Mathematics 741

ASSIGNMENT 3

due Tuesday, Nov. 12, 2019 – BEFORE CLASS

Please begin the solution to each question on a new page and then arrange the solutions in the correct order to assist marking.

1. Using the definition given in class of a *generalized eigenvector of grade g* of a matrix A prove: If $\underline{p}_{g+1}$ is a generalized eigenvector of grade $g+1$ of a matrix A with associated eigenvalue λ, then there exists a generalized eigenvector $\underline{p}_g$ of grade g satisfying
$$(A - \lambda I)\underline{p}_{g+1} = \underline{p}_g.$$

2. Let A be a constant $n \times n$ matrix. Define
$$\sigma = \max\{ \text{ real part of } \lambda : \text{ where } \lambda \text{ is an eigenvalue of } A\}.$$
Prove that given any $\epsilon > 0$, there is a K such that $\| e^{At} \| \leq K e^{(\sigma+\epsilon)t}$ for all $t \geq 0$. Give an example to show that in general it is not possible to find a K that works when $\epsilon = 0$.

3. Suppose that for a given continuous function $h(t)$ the system
$$x' = \begin{bmatrix} -2 & 2 \\ -1 & -3 \end{bmatrix} x + h(t)$$
has at least one solution $\psi(t)$ that satisfies
$$\sup\{\| \psi(t) \| : T \leq t < \infty\} < \infty.$$
Prove that **all** solutions satisfy this boundedness condition. State and prove a generalization of this result to the n-dimensional system $x' = Ax + g(t)$, where A is an $n \times n$ matrix with constant coefficients.

4. Consider $y'' + a(t)y = 0$, where $a(t)$ is continuous and periodic with minimum period $T > 0$. Prove that the product of the Floquet multipliers of the equivalent system is equal to 1.

5. Consider
$$\text{(P)} \qquad x' = A(t)x, \quad A(t+T) = A(t) \text{ for all } t, \text{ for some } T > 0,$$
where all of the components of $A(t)$ are continuous functions of t. Prove that a necessary and sufficient condition for (P) to have at least one nontrivial periodic solution of period T is that a Floquet multiplier must equal 1.

Extra Practice Problems (not to be submitted)

1. Solve the initial value problem $x' = Ax$, $x(2) = [1, 2, 3]^T$ where T denotes transpose and
$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. Find the general solution of $x'(t) = Ax(t) + b(t)$ where
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad b(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

THE END