Mathematics 741
Course Title: Methods of Applied Mathematics I

Some Background Material in Analysis

References:	Rudin,	Principles of Real Analysis
	Royden,	Real Analysis
	Goldberg,	Methods of Real Analysis

Let $\{f_k(x)\}_{k=1}^{\infty}$ be a sequence of real-valued functions defined on a set $\mathcal{S}$.

Definition: $\sum_{k=1}^{\infty} f_k(x)$ *converges pointwise* to $f(x)$ in $\mathcal{S}$ if for each $x \in \mathcal{S}$ given $\epsilon > 0$, $\exists$ (there exists) N_x such that $|\sum_{k=1}^{N} f_k(x) - f(x)| \leq \epsilon$ whenever $N > N_x$.

Definition: $\sum_{k=1}^{\infty} f_k(x)$ *converges uniformly* to $f(x)$ in $\mathcal{S}$ if given $\epsilon > 0\ \exists K$ such that $\forall$ (for all) $x \in \mathcal{S}$ and $\forall N \geq K$, $|\sum_{k=1}^{N} f_k(x) - f(x)| \leq \epsilon$.

Weierstrass Test

If there is a constant series $\sum_{k=1}^{\infty} p_k$ such that:

1. $|f_k(x)| \leq p_k\ \forall x \in \mathcal{S}$ and $k = 1, 2, \ldots$
2. $\sum_{k=1}^{\infty} p_k$ converges

then $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $\mathcal{S}$ to a function $f(x)$.

Theorem: If $\sum_{k=1}^{\infty} f_k(x)$ is a uniformly convergent series of continuous functions $f_k(x)$ defined on a set $\mathcal{S} \in \mathbb{R}^n$, then the function $f(x) = \sum_{k=1}^{\infty} f_k(x)$ is continuous on $\mathcal{S}$.

Definition: A function $\rho : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ is called a *metric* (distance function) if for every $(x, y) \in \mathcal{S} \times \mathcal{S}$:

1. $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ iff (if and only if) $x = y$,
2. $\rho(x, y) = \rho(y, x)$ (transitivity),
3. $\rho(x, y) \leq \rho(x, z) + \rho(x, y)\ \forall x \in \mathcal{S}$ (triangle inequality).

$(\mathcal{S}, \rho)$ is called a metric space.

Examples:

1. $\mathcal{S} = \mathbb{R}$, $\rho(x, y) = |x - y|$.
2. $\mathcal{S} = \mathbb{R}^2$, (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$, $\rho((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (Euclidean distance).
3. $\mathcal{S} = \mathbb{R}^2$, (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$, $\rho((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.
4. $\mathcal{S}$ is the set of continuous real-valued functions on $[a, b]$ an interval of $\mathbb{R}$,
   i.e., $\mathcal{S} = C([a, b], \mathbb{R})$. $x(t), y(t) \in \mathcal{S}$. $\rho(x, y) = \max_{a \leq t \leq b} |x(t) - y(t)|$ (Uniform metric).

Definition: A sequence of points $\{x_n\}_{n=1}^{\infty}$ in a metric space $(\mathcal{S}, \rho)$ is said to be a *Cauchy sequence* if $\forall\, \epsilon > 0\; \exists\, N$ such that $\rho(x_n, x_m) < \epsilon$ whenever $n \geq N$ and $m \geq N$.

Definition: An element $x \in \mathcal{S}$ is the *limit of a sequence* $\{x_n\}_{n=1}^{\infty}$ (i.e., $\lim_{n\to\infty} x_n = x$) if for every $\epsilon > 0$, $\exists N$ such that $\rho(x_n, x) < \epsilon\; \forall\, n \geq N$.

Definition: A metric space is *complete* if every Cauchy sequence has a limit is the space.

Examples: (1)-(4) are all complete metric spaces.
Let $\mathcal{S}$ be the set of all rational numbers in $[0, 1]$. Let $\rho(x, y) = |x - y|$ for $x, y \in \mathcal{S}$. Then the metric space $(\mathcal{S}, \rho)$ is **NOT** complete since a Cauchy sequence of rational numbers can converge to an irrational number.

Definition: Let $\mathcal{F}$ be a family of real valued functions defined on a set $\mathcal{D} \subseteq \mathbb{R}^n$. Then

(i) $\mathcal{F}$ is called *uniformly bounded* on $\mathcal{D}$ if $\exists$ a nonnegative constant M such that $\forall x \in \mathcal{D}$ and $\forall f \in \mathcal{F}$ $|f(x)| \leq M$.

(ii) $\mathcal{F}$ is called *equicontinuous* on $\mathcal{D}$ if $\forall\, \epsilon > 0$, $\exists$ a $\delta > 0$ (independent of x, y, f) such that $\forall x, y \in \mathcal{D}$ and $\forall f \in \mathcal{F}$, $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

Theorem: (**Ascoli-Arzela Lemma**)
Let $\mathcal{D}$ be a closed, bounded subset of $\mathbb{R}^n$ and let $\{f_m\}$ be a real valued sequence of functions in $C(\mathcal{D}, \mathbb{R})$. If $\{f_m\}$ is equicontinuous and uniformly bounded on $\mathcal{D}$, then there is a subsequence $\{m_k\}$ and a function $f \in C(\mathcal{D}, \mathbb{R})$ such that $\{f_{m_k}\}$ converges to f uniformly on $\mathcal{D}$.

Definition: A partially ordered set $(\mathcal{A}, \leq)$, consists of a set $\mathcal{A}$ and a relation $\leq$ on $\mathcal{A}$ such that for any a, b, and c in $\mathcal{A}$,

(i) $a \leq a$,

(ii) $a \leq b$ and $b \leq c$ implies that $a \leq c$,

(iii) $a \leq b$ and $b \leq a$ implies that $a = b$.

A *chain* is a subset $\mathcal{A}_0$ of $\mathcal{A}$ such that for all a and b in $\mathcal{A}_0$, either $a \leq b$ or $b \leq a$.
An *upper bound for a chain* $\mathcal{A}_0$ is an $a_0 \in \mathcal{A}$ such that $b \leq a_0\; \forall b \in \mathcal{A}_0$.
A *maximal element* for $\mathcal{A}$, if it exists, is an element a_1 of $\mathcal{A}$ such that $\forall b \in \mathcal{A}$, $a_1 \leq b$ implies that $a_1 = b$.

Zorn's Lemma: If each chain in a partially order set $(\mathcal{A}, \leq)$ has an upper bound, then $\mathcal{A}$ has a maximal element.

Definition: If $\mathcal{M}$ is a subset of a metric space $(\mathcal{S}, \rho)$ and $T : \mathcal{M} \to \mathcal{M}$, then T is a *contraction* on $\mathcal{M}$, if there is a nonnegative number $0 \leq \alpha < 1$ such that for all x and y in $\mathcal{M}$, $\rho(Tx, Ty) < \alpha\rho(x, y)$.

Theorem: (**Contraction Mapping or Banach Fixed Point Theorem**)
If $\mathcal{M}$ is a closed subset of a complete metric space $(\mathcal{S}, \rho)$ and $T : \mathcal{M} \to \mathcal{M}$ is a contraction, then T has a unique fixed point x^* (i.e., $\exists!$ (there exists a unique) $x^* \in \mathcal{M}$ such that $Tx^* = x^*$).
Furthermore, given any $x^0 \in \mathcal{M}$, x^* is the limit of the sequence of iterates $\{x^0, Tx^0, T^2x^0, T^3x^0, \ldots\}$, and

$$\rho(T^k x^0, x^*) \leq \frac{\alpha^k \rho(Tx^0, x^0)}{1 - \alpha}, \text{ where } \alpha \text{ is a contraction constant.}$$

PROOF: (**Method of Successive Approximations**)

Select any $x^0 \in \mathcal{M}$. We show that the sequence $\{T^n x^0\}_{n=1}^{\infty}$ converges to a fixed point of T. Define $d = \rho(Tx^0, x^0)$ and let $0 \leq \alpha < 1$ denote the contraction constant. We show that $\{T^n x^0\}_{n=1}^{\infty}$ is a Cauchy sequence.

$$\begin{aligned}
\rho(T^{k+1}x^0, T^k x^0) &\leq \alpha\rho(T^k x^0, T^{k-1}x^0) \\
&\leq \alpha^2\rho(T^{k-1}x^0, T^{k-2}x^0) \\
&\vdots \\
&\leq \alpha^k\rho(Tx^0, x^0) \\
&= \alpha^k d.
\end{aligned}$$

Assume that $m = n + k$ for integer $k > 0$.

$$\begin{aligned}
\rho(T^m x^0, T^n x^0) &= \rho(T^{n+k}x^0, T^n x^0) \\
&\leq \rho(T^{n+k}x^0, T^{n+k-1}x^0) + \rho(T^{n+k-1}x^0, T^{n+k-2}x^0) + \cdots + \rho(T^{n+1}x^0, T^n x^0) \\
&\leq \alpha^n d(\alpha^{k-1} + \alpha^{k-2} + \cdots + 1) \\
&\leq \alpha^n d\frac{1}{1-\alpha} \to 0 \text{ as } n \to \infty,
\end{aligned}$$

recalling that the geometric series: $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ if $0 \leq \alpha < 1$.
Therefore, $\{T^n x^0\}$ is a Cauchy sequence. Since $\mathcal{M}$ is a closed subset of a complete metric space, $\mathcal{M}$ is complete. Therefore, $\lim_{n\to\infty} T^{n+1}x^0 = x^*$ exists where $x^* \in \mathcal{M}$.

Since T is a contraction, T is continuous. (Prove this for homework).
Therefore, $Tx^* = T(\lim_{n\to\infty} T^n x^0) = \lim_{n\to\infty} T^{n+1}x^0 = x^*$, and so x^* is a fixed point.
Uniqueness: Suppose $\exists$ distinct fixed points x^* and y^*.
$\rho(x^*, y^*) = \rho(Tx^*, Ty^*) \leq \alpha\rho(x^*, y^*) < \rho(x^*, y^*)$, since $0 \leq \alpha < 1$, a contradiction.
Error estimate:

$$\begin{aligned}
\rho(T^k x^0, x^*) &\leq \rho(T^k x^0, T^{k+1}x^0) + \rho(T^{k+1}x^0, T^{k+2}x^0) + \cdots + \rho(T^{k+p}x^0, x^*) \\
&\leq \alpha^k d + \alpha^{k+1}d + \cdots + \alpha^{k+p-1} + \rho(T^{k+p}x^0, x^*) \\
&\leq d\left(\sum_{h=0}^{\infty} \alpha^{k+h}\right) \text{(letting } p \to \infty) \\
&= d\alpha^k\frac{1}{1-\alpha}.
\end{aligned}$$

HOMEWORK

1. Consider $T : [0, 1] \to [0, 1]$ where $Tx = x^2$. Take metric $\rho(x, y) = |x - y|$. Notice that $T0 = 0$ and $T1 = 1$. Explain why this does not contradict the theorem!
2. Consider $T : [-\frac{1}{4}, \frac{1}{4}] \to \mathbb{R}$ where $Tx = x^2$. Let $\rho(x, y) = |x - y|$.
   (a) Verify that all of the hypotheses of the Contraction Mapping Theorem are satisfied, and hence conclude that there is a unique fixed point. What is it?
   (b) Starting with $x^0 = \frac{1}{4}$, at most how many iteration does the error estimate predict you would need to get within 0.001 of the fixed point. How many iterations do you actually need?
3. Prove that if T is a contraction, then T is continuous.