat{L}] < x < \sum_j l_j$ then $\tau^* < 0$.*

2. *For each $m \geq 0$ there is exactly one solution τ_m^- of (10) on $(-\infty, 0)$ if $x < \sum_j l_j$ and no solution on $(-\infty, 0)$ if $x \geq \sum_j l_j$. Moreover, when $x < \sum_j l_j$ the sequence $\{\tau_m^-\}_{m\geq 0}$ is monotonically decreasing in m.*

3. *For each $m \geq 0$ there is exactly one solution τ_m^+ of (10) on $(0, \infty)$ if $x > 0$ and no solution on $(0, \infty)$ if $x \leq 0$. Moreover, when $x > 0$ the sequence $\{\tau_m^+\}_{m\geq 0}$ is monotonically increasing in m.*

At this point the methods of [2] and [10] differ. We consider first the method of [10] for computing $F^{(m)}, m = 0, 1$. They apply the argument of Daniels directly, but with the following strategy for choosing the saddlepoint. Whenever $x < E[\hat{L}]$, τ_m^- is chosen as the centre of the Taylor expansion for the integral in (6). Whenever $x > E[\hat{L}]$, instead τ_m^+ is chosen as the centre of the Taylor expansion for the integral in (7), and either (8) or (9) is used. Thus for example, when $x > E[\hat{L}]$ the approximation for $m = 1$ is

$$F^{(1)}(x) \sim x - E[L] + \frac{e^{\tau_1^+ x + \Psi(\tau_1^+)}}{\sqrt{2\pi\Psi^{(2)}(\tau_1^+)}}\left[1 + \frac{\Psi^{(4)}(\tau_1^+)}{8(\Psi^{(2)}(\tau_1^+))^2} - \frac{5(\Psi^{(3)}(\tau_1^+))^2}{24(\Psi^{(2)}(\tau_1^+))^3} + \cdots\right].$$

In [2], the $m = -1$ solution τ^*, suggested by large deviation theory, is chosen as the centre of the Taylor expansion, even for $m \neq -1$. The factor τ^{-m-1} is then included with the other non-exponentiated terms, leading to an asymptotic expansion with terms of the form

$$\int_{-\infty}^{\infty} e^{-w^2/2}(w + w_0)^{-m-1} w^k dw, w_0 = \tau^*/\sqrt{\Psi^{(2)}(\tau^*)}$$

These integrals can be evaluated in closed form, but are somewhat complicated, and more terms are needed for a given order of accuracy.

Numerical implementation of the saddlepoint method for portfolio credit problems thus boils down to efficient computation of the appropriate solutions of the saddlepoint condition (10). This is a relatively straightforward application of one-dimensional Newton-Raphson iteration, but must be done for a large number of values of (x, t, y). [10] report that for typical parameter values and up to 2^{10} obligors, saddlepoints were usually found in under 10 iterations, which suggests a saddlepoint expansion will run no more than about 10 times slower than the Edgeworth expansion with the same number of terms. However, both [10] and [2] observe that for the same computational effort the saddlepoint approximation is often far superior.

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