Randomized Structure Model of Credit Spreads

Chuang Yi[*] Alexander Tchernitser[†] Tom Hurd[‡]

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Abstract

We propose to randomize the initial condition of a generalized structure model, where the solvency ratio instead of the asset value is modeled explicitly. This initial randomization assumption is motivated by the fact that market players cannot observe the solvency ratio accurately. We find that positive short spreads can be produced due to imperfect observation on the risk factor. The two models we have considered, the Randomized Merton II (RM-II) and the Randomized Black-Cox II (RBC-II), both have explicit expressions for Probability of Default (PD), Loss Given Default (LGD) and Credit Spreads (CS). In the RM-II model, both PD and LGD are found to be of order of $\sqrt{T}$, as the maturity T approaches zero. It therefore provides an example that has no well-defined default intensity but still admits positive short spreads. In the RBC-II model, the positive short spread is generated through the positive default intensity of the model. Because explicit formulas are available, these two Randomized Structure (RS) models are easily implemented and calibrated to the market data. This is illustrated by a calibration exercise on Ford Motor Corp. Credit Default Swap (CDS) spread data.

[*]PH.D candidate in Financial Math, McMaster University. Email: yichuang@math.mcmaster.ca
[†]Director, Market Risk, Bank of Montreal.
[‡]Professor, Department of Mathematics and Statistics, McMaster University

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1 Introduction

Quantitative modeling of credit risk is becoming an essential tool to assess and control credit exposures for banks and other financial institutions. The first approach to assess credit risk, known as structure models, was introduced by Merton (1974). In this model, the firm's asset value V_t is specified to be a Geometric Brownian motion (GBM), and is perfectly observed by investors. The debt of the firm is assumed to be a constant, K, maturing at a future time, T. Default time τ is defined to equal T if the firm's asset value is insufficient to repay the promised debt K. It is well documented in the literature that the term structure of credit spreads (TSCS) generated by Merton's model are too low, especially for short maturities (see Black & Cox (1976) and Giesecke & Goldberg (2004)). The short spreads in Merton's model are zero, which is counterfactual to empirical data (see Giesecke & Goldberg (2004)). Generalizations include Black & Cox (1976) first passage time model, Longstaff & Schwartz (1995) treatment of stochastic interest rates, Leland & Toft (1996) endogenous default and optimal capital structure, Collin-Dufresne & Goldstein (2001) mean-reverting leverage ratio model and Fouque, Sircar & Solna (2006) consideration of stochastic volatility. However, none of the models mentioned above can produce positive short spreads.

Zhou (2001) and Chen & Kou (2006) introduce jumps to the dynamics of the firm asset value in first passage time setting and they successfully obtain positive short spreads. A more promising method, known as the incomplete information approach, is introduced by Duffie & Lando (2001) and Giesecke (2006). Duffie & Lando (2001) notice that the observations of the asset value may not be exact but are realized with some random noise. Giesecke (2006), on the other hand, argues that the default barrier is not constant but may also be stochastic. Both of their efforts succeed in raising the short spreads above zero. Without modeling the firm asset value or debt value, Coculescu, Geman & Jeanblanc (2006) define the default time to be the first passage time of some fundamental process through the default barrier. The real fundamental process is not observable and the observed process is modeled to be a Stochastic Differential Equation (SDE). Although both jump diffusion models and incomplete information models can produce positive short spreads, these models in the literature are too mathematically complicated to be implemented in practice. In general, neither jump diffusion models nor first passage models with incomplete information admit explicit expressions of credit spreads. These models then have to rely on numerical methods, which utilize either Laplace transforms, see Chen & Kou (2006), or Fortet's lemma, see Coculescu et al. (2006). However, calibration becomes problematic when it comes to implementation of these models. On one hand, the difficulty comes from the mathematical complexity of the models. On the other hand, the expenses to raise short spreads bring too many extra parameters. For example, Chen & Kou (2006) add extra four parameters in order to introduce double exponential jumps. Hence, most of the papers on these models are silent on the calibration issue.

In this paper, we propose to randomize the initial condition of a generalized structure model, where the solvency ratio instead of the asset value is modeled explicitly. This initial randomization assumption is motivated by the fact that market players cannot observe the solvency ratio accurately. The two models we have considered, the Randomized Merton II (RM-II) and the Randomized Black-Cox II (RBC-II), both have explicit expressions for Probability of Default (PD), Loss Given Default (LGD) and Credit Spreads (CS). Because explicit formulas are available, these two Randomized Structure (RS) models are easily implemented and calibrated to the market data.

The rest of this paper is organized as follows. In Section 1, we give a brief introduction of our motivation. In Section 2, we briefly review Merton's model. In Section 3, we introduce the idea of modeling the solvency ratio. The Black-Cox model is treated in a simplified version in Section 4. In Section 5, two versions of the

RM model are introduced with different assumptions on the initial distribution. In Section 6, two versions of the RBC model are introduced, where the default time is defined similarly as in the Black-Cox model. Section 7 provides a delayed information perspective on the RS model. In Section 8, a calibration exercise is conducted. We summarize this chapter in Section 9. All proofs are given in the appendix.

2 Merton's Model

Merton (1974) assumed that the firm's value V_t follows a Geometric Brownian Motion (GBM) under the risk-neutral measure, starting from a known constant V_0 at time zero. That is

$$dV_t = rV_t dt + \sigma V_t dW_t,$$

where r denotes constant interest rate, σ is the volatility, and W_t is a Standard Brownian Motion (SBM).

The firm is obliged to pay the debt holders a constant K at maturity T. Default happens at maturity T, when the firm has insufficient funds to pay back to the debt holders at that time, namely when $V_T < K$. Thus, the probability of default $PD(T)$, as a function of maturity T, can easily be calculated through

$$\begin{aligned} PD(T) &:= P(V_T < K) \\ &= \Phi\left(-\frac{\log \frac{V_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right), \end{aligned} \tag{1}$$

where Φ stands for the Cumulative Distribution Function (cdf) of a standard normal distribution.

Following Altman, Resti & Sironi (2004), the expected recovery rate $RR(T)$ (under the risk-neutral measure), as a function of maturity T, given default at maturity T, can be evaluated as[1]

$$
\begin{aligned}
RR(T) &:= \mathbb{E}[\frac{V_T}{K}|V_T < K] \\
&= \frac{V_0}{K} e^{rT} \frac{\Phi\left(-\frac{\log\frac{V_0}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)}{PD(T)},
\end{aligned}
$$

where $PD(T)$ is the probability of default given in (1).

Expected LGD (under the risk-neutral measure), $LGD(T)$, as a function of maturity T, is defined to be $LGD(T) := 1 - RR(T)$.

Under the assumption of a constant interest rate, the TSCS in Merton's model can be expressed as

$$
\begin{aligned}
CS(T) &:= -\frac{1}{T}\log[1 - PD(T) \times LGD(T)] \qquad (2) \\
&= -\frac{1}{T}\log\left(\Phi(\frac{\log\frac{V_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) + \frac{V_0}{K}e^{rT}\Phi(-\frac{\log\frac{V_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})\right),
\end{aligned}
$$

which is a function of maturity T, interest rate r, asset volatility σ and the initial leverage ratio $\frac{K}{V_0}$. Setting interest rate r to 0.02 and initial leverage ratio $\frac{K}{V_0}$ to 0.8, Figure 1 plots Merton's TSCS given by equation (2), for varying volatility σ.

The short spread is defined to be the right limit of CS as maturity T goes to zero.

$$
CS(+0) := \lim_{T\to+0} CS(T) = \lim_{T\to+0} \frac{PD(T) * LGD(T)}{T}.
$$

If $V_0 > K$, using L'Hospital's rule, one can show that Merton's short spread is always zero. If $V_0 < K$, Merton's short spread is positive infinity.

3 Modeling The Solvency Ratio

Merton (1974) modeled the firm's asset value V_t explicitly as a GBM and the debt K_t as a constant K. As a consequence, the solvency ratio in Merton's model, i.e. $\log(V_t/K_t)$, is a Drifted Brownian Motion (DBM). Instead of specifying the dynamics

[1]This recovery rate is the recovery of face value of the bond. Other recovery rate assumptions include recovery of treasury and recovery of market value, see Duffie & Singleton (1999) for a discussion.

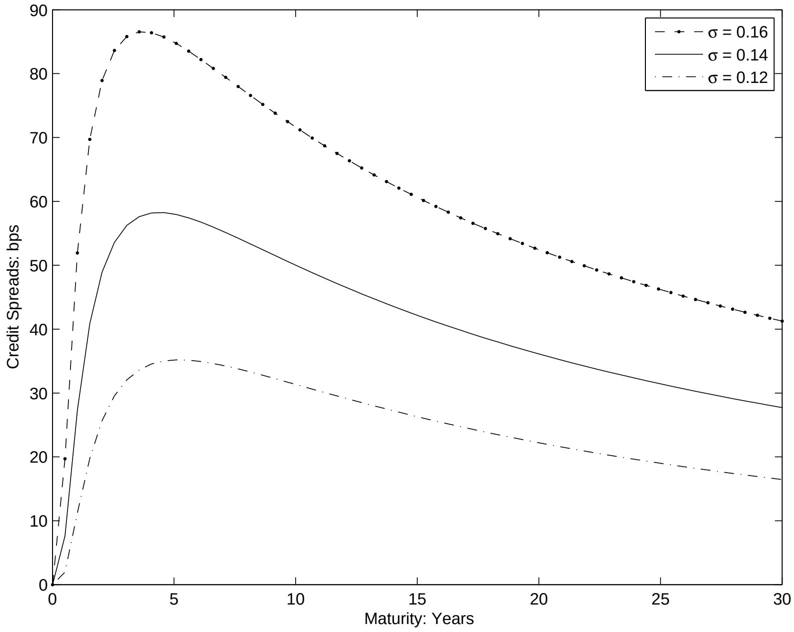

Figure 1: Merton's term structure of credit spreads, varying asset volatility σ. We set $\frac{K}{V_0} = 0.8$ and $r = 0.02$.

of the firm's asset value and the debt value separately, we can model the solvency ratio itself directly.

Assume that the solvency ratio X_t follows a DBM under the risk-neutral measure

$$X_t = x_0 + \mu t + \sigma W_t, \tag{3}$$

where x_0 is a constant. Default happens at time T if $X_T < 0$. Then, the probability of default $PD(T)$, expected recovery rate $RR(T)$ and the credit spread $CS(T)$ are given by

$$\begin{aligned} PD(T) &:= P(X_T < 0) = \Phi\left(-\frac{x_0 + \mu T}{\sigma\sqrt{T}}\right), \\ RR(T) &:= \mathbb{E}[e^{X_T} | X_T < 0] = \frac{\Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right) e^{x_0+\mu T+\frac{1}{2}\sigma^2 T}}{\Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)}, \\ CS(T) &= -\frac{1}{T}\log\left(\Phi(\frac{x_0 + \mu T}{\sigma\sqrt{T}}) + \Phi(-\frac{x_0 + \mu T + \sigma^2 T}{\sigma\sqrt{T}}) e^{x_0+\mu T+\frac{1}{2}\sigma^2 T}\right). \end{aligned}$$

This is exactly Merton's model with the following parameter constrains:

$$\begin{aligned} x_0 &= \log(V_0/K), \\ \mu &= r - \frac{1}{2}\sigma^2 > -\frac{1}{2}\sigma^2. \end{aligned}$$

Our setting here is more general than Merton (1974), since we do not specify either asset or debt processes. The debt can be a constant, a random variable, such as Giesecke (2006) or a stochastic process, such as Collin-Dufresne & Goldstein (2001). Practically speaking, corporate restructuring is allowed in this model, but not in Merton's model. By Ito's lemma, it is easy to see that the drift of the solvency ratio equals the default free interest rate minus half of the squared volatility of the solvency ratio in Merton (1974). This implies that the drift of the solvency ratio has to be larger than negative half of the squared volatility of the solvency ratio in Merton's model, considering positive interest rates. However, we do not impose any restriction on the relationship between the drift and volatility of the solvency ratio. We have a broader set of admissible parameters than in Merton's case.

4 The Black-Cox Model

In Merton's model, the default event can only happen at the maturity. However, in reality, defaults could happen before the maturity of an indenture. Black & Cox (1976) then proposed the well known first passage time model. Instead of describing what Black and Cox have done exactly in the 1976 paper, we give a simplified version of the model which maintains the essence of the original one.

For a given company, let its solvency ratio X_t be a DBM given by Equation (3) (under the risk-neutral measure). In addition, we impose positivity assumption on x_0 to ensure that no default has happened up to now. The risk-neutral default time τ is defined as the first time X_t crosses the zero boundary, i.e.

$$\tau = \inf\{t \geq 0; X_t = 0\}. \tag{4}$$

For a given future time $T > 0$, the default probability $P(\tau < T | X_0 = x_0)$ can be calculated using the reflection principle of Brownian motion and it is given by

$$P(\tau < T | X_0 = x_0) = \Phi\left(-\frac{x_0 + \mu T}{\sigma\sqrt{T}}\right) + e^{-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0 - \mu T}{\sigma\sqrt{T}}\right). \qquad (5)$$

Detailed derivation could be found in Steele (2004). Note that the first term is exactly the Merton's default probability (the probability of default at T). The second term comes from possibilities of default before T.

Assuming constant risk-neutral $LGD = l$, then the TSCS is given by

$$CS(T) = -\frac{1}{T}\log[1 - lP(\tau < T | X_0 = x_0)]. \qquad (6)$$

This TSCS has similar shapes as in Figure 1. Define the default intensity λ at time zero as

$$\lambda := \frac{\partial P(\tau < T | X_0 = x_0)}{\partial T}|_{T=0}. \qquad (7)$$

Using L'Hospital's rule, one can show that $\lambda = 0$ in the Black-Cox model. Consequently, the short spread in this model is always zero.

5 Randomized Merton Model

In the classical structure models discussed previously, the solvency ratio X_t has a constant initial value $X_0 = x_0$. This means that we can fully observe the solvency ratio at current time. However, in reality, the current solvency ratio cannot be exactly observed by the market players. It is therefore reasonable to randomize the initial value X_0.

Assume that the solvency ratio X_t follows a DBM under the risk-neutral measure

$$X_t = X_0 + \mu t + \sigma W_t, \qquad (8)$$

with a random initial value X_0. At time zero, we cannot observe the initial value X_0 accurately, but instead, we observe X_0 plus some random noise. We also assume that X_0 and W_t are independent for all $t > 0$. This is a reasonable assumption, since the noise should not affect the evolution of the solvency ratio process. However, it does contaminate the information observed by market players.

In the following two subsections, the solvency ratio is assumed to follow Equation (8). The default probability is defined as in Merton's model. The interest rate is assumed to be constant and the credit spread is calculated using Equation (2). Two models with different assumptions on the initial randomization are studied respectively. We focus on how the the short spread is influenced by the randomization of the initial value.

5.1 Randomized Merton I (RM-I)

In this RM-I model, we assume the following distribution for X_0

- RM-I Assumption on X_0: $X_0 \sim N(x_0, \sigma_0^2)$.

This is a natural assumption, since the DBM is normally distributed. It follows that $X_T \sim N(x_0 + \mu T, \sigma_0^2 + \sigma^2 T)$. As in Merton's model, we define the default time to be T, if $X_T < 0$. The default probability $PD(T)$, the expected recovery rate $RR(T)$ and the credit spreads $CS(T)$ are given by

$$
\begin{aligned}
PD(T) &:= P(X_T < 0) = \Phi\left(-\frac{x_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right), \\
RR(T) &:= \mathbb{E}[e^{X_T} | X_T < 0] = \frac{e^{x_0 + \mu T + \frac{1}{2}(\sigma_0^2 + \sigma^2 T)} \Phi\left(-\frac{x_0 + \mu T + \sigma_0^2 + \sigma^2 T}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right)}{\Phi\left(-\frac{x_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right)}, \\
CS(T) &= -\frac{1}{T} \log\left(\Phi(\frac{x_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}) + \Phi(-\frac{x_0 + \mu T + \sigma^2 T}{\sqrt{\sigma_0^2 + \sigma^2 T}}) e^{x_0 + \mu T + \frac{1}{2}(\sigma_0^2 + \sigma^2 T)}\right).
\end{aligned}
$$

When $\sigma_0 = 0$, this becomes the original Merton's model. When $T = 0$, both $PD(0)$ and $RR(0)$ are positive constants. As a result, the short spread becomes positive infinity. Therefore the RM-I model is inappriate for pricing the short spread.

5.2 Randomized Merton II (RM-II)

In the RM-I model, infinite short spread is due to nonzero default probability at time zero, which in turn is due to the positive probability that $X_0 < 0$. In this RM-II model, we assume the following distribution for X_0, which has no mass on $(-\infty, 0)$.

- RM-II Assumption on X_0: its Probability Density Function (pdf) $f(x_0; y_0, \sigma_0)$ is given by

$$f(x_0; y_0, \sigma_0) = \begin{cases} \phi(x_0; y_0, \sigma_0)/\Phi(y_0/\sigma_0) & if \quad x_0 \geq 0 \\ 0 & if \quad x_0 < 0. \end{cases} \tag{9}$$

where function $\phi_(x; \mu, \sigma)$ denotes the pdf of $N(\mu, \sigma^2)$ given by

$$\phi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{10}$$

This initial randomization will ensure zero default probability at time zero, namely $P(X_0 < 0) = 0$.

Remark 5.1 *As time progresses from zero to $t \in (0, T)$, the solvency ratio X_t can be negative without triggering a default in Merton's model. Therefore, the nonnegative assumption on X_0 is not a consistent assumption for a dynamic model. Nevertheless, this assumption is reasonable for a static model which can be used to price the current short spread.*

The default probability $PD(T)$ and the recovery rate $RR(T)$ can be calculated by conditioning

$$\begin{aligned} PD(T) &:= \mathbb{E}[P(X_T < 0|X_0)], \qquad (11) \\ RR(T) &:= \frac{\mathbb{E}[\mathbb{E}[e^{X_T} 1_{\{X_T<0\}}|X_0]]}{\mathbb{E}[P(X_T < 0|X_0)]}. \qquad (12) \end{aligned}$$

The following Proposition gives explicit formulas for $PD(T)$, $RR(T)$ and $CS(T)$, as well as their asymptotics when $T \to +0$.

Proposition 5.1 *In the RM-II model, the default probability $PD(T)$, the recovery rate $RR(T)$ and the credit spreads $CS(T)$ have the following representations*[2]

$$\begin{aligned} PD(T) &= \frac{A}{\Phi(y_0/\sigma_0)}, \qquad (13) \\ RR(T) &= \frac{Be^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{A}, \qquad (14) \\ CS(T) &= -\frac{1}{T} \log \left(\frac{\Phi(y_0/\sigma_0) - A + Be^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{\Phi(y_0/\sigma_0)} \right), \qquad (15) \end{aligned}$$

[2]Pykhtin (2003) obtained a similar expression for the recovery rate in his recovery risk model.

where the function $\Phi_2(x_1, x_2, \rho)$ denotes the cdf of a bivariate normal distribution with marginal distributions being standard normal and correlation coefficent ρ and[3]

$$\begin{aligned} A &= \Phi_2\left(-\frac{y_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right), \\ B &= \Phi_2\left(-\frac{y_0 + \mu T + \sigma_0^2 + \sigma^2 T}{\sqrt{\sigma_0^2 + \sigma^2 T}}, \frac{y_0}{\sigma_0} + \sigma_0, -\frac{\sigma_0}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right). \end{aligned}$$

Moreover, we have

$$\lim_{T \to +0} \frac{PD(T)}{\sqrt{T}} = \frac{\sigma f(0; y_0, \sigma_0)}{\sqrt{2\pi}}, \tag{16}$$

$$\lim_{T \to +0} \frac{LGD(T)}{\sqrt{T}} = \frac{\sigma \sqrt{2\pi}}{4}, \tag{17}$$

$$\lim_{T \to +0} CS(T) = \frac{\sigma^2 f(0; y_0, \sigma_0)}{4}. \tag{18}$$

From the above proposition, we can see that $PD(T)$ indeed vanishes to zero as maturity T approaches zero. When there is no random noise of the initial observation, i.e. $\sigma_0 = 0$, the RM-II model reduces to Merton's model. Both $PD(T)$ and $LGD(T)$ are found to have an order of $\sqrt{T}$, as $T \to +0$. As a result, the default intensity does not exist in the RM-II model, but it can still generate positive short spread. This positive short spread has an explicit formula given by Equation (18).

The short spread only depends on σ, y_0 and σ_0, and it does not depend on the drift μ. However, if we allow y_0 to be a function of μ, the short spread may depend on μ too, as we can see in section 7. Equation (18) implies that the short spread increases when σ increases while holding other parameters constant. If we fix σ and σ_0, the short spread is a decreasing function of y_0/σ_0. The ratio y_0/σ_0 can be regarded as Distance to Default (DD). More uncertainty about the observed solvency ratio indicates higher risk and hence the short spread should be higher. This uncertainty should be measured by DD instead of σ_0. This result may also imply that a firm's credit spreads will fall after its annual report. This awaits empirical results from testing the model. The situation for σ_0 is more complicated. When σ_0 increases

[3] A series expansion of these functions are given by Vasicek (1998). We thank Michael Gordy for pointing out this.

from zero, the short spread first increases to a maximum and then decreases. The maximum is reached at a σ_0^{max}, which solves the following equation

$$(y_0^2\sigma_0^2 - 1)\Phi(y_0/\sigma_0) + y_0\phi_0 = 0.$$

This equation is obtained by setting the first order derivative of $CS(+0)$ with respect to σ_0 to be zero.

Figures 2 and 3 show term structure of credit spreads for varying σ_0 and μ respectively, while holding other parameters constant. The short spread of the RM-II model is clearly above zero as seen from both figures. Figure 2 also shows that $CS(T)$ may decrease when σ_0 increases. Some people may argue that this is counter-intuitive, since more uncertainty about the current observation should require to pay more for the protection of default. Hence the credit spread should be higher for larger σ_0. This argument is only correct if we replace the risk measure σ_0 by DD (i.e. y_0/σ_0). The credit spread is indeed a monotone decreasing function of DD. Figure 3 also demonstrates that the RM-II model is capable of generating upwarding term structure of credit spreads by choosing sufficient negative μ. Merton's model cannot produce upward increasing term structure of credit spreads because of the nonnegativity restriction on the constant interest rate. In the RM-II, however, we do not specify the dynamics of the asset value V_t. We allow $\mu + \frac{1}{2}\sigma^2$ to be negative in the RM-II model, because $\mu + \frac{1}{2}\sigma^2$ does not necessarily represent the interest rate. As a result, the RM-II model is able to generate varying shapes of term structure of credit spreads.

Figure 4 plots the short spread defined in Equation (18) as a function of y_0, σ_0 and DD. The middle picture in Figure 4 shows a hump-shaped curve of the short spread as a function of σ_0. The maximum short spread is achieved at $\sigma_0^{max} = 0.4167$, in the case when $\sigma = 0.12$ and $y_0 = 0.35$.

We have used the techniques of randomizing the initial condition of the solvency ratio and have studied two models with Merton's definition of default. The short spread in RM-I model is infinity while the RM-II model successfully generates positive short spreads. We conclude that the RM-I model is inapproriate for modeling short spreads and the RM-II model is recommended.

In order to prove Proposition 5.1, the following two lemmas are needed.

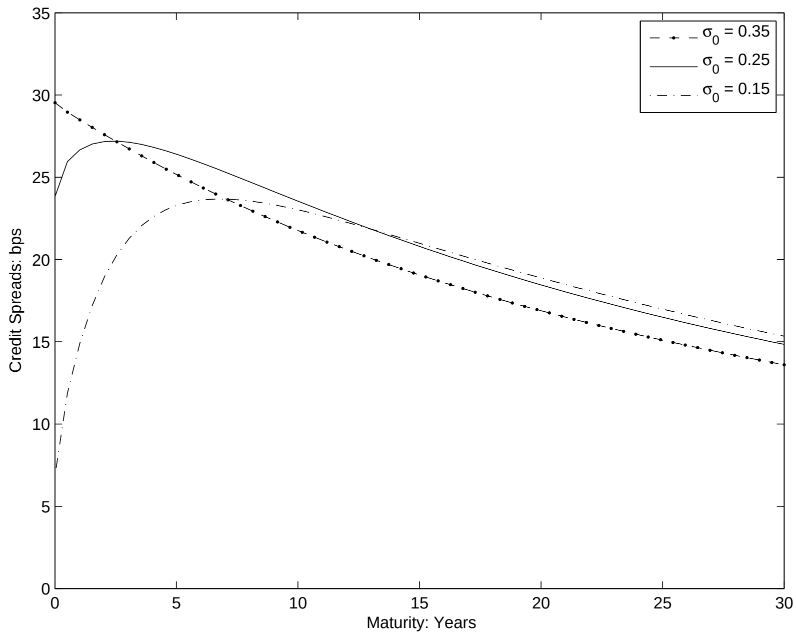

Figure 2: Term structure of credit spreads of RM-II model for varying σ_0: $\mu = 0.01$, $\sigma = 0.12$, $y_0 = 0.35$.

Lemma 5.2 *Let (X, Y) be a bivariate normal with correlation coefficient ρ and marginals $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$. Then the following equation holds*

$$\Phi_2\left(-\frac{\mu_x}{\sigma_x}, -\frac{\mu_y}{\sigma_y}, \rho\right) = \int_0^{+\infty} \Phi\left(\frac{y\rho\sigma_x/\sigma_y + \mu_y\rho\sigma_x/\sigma_y - \mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y; -\mu_y, \sigma_x)dy. \quad (19)$$

Lemma 5.3 *As $T \to +0$ the following expansion holds*

$$\Phi_2\left(-\frac{y_0 + \mu T}{\sqrt{\sigma_0^2 + \sigma^2 T}}, \frac{y_0}{\sigma_0}, \frac{-\sigma_0}{\sqrt{\sigma_0^2 + \sigma^2 T}}\right) = \frac{\sigma\phi(0; y_0, \sigma_0)\sqrt{T}}{\sqrt{2\pi}} + \frac{y_0\sigma^2\phi(0; y_0, \sigma_0)T}{4\sigma_0^2} + O(T^{3/2}). \quad (20)$$

6 Randomized Black-Cox (RBC) Model

As mentioned in the Remark 5.1, the assumption that $X_0 > 0$ is inconsistant for a dynamic model in Merton's definition of default. However, it is natural to make this assumption in the Black-Cox setting.

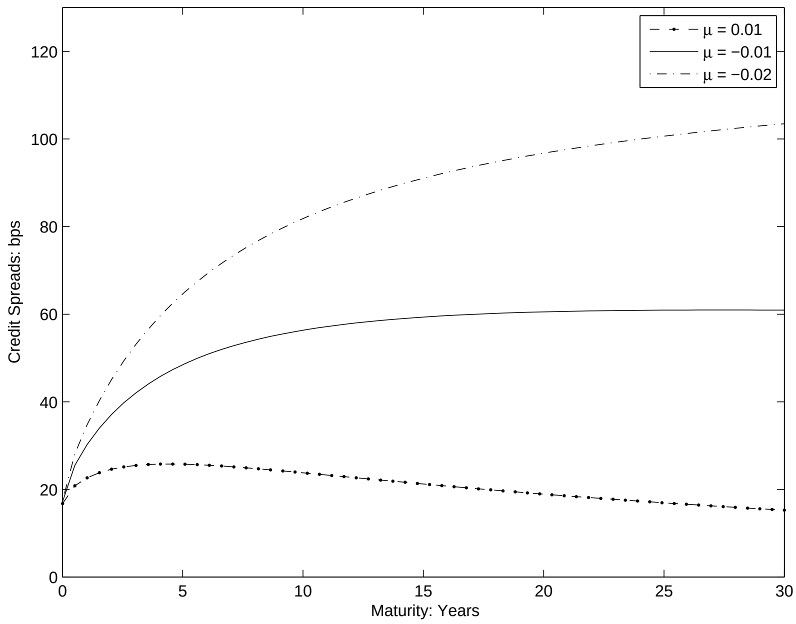

Figure 3: Term structure of credit spreads of RM-II model for varying μ: $\sigma = 0.12$, $y_0 = 0.35$, $\sigma_0 = 0.20$.

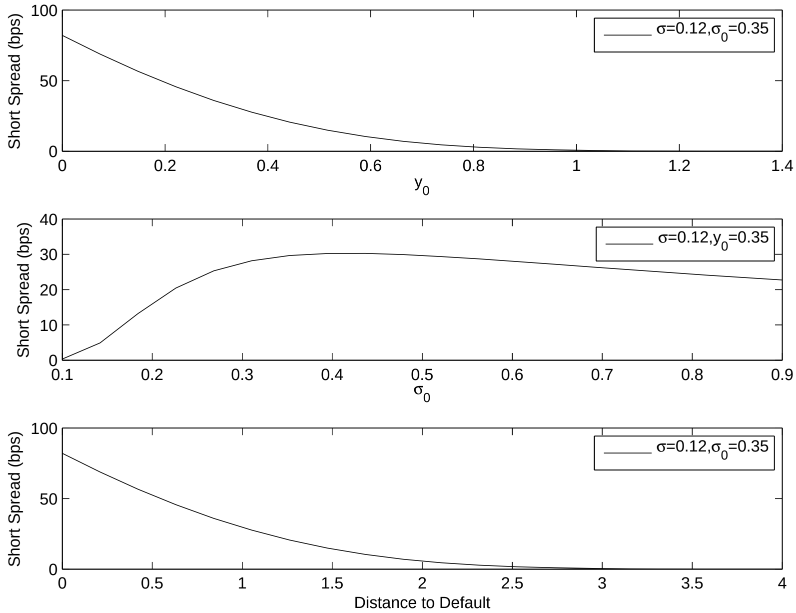

Figure 4: RM-II short spread as a function of y_0, σ_0 and DD.

In this section, we apply the randomization technique to the Black-Cox model and study two different models which have different assumptions on the initial distribution. In the following two subsections, the solvency ratio is still assumed to follow Equation (8). The default time is assumed to be the first passage time defined by Equation (4). The interest rate and the expected recovery rate are assumed to be constant and the credit spreads are calculated using Equation (6).

6.1 Randomized Black-Cox I (RBC-I)

In this RBC-I model, we make the following assumption about X_0

- RBC-I Assumption on X_0: assume the pdf of X_0 is given by Equation (9).

The probability of default $P(\tau < T)$ can be calculated through

$$P(\tau < T) = \mathbb{E}[P(\tau < T|X_0)], \tag{21}$$

where $P(\tau < T|X_0 = x_0)$ is given by Equation (5). The following Proposition gives an explicit expression of the default probability.

Proposition 6.1 *For the RBC-I model, the default probability has the following expression*

$$\begin{aligned} P(\tau < T) \;=\;& \frac{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}{\Phi(y_0/\sigma_0)} \\ &+\frac{\Phi_2\left(-\frac{y_0-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right) e^{-2\mu y_0/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}}{\Phi(y_0/\sigma_0)}. \end{aligned} \tag{22}$$

Moreover, we have[4]

$$\lim_{T\to+0} \frac{P(\tau < T)}{\sqrt{T}} = \frac{2\sigma f(0; y_0, \sigma_0)}{\sqrt{2\pi}}. \tag{23}$$

This Proposition implies that $P(\tau < T)$ has an order of $\sqrt{T}$, as $T \to +0$. For constant LGD assumption, the short spread in the RBC-I model becomes infinity.

[4]The Right Hand Side (RHS) of Equation (23) is exactly twice of the RHS of Equation (16). This implies that the ratio of the first passage default probability and the Merton's default probability is 2 when $T \to +0$, see Yi (2006) for a discussion.

Similar to the RM-I, the RBC-I model is inappropriate for modeling the short spread. Different choices of the distribution of the initial state X_0 will yield different order of convergence for $P(\tau < T)$, as $T \to +0$. The next session provides a better alternative.

6.2 Randomized Black-Cox II (RBC-II)

In this RBC-II model, we propose the following distribution for X_0:

- RBC-II Assumption on X_0: its pdf $f(x_0; a, v_0, \sigma_0)$ is assumed to be

$$f(x_0; a, v_0, \sigma_0) = \begin{cases} \frac{\phi(x_0; a+v_0, \sigma_0) - e^{-2av_0/\sigma_0^2}\phi(x_0; v_0-a, \sigma_0)}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)} & if \quad x_0 \geq 0 \\ 0 & if \quad x_0 < 0 \end{cases} \tag{24}$$

  where $\sigma_0 > 0$ and $a > |v_0|$.

Direct integration shows that $f(x_0; a, v_0, \sigma_0)$ is indeed a valid pdf. Figure 5 shows an example of the pdf.

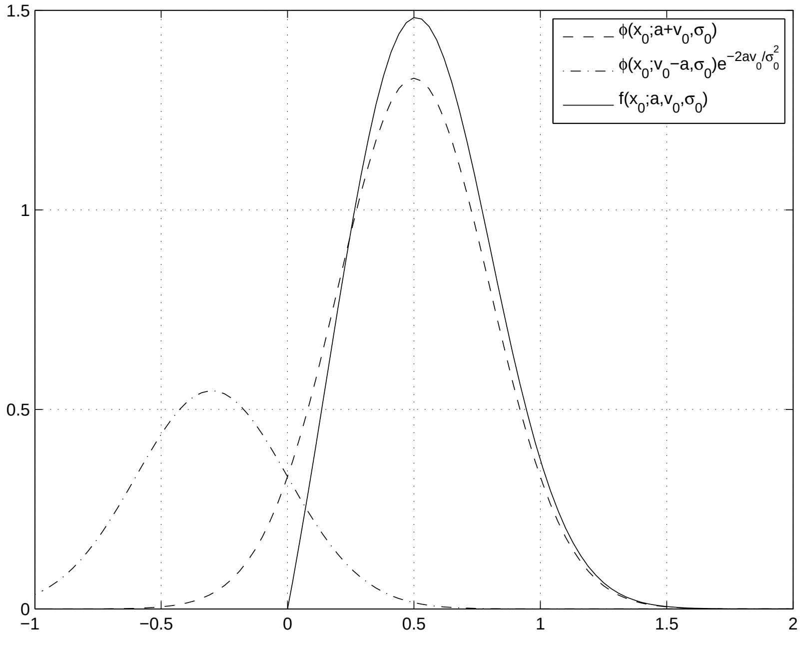

Figure 5: Probability Density Function $f(x_0; a, v_0, \sigma_0)$ with parameters $a = 0.4$, $v_0 = 0.1$ and $\sigma_0 = 0.3$.

The motivation of this distribution is that the conditional first passage probability $P(t < \tau < T|\tau > t)$ can be written as an integral of a pdf which has a form of $f(x_0; a, v_0, \sigma_0)$ defined above. Using Lemmas 5.2 and 5.3, the explicit formula for $P(\tau < T)$ and its asymptotics (as $T \to +0$) are given by the following Proposition.

Proposition 6.2 *For the RBC-II model, the default probability has the following expression*

$$P(\tau < T) = \frac{A + B - C - D}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},$$

where

$$\begin{aligned}
A &= \Phi_2\left(-\frac{a+v_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{a+v_0}{\sigma_0}, \rho\right), \\
B &= \Phi_2\left(-\frac{a+v_0-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{a+v_0-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho\right) e^{2\mu^2\sigma_0^2/\sigma^4 - 2\mu(a+v_0)/\sigma^2}, \\
C &= \Phi_2\left(-\frac{v_0-a+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{v_0-a}{\sigma_0}, \rho\right) e^{-2av_0/\sigma_0^2}, \\
D &= \Phi_2\left(-\frac{v_0-a-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{v_0-a-2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho\right) e^{2\mu^2\sigma_0^2/\sigma^4 - 2av_0/\sigma_0^2 - 2\mu(v_0-a)/\sigma^2}, \\
\rho &= \frac{-\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\lim_{T\to+0} \frac{P(\tau < T)}{T} &= \frac{a\sigma^2\phi(0; a+v_0, \sigma_0)/\sigma_0^2}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)} \\
&= \frac{\sigma^2}{2}\frac{\partial f(x_0; a, v_0, \sigma_0)}{\partial x_0}|_{x_0=0}.
\end{aligned}$$

The above Proposition ensures positive intensity for the RBC-II model. Consequently, positive short spreads are generated in the RBC-II model. Note that we have obtained an equivalent expression of the intensity as in Duffie & Lando (2001). In their model, the log of asset value follows DBM with a constant initial value z_0: $Z_t = z_0 + mt + \sigma W_t$. The default time τ is defined to be $\tau := \inf\{s > 0 : Z_s = 0\}$. Consider fixed time $t > 0$, assume the only information available is $\mathcal{H}_t := \{1_{\{\tau > s\}} : s \leq t\}$. Conditional on $\tau > t$, Z_t has a conditional density $f(\cdot)$ which is bounded and has bounded derivative with $f(0) = 0$ and $f'(0)$ is defined from the right. Duffie & Lando (2001) stated that, the default intensity $\lambda := \lim_{h \to +0} P(t < \tau \le t+h | \tau > t)/h$, is given by $\frac{1}{2}\sigma^2 f'(0)$. However, in their model, there is perfect information at time zero and they can only establish the existence of an intensity for $t > 0$. By randomizing the initial value of the solvency ratio, the RBC-II model avoids this difficulty.

Example 6.1 *When $\mu/\sigma^2 = v_0/\sigma_0^2$.*
In this case, the probability of default is reduced to

$$P(\tau < T) = \frac{\Phi\left(-\frac{a+v_0+\mu T}{\sqrt{\sigma^2 T+\sigma_0^2}}\right) + e^{-2a\mu/\sigma^2}\Phi\left(-\frac{a-v_0-\mu T}{\sqrt{\sigma^2 T+\sigma_0^2}}\right) - \Phi\left(-\frac{a+v_0}{\sigma_0}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)}{\Phi\left(\frac{a+v_0}{\sigma_0}\right) - e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},$$

where Φ_2 functions disappear and only Φ functions are involved. The reduction of dimensionalities of the integrals can be proved either by Vasicek's expansions, or by separation of integration techniques.

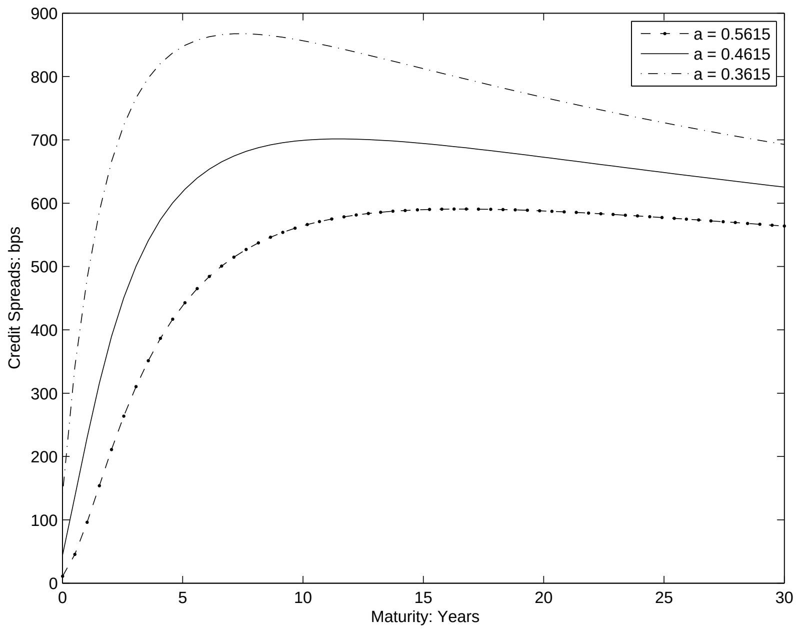

Figure 6: Term structure of credit spreads of RBC-II model for varying a: $\mu = -0.0417$, $\sigma = 0.2030$, $v_0 = 0.2402$, $\sigma_0 = 0.2162$ and $l = 1$.

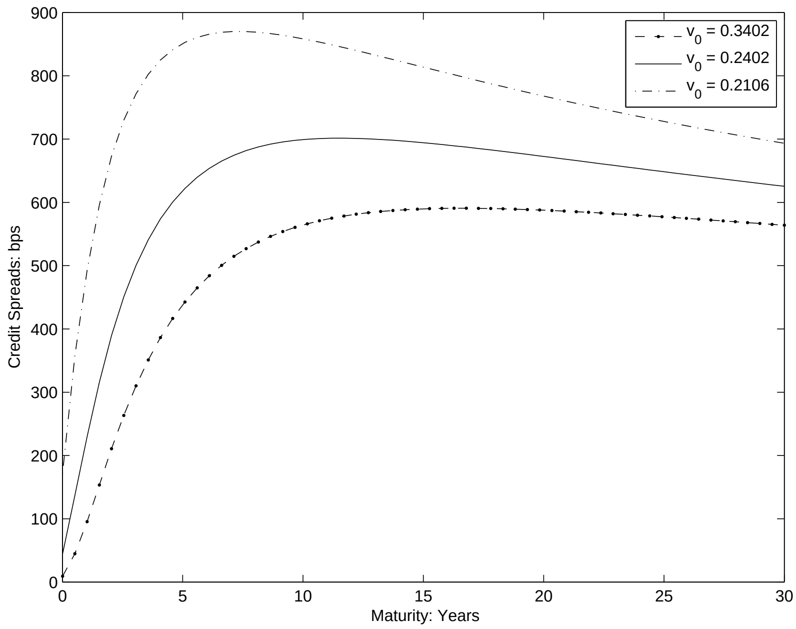

Figure 7: Term structure of credit spreads of RBC-II model for varying v_0: $\mu = -0.0417$, $\sigma = 0.2030$, $a = 0.4615$, $\sigma_0 = 0.2162$ and $l = 1$.

Figures 6 and 7 show that the TSCS increases as a or v_0 decreases. Since for smaller a or v_0, the initial distribution of X_0 has more mass close to zero. It is therefore more likely to default which in turn implies higher credit spreads. Figure 8 shows that the credit spread does not depend on σ_0 in a monotonic way. For some maturities the credit spread increases with σ_0 while it may decrease with σ_0 for other maturities. Figure 9 shows the short spread as a function of parameters a, v_0, σ_0 and σ. The short spread increases as a decreases, or v_0 decreases or σ increases. As similar to the RM-II model, the short spread of the RBC-II model first increases to a maximum and then decreases, as σ_0 increases from zero. This maximum achieved at some σ_0 which can be solved by setting $\partial CS(+0)/\partial \sigma_0 = 0$.

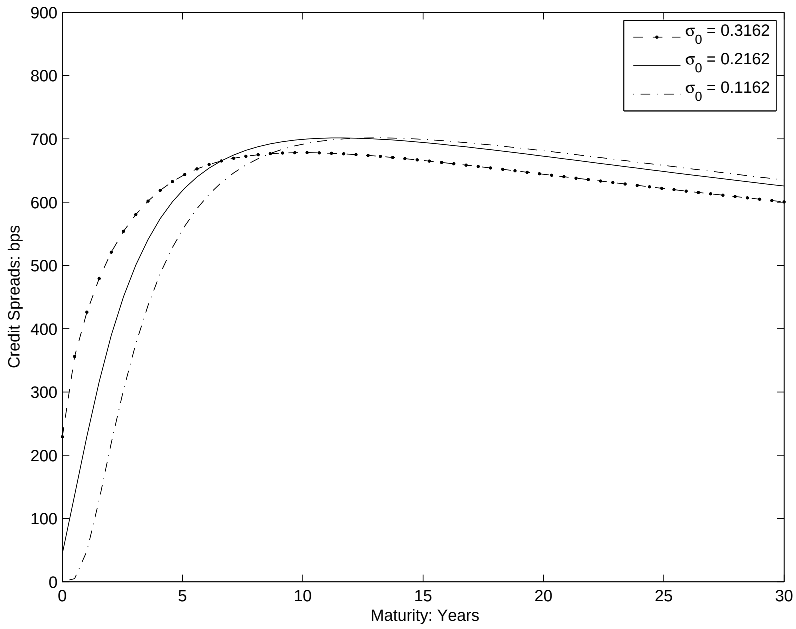

Figure 8: Term structure of credit spreads of RBC-II model for varying σ_0: $\mu = -0.0417$, $\sigma = 0.2030$, $v_0 = 0.2402$, $a = 0.4615$ and $l = 1$.

7 Delayed Information vs. Randomized Structure Model

Randomization of the initial value in the RS model may look awkward at first glance. This section gives a natural construction of the RS model through delayed information.

7.1 Delayed Information vs. RM-II

Assume the solvency ratio X_t is given by Equation (3) with constant initial value $X_0 = x_0$. Let time t be the current time and T be a future time. Let ϵ be a small positive number. At the current time t, we assume two sets of information available. First, we assume $X_t > 0$. Second, X_t is not observed, but the delayed solvency ratio $X_{t-\epsilon} = a$ is realized at time t. Then, conditional on $X_{t-\epsilon} = a$, we have

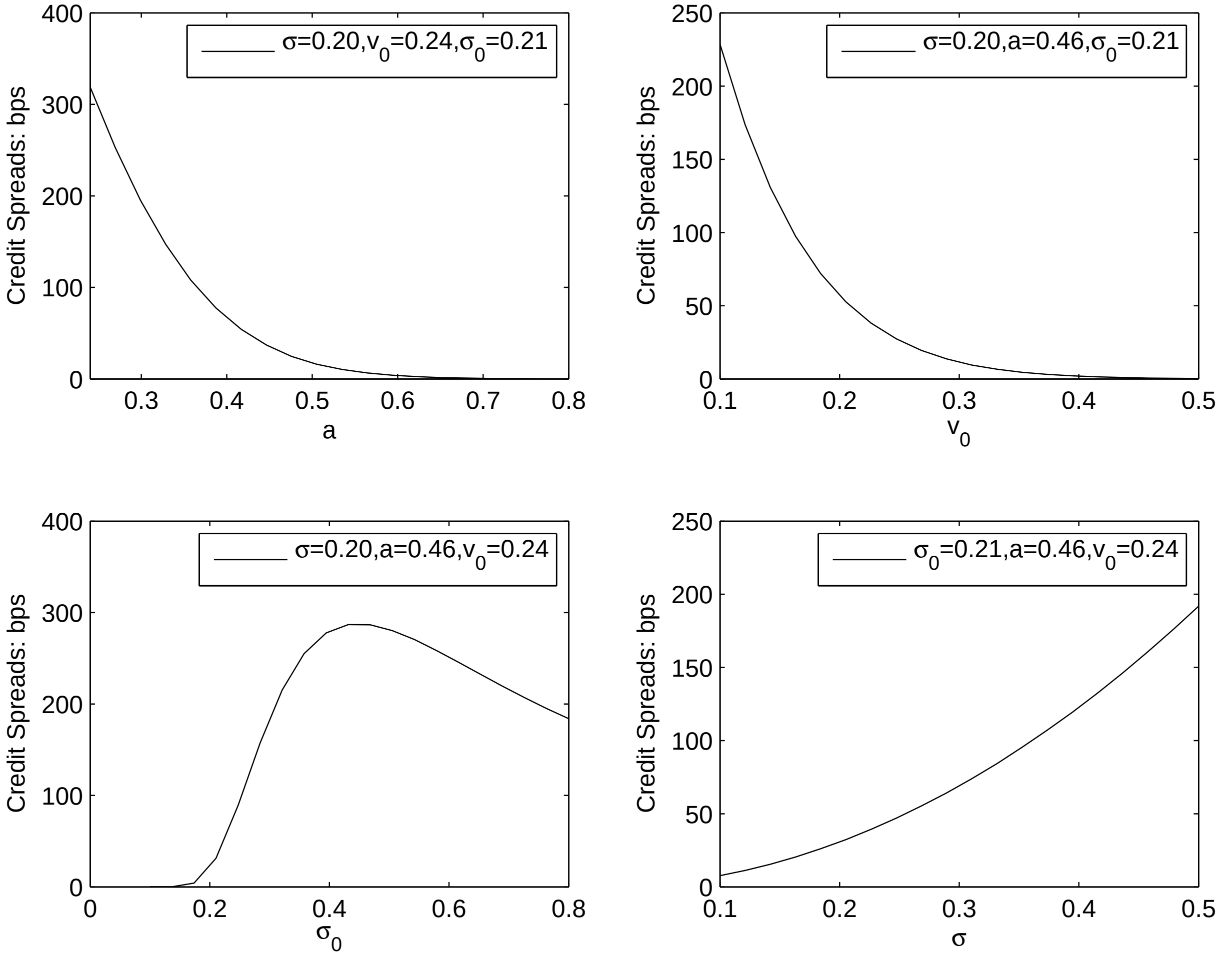

Figure 9: RBC-II short spread as a function of a, v_0, σ_0 and σ.

$X_t \sim N(a + \mu\epsilon, \sigma^2\epsilon)$. This is because

$$X_t = X_{t-\epsilon} + \mu\epsilon + \sigma(W_t - W_{t-\epsilon}).$$

The conditional default probability can be calculated through

$$\begin{aligned} P(X_T < 0 | X_t > 0, X_{t-\epsilon} = a) &= \frac{P(X_T < 0, -X_t \leq 0 | X_{t-\epsilon} = a)}{P(-X_t < 0 | X_{t-\epsilon} = a)} \\ &= \frac{\Phi_2\left(-\frac{a+\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}, \frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}, -\frac{\sqrt{\epsilon}}{\sqrt{T-t+\epsilon}}\right)}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right)}. \end{aligned}$$

This is equivalent to the RM-II model if we set $y_0 = a + \mu\epsilon$ and $\sigma_0 = \sigma\sqrt{\epsilon}$. This construction suggests that uncertainty about the current solvency ratio may come from the delayed realization of the solvency ratio. The longer the observation is delayed (i.e. for larger ϵ), the more uncertainty is the current solvency ratio (i.e. larger σ_0). The short spread derived from this construction becomes

$$\frac{\frac{\sigma^2}{4\sqrt{2\pi\sigma^2\epsilon}} e^{-\frac{(a+\mu\epsilon)^2}{2\sigma^2\epsilon}}}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right)}.$$

The short spread derived from this delayed information depends on the drift parameter μ through y_0. This is because y_0 is a linear function of μ. Figure 10 shows that the short spread increases as a or μ decreases, or σ or ϵ increases. This indicates that the longer the information is delayed, the higher the short spread.

7.2 Delayed Information vs. RBC-II

This delayed information approach can easily be extended to first passage models, where the default time τ is defined as in Equation (4). As before, X_t denotes the solvency ratio process which we assume staionary increments. At the current time t, we assume the following information are available. First, default has not happened up to now, namely $\tau > t$ is known. Second, at current time t, we can only observe the path of the solvency ratio up to a previous time $t - \epsilon$, particulary $X_{t-\epsilon} = a > 0$ is realized at time t but X_t is not. Let $\mathcal{F}_{t-\epsilon}$ denote the filtration generated by the solvency ratio process up to time $t-\epsilon$. Then, the default time for a given future time $T > t$ can be calculated through

$$\begin{aligned} P(\tau < T|\mathcal{F}_{t-\epsilon}, \tau > t) &= \frac{P(t < \tau < T|\mathcal{F}_{t-\epsilon})}{P(\tau > t|\mathcal{F}_{t-\epsilon})} \\ &= \frac{P(\epsilon < \tau < T - t + \epsilon|X_0 = a)}{P(\tau > \epsilon|X_0 = a)} \\ &= \frac{P(\tau < T - t + \epsilon|X_0 = a) - P(\tau < \epsilon|X_0 = a)}{1 - P(\tau < \epsilon|X_0 = a)}. \end{aligned} \tag{25}$$

The above formula can be calculated explicitly if $P(\tau < \epsilon|X_0 = a)$ has explicit expression. The default intensity is then given by

$$\lambda_t = \frac{\partial P(\tau < \eta + \epsilon|X_0 = a)/\partial\eta|_{\eta=0}}{P(\tau > \epsilon|X_0 = a)}. \tag{26}$$

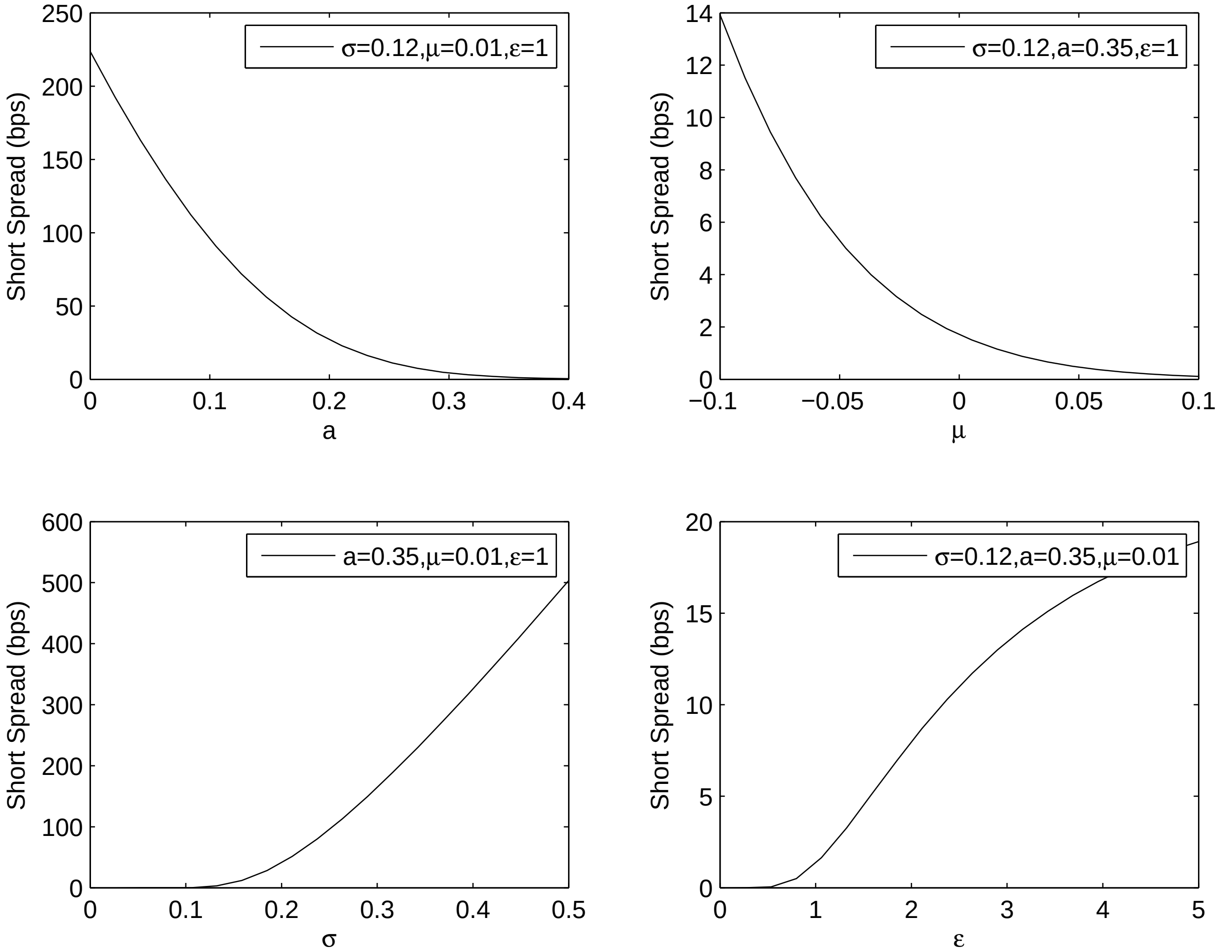

Figure 10: The short spread curve derived from delayed information as a function of a, μ, σ and ϵ.

Note that the numerator is exactly the pdf of the first passage time taking value at ϵ. We thus have obtained positive intensities through delayed information.

Consider the case when X_t is a DBM. We know that $P(\tau < \epsilon | X_0 = a)$ has explicit formula given by Equation (5). Then the default probability is given by

$$
\begin{aligned}
&P(\tau < T | \mathcal{F}_{t-\epsilon}, \tau > t) = \\
&\frac{\Phi\left(-\frac{a+\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}\right) + e^{-2a\mu/\sigma^2}\Phi\left(-\frac{a-\mu(T-t+\epsilon)}{\sigma\sqrt{T-t+\epsilon}}\right) - \Phi\left(-\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon-a}{\sigma\sqrt{\epsilon}}\right)}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon-a}{\sigma\sqrt{\epsilon}}\right)}.
\end{aligned}
$$

This is equivalent to the RBC-II model discussed in Example 6.1 with $v_0 = \mu\epsilon$ and

$\sigma_0^2 = \sigma^2\epsilon$. These parameter constrains satisfy $\mu/\sigma^2 = v_0/\sigma_0^2$. The default intensity λ_t has the following elegant expression:

$$\lambda_t = \frac{ae^{-\frac{(a+\mu\epsilon)^2}{2\sigma^2\epsilon}}/\sqrt{2\pi\sigma^2\epsilon^3}}{\Phi\left(\frac{a+\mu\epsilon}{\sigma\sqrt{\epsilon}}\right) - e^{-2a\mu/\sigma^2}\Phi\left(\frac{\mu\epsilon-a}{\sigma\sqrt{\epsilon}}\right)} \tag{27}$$

Note that this construction only makes sense when $t > \epsilon > 0$, since $\mathcal{F}_{-\epsilon}$ is not well-defined here. We can therefore only establish the existence of an intensity for $t > 0$ through the construction of delayed information. The Duffie-Lando approach is thus equivalent to the delayed information approach.

8 Calibration Excercise

In this section, we demonstrate the incomplete information effects on TSCS by fitting the theoretical CS in both the RM-II model and the RBC-II model to the observed CDS spread curve. A thorough investigation of an optimal calibration procedure is beyond the scope of this paper.

In the RM-II model, the parameters of interest are μ, σ, y_0 and σ_0. In the RBC-II model, the parameters of interest are μ, σ, a, v_0 and σ_0. In this calibration exercise, we set the $LGD = 1$ for the RBC-II model and the Black-Cox model.

For comparison, we also fit the Merton's CS and the Black-Cox's CS to the market data. The parameters of interest in both the Merton and the Black-Cox models are μ, σ and y_0. However, in fact, only two parameters need to be calibrated in Merton's model because μ and σ are related through $r = \mu + 0.5\sigma^2$, where the short rate r is proxied by the 1-month U.S Treasury yield. Parameters are calibrated by minimizing the cross-sectional Mean Absolute Error(MAE).

We take the CDS curve of Ford Motor Corp. on March 16, 2007, with 0.25-, 1-, 2-, 3-, 4-, 5-, 7-, and 10-year maturities.[5] The short rate r is taken to be 5.18%, which is the 1-month U.S Treasury yield on March 16, 2007.

[5]The 3-month CDS is not directly available from Bloomberg. It is interpolated from its nearest two points available, namely 1-year and 2-year CDS data.

Figures 11 and 12 show the calibrated results. It can be seen from the picture that both RM-II and RBC-II models outperform the classical structure models in fitting the CDS spreads, especially for short maturities. We also find that the RBC-II model fits better than the RM-II model. In fact, the MAE for the RBC-II, the RM-II, the Black-Cox and the Merton's model are 7 basis points (bps), 15 bps, 68 bps and 30 bps respectively. For the short end maturities, such as 3-month, both Merton's and Black-Cox's spreads are close to zero (0.3709 bps and 4 bps respectively), but catering for incomplete information allows the RM-II and RBC-II to generate a positive value of 83.327 bps and 89 bps respectively.

The calibrated parameters are illustrated in Table 1. Note that $\mu + 0.5\sigma^2$ is negative (-0.1033) in the RM-II model, which does not represent the interest rate.

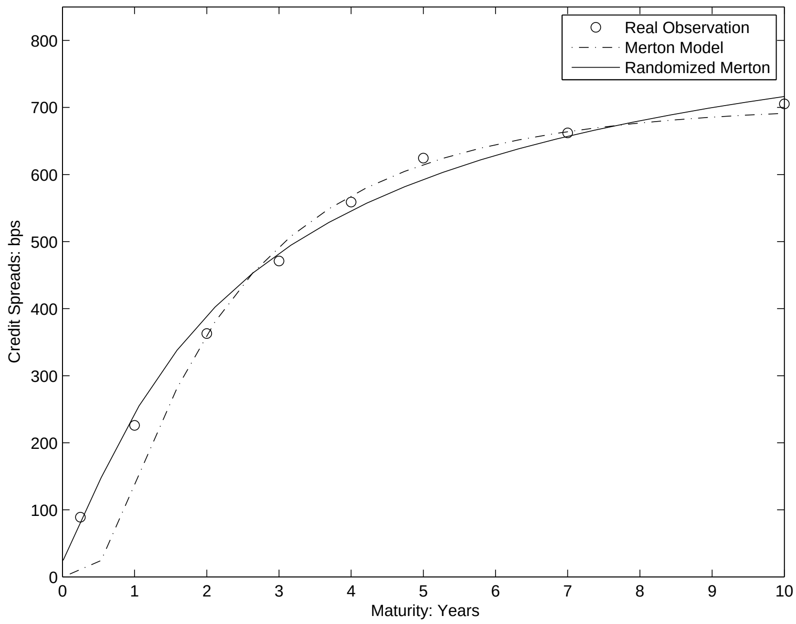

Figure 11: RM-II model vs. Merton model fit to Ford Motor CDS curve on March 16, 2007.

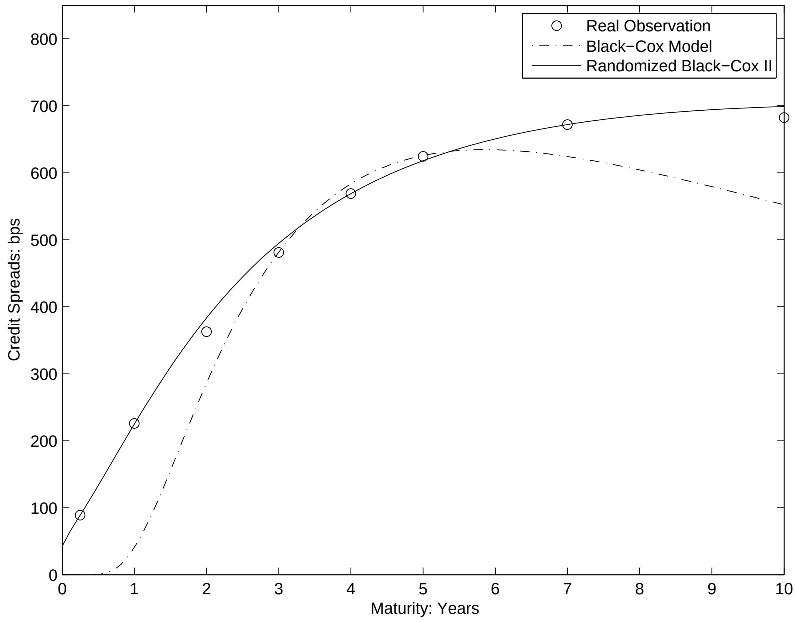

Figure 12: RBC-II model vs. Black-Cox model fit to Ford Motor CDS curve on March 16, 2007.

Model	μ	σ	y_0	σ_0	v_0	a	MAE (bps)
Merton	-0.2449	0.7703	1.4852				30
RM-II	-0.1432	0.2825	0.4926	0.2045			15
Black-Cox	-0.3220	0.6288	1.9588				68
RBC-II	-0.0417	0.2030		0.2162	0.2402	0.4615	7

Table 1: Calibrated parameters for the three models.

9 Summary

Motivated by Merton's model, we proposed to treat the solvency ratio directly as the risk factor. Based on the classical structure models, we then proposed to randomize the initial value of the solvency ratio to take account of imperfect information. We have mainly looked at four different RS models, which cover two types of definition of default and different assumptions on the initial randomization.

From the RM-II and the RBC-II models, we found that positive short spreads can be produced due to imperfect observation on the solvency ratio. We also found that various shapes of the term structure of credit spreads can be generated. The PD, the LGD and the CS are given in explicit formulae for both models, which have explicit expressions for their short spreads. In the RM-II model, we found that both $PD(T)/\sqrt{T}$ and $LGD(T)/\sqrt{T}$ converge to positive constants as $T \to +0$. The default intensity is thus not defined in the RM-II model, but positive short spreads can still be produced. In the RBC-II model, we found that $PD(\tau < T)/T$ converges to a positive constant. Therefore, the default intensity does exist in the RBC-II model and it generates positive short spreads under the assumption of constant LGD.

Merton's model becomes a special case of the RM-II model while the Black-Cox model is a special case of the RBC-II model. Numerical analysis and a calibration exercise illustrate that the randomized structure models outperforms the classical structure model in fitting the term structure of credit spreads, especially for short maturities. These two RS models generalize the classical structure models in two folds. First, instead of modeling the asset value and debt separately, we modeled the solvency ratio directly as a DBM. Second, imperfect information is considered and positive short spreads are generated. From the RM-I and the RBC-I, we also noticed that the short spread may become infinity if the random initial distribution has too much mass close to the default barrier.

We next provided a delayed information perspective on the RS models. The models constructed through delayed information are special cases in our general RS models in two ways. First, number of parameters are reduced due to some parameter constrains. Second, positive short spreads can be generated for only $t > 0$.

10 Appendix I

- **Proof of Proposition 5.1:**

From Equations (9-12), we have

$$
\begin{aligned}
PD(T) &= \frac{1}{\Phi(y_0/\sigma_0)} \int_0^{+\infty} P(X_T < 0 | X_0 = x_0)\phi(x_0; y_0, \sigma_0) dx_0 \\
&= \frac{1}{\Phi(y_0/\sigma_0)} \int_0^{+\infty} \Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0, \sigma_0) dx_0, \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}{\Phi(y_0/\sigma_0)}, \\
RR(T) &= \frac{\int_0^{+\infty} e^{x_0+\mu T+\frac{1}{2}\sigma^2 T}\Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0, \sigma_0) dx_0}{\Phi(y_0/\sigma_0) PD(T)} \\
&= \frac{e^{y_0+\mu T+\frac{1}{2}(\sigma_0^2+\sigma^2 T)}\int_0^{+\infty} \Phi\left(-\frac{x_0+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right)\phi(x_0; y_0+\sigma_0^2, \sigma_0) dx_0}{\Phi(y_0/\sigma_0) PD(T)} \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T+\sigma_0^2+\sigma^2 T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}+\sigma_0, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right) e^{y_0+\mu T+\frac{1}{2}\sigma^2 T+\frac{1}{2}\sigma_0^2}}{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2 T}}, \frac{y_0}{\sigma_0}, -\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 T}}\right)}.
\end{aligned}
$$

We have used Lemma 5.2 for the last steps of each calculation. Then the formula for $CS(T)$ comes in handy. For the asymptotics, Equation (16) is a direct result from Lemma 5.3. Note that $LGD(T) = 1 - RR(T)$, Equation (17) is obtained by applying Lemma 5.3 to $1 - RR(T)$. Equation (18) is a direct result of Equations (16) and (17).

- **Proof of Lemma 5.2:**

The Left Hand Side (LHS) of Equation 19 is the probability that both X and Y are less than zero, i.e. $P(X < 0, Y < 0)$. We can also calculate this probability by conditioning

$$
\begin{aligned}
P(X < 0, Y < 0) &= \int_{-\infty}^{0} P(X < 0 | Y = y)\phi(y; \mu_y, \sigma_y) dy \\
&= \int_{-\infty}^{0} \Phi\left(\frac{-y\rho\sigma_x/\sigma_y + \mu_y\rho\sigma_x/\sigma_y - \mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y; \mu_y, \sigma_y) dy \\
&= \int_{0}^{+\infty} \Phi\left(\frac{y\rho\sigma_x/\sigma_y + \mu_y\rho\sigma_x/\sigma_y - \mu_x}{\sigma_x\sqrt{1-\rho^2}}\right)\phi(y; -\mu_y, \sigma_y) dy
\end{aligned}
$$

For the second step, we have used the fact that X is still normally distributed conditional on $Y = y$, i.e. $X|Y = y \sim N(y\rho\sigma_x/\sigma_y - \mu_y\rho\sigma_x/\sigma_y + \mu_x, \sigma_x\sqrt{1-\rho^2})$.

- **Proof of Lemma 5.3:**

From Lemma 5.2, the LHS of Equation (20) can be written as

$$
\begin{aligned}
LHS &= \frac{1}{\sqrt{2\pi\sigma_0^2}}\int_0^{+\infty}\Phi\left(-\frac{x+\mu T}{\sigma\sqrt{T}}\right)e^{-\frac{(x-y_0)^2}{2\sigma_0^2}}dx \\
&= \frac{\sigma\sqrt{T}}{\sqrt{2\pi\sigma_0^2}}\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)e^{-\frac{(z\sigma\sqrt{T}-\mu T-y_0)^2}{2\sigma_0^2}}dz \\
&= \sigma\sqrt{T}\phi(0;y_0,\sigma_0)\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)e^{-\frac{-2y_0z\sigma\sqrt{T}+O(T)}{2\sigma_0^2}}dz \\
&= \sigma\sqrt{T}\phi(0;y_0,\sigma_0)\int_{\mu\sqrt{T}/\sigma}^{+\infty}\Phi(-z)[1+y_0z\sigma\sqrt{T}/\sigma_0^2+O(T)]dz \\
&\to \frac{\sigma\phi(0;y_0,\sigma_0)\sqrt{T}}{\sqrt{2\pi}}+\frac{y_0\sigma^2\phi(0;y_0,\sigma_0)T}{4\sigma_0^2}+O(T^{3/2}).
\end{aligned}
$$

For the last step, we have used the following two equalities

$$
\int_0^{+\infty}\Phi(-z)dz=\frac{1}{\sqrt{2\pi}},\quad \int_0^{+\infty}z\Phi(-z)dz=\frac{1}{4}.
$$

- **Proof of Proposition 6.1:**

Recall that $P(\tau<T|X_0=x_0)$ is given by Equation (5), we have

$$
\begin{aligned}
P(\tau<T) &= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty}P(\tau<T|X_0=x_0)\phi(x_0;y_0,\sigma_0)dx_0 \\
&= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty}\Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;y_0,\sigma_0)dx_0 \\
&\quad +\frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty}e^{-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;y_0,\sigma_0)dx_0 \\
&= \frac{1}{\Phi(y_0/\sigma_0)}\int_0^{+\infty}\Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;y_0,\sigma_0)dx_0 \\
&\quad +\frac{e^{-2y_0\mu/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}}{\Phi(y_0/\sigma_0)}\int_0^{+\infty}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;y_0-2\mu\sigma_0^2/\sigma^2,\sigma_0)dx_0 \\
&= \frac{\Phi_2\left(-\frac{y_0+\mu T}{\sqrt{\sigma_0^2+\sigma^2T}},\frac{y_0}{\sigma_0},-\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2T}}\right)}{\Phi(y_0/\sigma_0)} \\
&\quad +\frac{\Phi_2\left(-\frac{y_0-2\mu\sigma_0^2/\sigma^2-\mu T}{\sqrt{\sigma_0^2+\sigma^2T}},\frac{y_0-2\mu\sigma_0^2/\sigma^2}{\sigma_0},-\frac{\sigma_0}{\sqrt{\sigma_0^2+\sigma^2T}}\right)e^{-2\mu y_0/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}}{\Phi(y_0/\sigma_0)}.
\end{aligned}
$$

Lemma 5.2 is used for the last step. From Lemma 5.3, we have

$$
\begin{aligned}
\lim_{T\to+0}\frac{P(\tau<T)}{\sqrt{T}} &= \frac{\sigma[\phi(0;y_0,\sigma_0)+\phi(0;y_0-2\mu\sigma_0^2/\sigma^2,\sigma_0)e^{-2\mu y_0/\sigma^2+2\mu^2\sigma_0^2/\sigma^4}]}{\sqrt{2\pi}\Phi(y_0/\sigma_0)} \\
&= \frac{2\sigma\phi(0;y_0,\sigma_0)}{\sqrt{2\pi}\Phi(y_0/\sigma_0)}.
\end{aligned}
$$

- **Proof of Proposition 6.2**

$$
\begin{aligned}
P(\tau<T) &= \int_{-\infty}^{+\infty} P(\tau<T|X_0=x_0)f(x_0;a,v_0,\sigma_0)dx_0 \\
&= \frac{A+B-C-D}{\Phi\left(\frac{a+v_0}{\sigma_0}\right)-e^{-2av_0/\sigma_0^2}\Phi\left(\frac{v_0-a}{\sigma_0}\right)},
\end{aligned}
$$

where A, B, C and D are given by

$$
\begin{aligned}
A &= \int_0^{+\infty}\Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;a+v_0,\sigma_0)dx_0, \\
B &= \int_0^{+\infty}e^{-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;a+v_0,\sigma_0)dx_0, \\
C &= \int_0^{+\infty}e^{-2av_0/\sigma_0^2}\Phi\left(-\frac{x_0+\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;v_0-a,\sigma_0)dx_0, \\
D &= \int_0^{+\infty}e^{-2av_0/\sigma_0^2-2x_0\mu/\sigma^2}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;v_0-a,\sigma_0)dx_0.
\end{aligned}
$$

Note that C and D can also be written as

$$
\begin{aligned}
C &= e^{-2\mu(a+v_0)/\sigma^2+2\mu^2\sigma_0^2/\sigma^4} \\
&\quad * \int_0^{+\infty}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;a+v_0-2\mu\sigma_0^2/\sigma^2,\sigma_0)dx_0, \\
D &= e^{-2av_0/\sigma_0^2-2\mu(v_0-a)/\sigma^2+2\mu^2\sigma_0^2/\sigma^4} \\
&\quad * \int_0^{+\infty}\Phi\left(-\frac{x_0-\mu T}{\sigma\sqrt{T}}\right)\phi(x_0;v_0-a-2\mu\sigma_0^2/\sigma^2,\sigma_0)dx_0.
\end{aligned}
$$

Explicit expressions for A, B, C and D are finally obtained by invoking Lemma 5.2. The asymptotic equation of $P(\tau<T)$ is obtained by using Lemma 5.3 and the following identities:

$$
\begin{aligned}
\phi(0;a+v_0,\sigma_0) &= \phi(0;v_0-a,\sigma_0)e^{-2av_0/\sigma_0^2} \\
&= e^{2\mu^2\sigma_0^2/\sigma^4-2\mu(a+v_0)/\sigma^2}\phi(0;a+v_0-2\mu\sigma_0^2/\sigma^2,\sigma_0) \\
&= e^{2\mu^2\sigma_0^2/\sigma^4-2av_0/\sigma_0^2-2\mu(v_0-a)/\sigma^2}\phi(0;v_0-a-2\mu\sigma_0^2/\sigma^2,\sigma_0).
\end{aligned}
$$

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