Mathematics 741

ASSIGNMENT 4

due Tuesday, December 3, 2019 – BEFORE CLASS

Please begin the solution to each question on a new page.

1. Given
$$\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

   Assume that the eigenvalues of A are $\alpha, \beta \in \mathbb{R}, \alpha < \beta < 0$. Let $\underline{p}_\alpha$ and $\underline{p}_\beta$ denote the associated eigenvectors. Show that the phase portrait is a node and discuss the angle of approach to the origin paying attention to the slope of the orbits as t tends to $+\infty$ and $-\infty$. Draw a representative phase portrait. Be sure to label your graph clearly including $\underline{p}_\alpha$ and $\underline{p}_\beta$.

   Apply the result just proved to draw the phase portrait given that $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. Justify carefully.

2. Consider $\underline{x}' = A\underline{x}$, where $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$.

   In each case:

   (i) Sketch the phase portrait. Pay attention to the slope of the orbits as t tends to $+\infty$ and $-\infty$.

   (ii) Determine the set of equilibrium points. Discuss whether each equilibrium point is stable, unstable, or asymptotically stable.

3. Consider the planar period system $x'(t) = A(t)x(t)$ where the period $T = \pi$:
$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2(t) & 1 - \frac{3}{2}\cos(t)\sin(t) \\ -1 - \frac{3}{2}\sin(t)\cos(t) & -1 + \frac{3}{2}\sin^2(t) \end{bmatrix}.$$

   (a) Show that $x(t) = \begin{bmatrix} -e^{\frac{t}{2}}\cos(t) \\ e^{\frac{t}{2}}\sin(t) \end{bmatrix}$, is a solution and that this solution is unbounded as $t \to \infty$.

   (b) Find the Floquet multipliers.

   (c) Is the zero solution of this system stable or unstable? Explain.

   (d) What can you conclude about what the eigenvalues of $A(t)$ tell you about the stability of the origin?

4. Find a Liapunov function of the form $v(x, y) = \alpha x^a + \beta y^b$ to determine the local stability and if possible, global asymptotic stability of the trivial solution of
$$\begin{aligned} x' &= y^5 - x^3, && (1) \\ y' &= -x^9. && (2) \end{aligned}$$

THE END