Math 4E03 **Assignment 2** **due March 19, 2021**

In this assignment $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ the ring of integers and $\mathbb{F}_p$ the field of integers modulo p where p is a prime number.

1. Let $\mathbb{K}$ be a field and let R be a commutative ring. Prove that if $\varphi : \mathbb{K} \to R$ is a ring homomorphism then either φ is injective (one-to-one) or $\varphi(\mathbb{K}) = \{0\}$.

2. This problem exercises your factorization skills.
(i) Factor $x^4 + 3$ into irreducible factors in $\mathbb{F}_7[x]$.
(ii) Factor $x^3 + 2x^2 + 10x + 6$ into linear factors in $\mathbb{F}_{11}[x]$.

3. Show that $x^4 - 22x^2 + 1$ is irreducible over $\mathbb{Q}$.

4. Which pf the following polynomials in $\mathbb{Z}[x]$ satisfy an Eisenstein criterion for irreducibility over $\mathbb{Q}$.
(i) $4x^{10} - 27x^3 + 24x - 18$.
(ii) $x^6 + 12x^5 - 30x + 60$.

5. Find all values of $c \in \mathbb{F}_5$ for which $\mathbb{F}_5[x]/(x^2 + cx + 1)$ is a field.

6. In this problem, we take a look at an *iterated quadratic extension* of $\mathbb{Q}$.
(i) Prove that $\mathbb{Q}(\sqrt{5} + \sqrt{7}) = \mathbb{Q}(\sqrt{5}, \sqrt{7})$.
(ii) Use part (i) to show that $[\mathbb{Q}(\sqrt{5} + \sqrt{7}) : \mathbb{Q}] = 4$.
(iii) Determine the minimal polynomial of $\sqrt{5} + \sqrt{7}$ over $\mathbb{Q}$.
$c \in \mathbb{F}_5$ for which $\mathbb{F}_5[x]/(x^2 + cx + 1)$ is a field.