Jordan Canonical Form

Theorem:(Jordan Canonical Form) Any constant $n \times n$ matrix A is similar to a matrix J in Jordan canonical form. That is, there exists an invertible matrix P such that the $n \times n$ matrix $J = P^{-1}AP$ is in the canonical form

$$J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_s \end{bmatrix}.$$

where each Jordan block matrix J_k is an $n_k \times n_k$ matrix of the form

$$J_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}, \quad (k = 1, 2, \ldots, s).$$

The sum $n_1 + n_2 + \cdots + n_s = n$. The numbers λ_k (k=1,2,...,s) are the eigenvalues of A. If $p \neq q$ and λ_p appears on the diagonal of J_p and λ_q appears on the diagonal of J_q, then λ_p need **not** be different from λ_q. In fact, if m_j denotes the geometric multiplicity of the eigenvalue λ_j of A, then λ_j will appear on the diagonal of exactly m_j blocks of J of the form J_j of differing sizes $(n_{j_1} \times n_{j_1}), \ldots, (n_{j_{m_j}} \times n_{j_{m_j}})$ and the sum $n_{j_1} + \ldots + n_{j_{m_j}} = r_j$, where r_j denotes the algebraic multiplicity of the eigenvalue λ_j.

The linearly independent columns of the matrix P such that $P^{-1}AP = J$ are chosen as follows:

Each column of P that corresponds to the first column of each Jordan block $J_k, \;\; k = 1, \ldots, s$ is an eigenvector of A corresponding to the eigenvalue λ_k. If we call these eigenvectors $p_{k,1}$, the remaining columns of P (if any) are made up of generalized eigenvectors of A arranged in order of increasing grade and related to each other by

$$(A - \lambda_k I)p_{k,g+1} = p_{k,g}, \quad g = 1, 2, .., n_k - 1,$$

where $p_{k,g}$ denotes a generalized eigenvector of grade g corresponding to λ_k.