What is a knot?

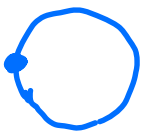
Practical answer: Take a reasonably long piece of string (say 30 cm) and tie a knot in it. Then glue the ends together so it cannot be untied. Now you have a true knot.

Thinking of the string as a single curve, it forms a knot if it is a simple closed curve in space.

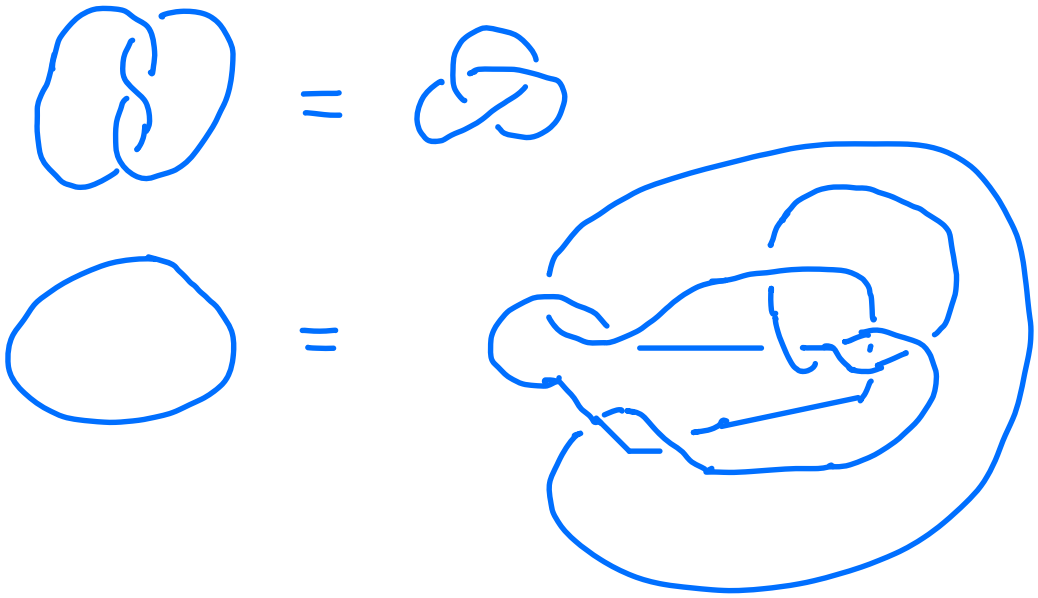
$f: [0,1] \to \mathbb{R}^3$ continuous

- $f(0) = f(1)$
- $f(x) \neq f(y)$ if $x \neq y$ in $(0,1)$

Ex. The unknot is the simple closed curve



Some knots look different but are actually the same.



Some knots look the same but are actually different.

left and right handed trefoils.

When can we change a knot A into knot B?

No cutting is allowed, just continuous deformation (called isotopy). If one knot can be manipulated by hand into another, then the two knots are said to be <u>equivalent</u>.

The physical form of the knot will not play a role, one knot might be made of 30 cm of rope and another from 10 cm of rope but they can nonetheless be equivalent. Likewise, the thickness of the rope will not play a role.

To visualize this, imagine that the string is elastic and can be stretched or shrunk (like silly putty).

As mathematical objects, knots belong in the realm of <u>topology</u>.

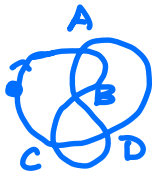
Origins of knot theory

Stretch back to the famous C.F. Gauss (1777-1855) who studied "plane curves" and developed a notation for recording the crossings of plane curves.

"Gauss word"

Gauss Word

ABCABC



ABCDBADC

Lord Kelvin developed a theory of "vortex atoms," inspired by Helmholtz

who realized at the core of every vortex is a fillament. (like the eye of a hurricane). Kelvin asked "can atoms themselves be vortices in the ether of space?"

This idea, though beautifully compelling, was eventually abandoned, as it was discredited by Einstein's theory of relativity and Maxwell's equations for electricity and magnetism. Neither theory supports the idea of ether.

Nevertheless, interest in knots had been struck, and Peter Guthrie Tait, a friend and colleague of Kelvin, produced a systematic tabulation of knots up to 10 crossings. (Tait, Little, and Kirkman)

This early work was amazing accurate even though it was entirely empiracle and they lacked the mathematical tools to make the results rigorous.

The mathematical theory of knots was developed in the early 20th century.

M. Dehn (1914) RH ≠ LH trefoils

J. Alexander (1928) polynomial

K. Reidemeister (1932) Knotentheorie
first book on knot theory

H. Seifert geometric methods

H. Schubert 1947 prime decomposition

C. Papakyriakopoulous 1957 proved Dehn's lemma

F. Waldhausen 1968 algebraic classification of knots

1960s K. Murasugi alternating knots

1970s J. Conway rational knots

1980s V. Jones polynomial invariant

W. Thurston hyperbolic knots

1988 Gordon-Luecke knots are determined by their complement.

Hoste-Thistlethwaite-Weeks
"The first 1,701,936 knots"
Math. Intell. (1998)