Mathematics 741

ASSIGNMENT 2

due Tuesday, Oct. 29, 2019 – BEFORE CLASS

Please begin the solution to each question on a new page and then arrange the solutions in the correct order to assist marking.

1. Let $x(t)$ and $y(t)$ denote the density at time t of a hare and a fox population, respectively, where fox eat hare. Assuming that the populations are "homogeneous" in space (i.e., "well-mixed") and that there are no other predators or prey. A simple model of this prey-predator system is:

$$\begin{aligned} x' &= rx(1 - x/K) - yf(x) \\ y' &= -dy + \eta y f(x), \end{aligned}$$

   where $r, K, \eta,$ and d are positive constants and where $x(0) > 0$ and $y(0) > 0$. Define $\mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$. Assume also that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable function for all $x \in \mathbb{R}_+$ and that $f(0) = 0$. Using existence and uniqueness theory and results on the continuation of solutions (without trying to solve) prove that

   (a) Neither the hare nor the fox population can die out within a *finite* period of time and hence conclude that all solutions remain positive.

   (b) Solutions are defined for all time $t \geq 0$.

2. Consider the planar I.V.P.

$$\begin{aligned} x' &= xf(y), \\ y' &= g(y), \end{aligned}$$

$$x(t_0) = x_0, \ y(t_0) = y_0,$$

   where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : \mathbb{R} \to \mathbb{R}$ is a uniformly Lipschitz function. Prove that the solution of this I.V.P. is unique.

3. Assume that $f : \mathcal{D} \to \mathbb{R}^n$ where $\mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^n$ is open, convex, and connected. Assume also that $f \in \mathcal{C}(\mathcal{D}) \cap Lip_x(\mathcal{D})$.

   Consider the initial value problems,

$$\begin{aligned} x'(t) &= f(t, x), \quad x(t_0) = u_1, \qquad (1) \\ x'(t) &= f(t, x), \quad x(t_0) = u_2. \qquad (2) \end{aligned}$$

   where (t_0, u_1) and (t_0, u_2) are in $\mathcal{D}$. Let $\phi_1(t)$ and $\phi_2(t)$ be the solutions of (1) and (2), respectively, with common interval of existence $\mathcal{J}$. Then, prove that

$$||\phi_1(t) - \phi_2(t)|| \leq ||u_1 - u_2|| e^{L|t - t_0|}, \quad \text{for all} \quad t \in \mathcal{J},$$

   where L is the *Lip* constant for f with respect to x on $\mathcal{D}$.

   Did you require all of the assumptions? Explain.

4. Assume that $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^n$, $f, g \in \mathcal{C}(\mathcal{D})$ where $\mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^n$ is open, convex, and connected and $(t_0, x_0) \in \mathcal{D}$. Assume also that g is uniformly $Lip_x(\mathcal{D})$ and $||f(t,x) - g(t,x)|| \leq \epsilon$ for some $\epsilon > 0$ for all $(t,x) \in \mathcal{D}$.

   Consider the initial value problems,

$$
\begin{aligned}
x'(t) &= f(t,x), \quad x(t_0) = x_0, && (3)\\
x'(t) &= g(t,x), \quad x(t_0) = x_0. && (4)
\end{aligned}
$$

   Let $\phi_1(t)$ and $\phi_2(t)$ be solutions of (3) and (4), respectively, with common interval of existence $\mathcal{J}$. Then, prove that

$$
||\phi_1(t) - \phi_2(t)|| \leq \frac{\epsilon}{L}\left(e^{L|t-t_0|} - 1\right), \quad \text{for all} \quad t \in \mathcal{J},
$$

   where L is the Lip constant for g with respect to x on $\mathcal{D}$.

   Did you require all of the assumptions? Explain.

5. Consider the translation of a solution in time.

   (a) Let $f \in C(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$, and let f be smooth enough so that the solution of the I.V.P.

$$
x' = f(x), \quad x(t_0) = \xi,
$$

   denoted $\phi(t, t_0, \xi)$ is unique for each $(t_0, \xi) \in (\mathbb{R} \times \mathcal{D})$. If $\phi(t, 0, \xi)$ denotes the solution of the I.V.P.

$$
x' = f(x), \quad x(0) = \xi,
$$

   prove that $\phi(t, t_0, \xi) = \phi(t - t_0, 0, \xi)$ for all $\xi \in \mathcal{D}$, all $t_0 \in \mathbb{R}$ and all t such that ϕ is defined.

   (b) Consider the solution of the I.V.P.

$$
x' = tx,\ x(0) = 1, \text{ denoted } \phi(t, 0, 1), \text{ i.e., } \phi(0, 0, 1) = 1,
$$

   and the solution of the I.V.P.

$$
x' = tx,\ x(2) = 1, \text{ denoted } \phi(t, 2, 1), \text{ i.e., } \phi(2, 2, 1) = 1.
$$

   Solve the two I.V.P.s and check whether $\phi(t, 2, 1) = \phi(t - 2, 0, 1)$. Does this contradict what you proved in the first part of this question? Explain.

THE END