Mathematics 741

ASSIGNMENT 1

due Tuesday, October 8, 2019 – BEFORE CLASS BEGINS

Please begin the solution to each question on a new page and then arrange the solutions in the correct order to assist marking.

1. Use the **Method of Successive Approximations** to solve the IVP:

$$x'(t) = x, x(0) = 1.$$

What function is your sequence of approximations converging to? Is this the correct solution?

2. Let r, k, and f be real-valued, continuous functions that satisfy $r(t) \geq 0,\ k(t) \geq 0$, and

$$r(t) \leq f(t) + \int_a^t k(s)r(s)\ ds, \quad a \leq t \leq b,$$

for some fixed constants a and b. Use the **Method of Successive Approximations** to prove that

$$r(t) \leq \int_a^t f(s)k(s)\exp\left[\int_s^t k(u)\ du\right]\ ds + f(t) \quad a \leq t \leq b.$$

3. Consider $T : [-\frac{1}{2}, \frac{1}{2}] \longrightarrow \mathbb{R}$ where $Tx = x^3$. Let $\rho(x, y) = |x - y|$.

   (a) Verify that **ALL** of the hypotheses of the Contraction Mapping Theorem are satisfied, and hence conclude that there is a unique fixed point.

   (b) What is the fixed point?

   (c) Starting with $x_0 = \frac{1}{4}$, at most how many iterations does the error estimate predict that you would require to get within 0.001 of the fixed point?

   (d) How many iterations do you actually need to get within 0.001 of the fixed point if $x_0 = \frac{1}{4}$?

4. Assume that $(t, x) \in \mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^n$ is a non-empty, open, and connected set, and that $f : \mathcal{D} \to \mathbb{R}^n$.

   (a) Prove that $C^1(\mathcal{D}) \subseteq locallyLip_x(\mathcal{D}) \subseteq C(\mathcal{D})$ with respect to x.

   (b) Show by counterexamples that the inclusions are proper.

5. Consider the initial value problems:

   (a) $\frac{dy}{dt} = (t^2 + y^4)^{\frac{1}{2}}, \quad y(t_0) = y_0,$

   (b) $\frac{dy}{dt} = \frac{1}{(1-t)^2}\sin((t-1)^4 y), \quad y(t_0) = y_0.$

   (a) Discuss whether these functions are uniformly or locally Lipschitz with respect to x on $\mathbb{R} \times \mathbb{R}$ or neither.

   (b) In each case, justifying all of your assertions carefully, discuss existence, uniqueness, and continuation of solutions (i.e. the maximal interval of existence of each solution) through each point $(t_0, y_0) \in \mathbb{R}^2$. (DO NOT SOLVE) *Hint: Discuss sets of initial conditions with the same properties together.*

Extra Practice Problems (not to be submitted)

1. Consider the function $f(t, x) = t\sin(x)$ where $(t, x) \in D = \mathbb{R} \times \mathbb{R}$. Determine whether this function is locally Lipschitz with respect to x on D , uniformly Lipschitz with respect to x on D, or neither.

2. Consider the function $f(t, x) = x\sin(t)$ where $(t, x) \in D = \mathbb{R} \times \mathbb{R}$. Determine whether this function is locally Lipschitz with respect to x on D, uniformly Lipschitz with respect to x on D, or neither.

THE END