Recall that we defined a knot as a simple closed curve in $\mathbb{R}^3$

$f: [0,1] \longrightarrow \mathbb{R}^3$ continuous

$f(0) = f(1)$

$f(x) \neq f(y)$ if $x \neq y$ in $(0,1)$

<u>Problem</u> wild embeddings

size of knot decreases by $\frac{1}{2^n}$

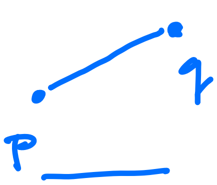
<u>WILD KNOT</u>

If wild embeddings are allowed, then a knot can shrink away.

Such an embedding is continuous but not smooth. We can avoid wild embeddings by requiring the map $f: [0,1] \to \mathbb{R}^3$ to be smooth.

Alternatively, we can avoid wild embeddings by working with <u>polygonal curves</u>.

If $p, q \in \mathbb{R}^3$ are distinct points, then let $\overline{pq}$ be the line segment from p to q.



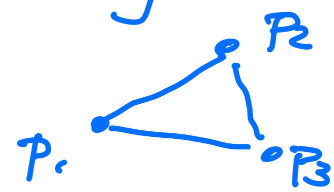
Given points $p_1, p_2, \ldots, p_n \in \mathbb{R}^3$ let $(p_1, \ldots, p_n)$ be the union of the line segments $\overline{p_1p_2}, \overline{p_2p_3}, \ldots \overline{p_{n-1},p_n}, \overline{p_np_1}$.

This is a <u>closed polygonal curve</u>.
We say the closed polygonal curve $(p_1, \dots, p_n)$ is <u>simple</u> if it has no self-intersection points.

<u>Specifically</u>, if each line segment $p_i p_{i+1}$ intersects only the <u>adjacent</u> line segments $\overline{p_{i-1} p_i}$ and $\overline{p_{i+1} p_{i+2}}$ and only at the points p_i, p_{i+1}, resp.

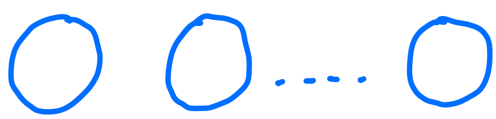
<u>Defn</u> A <u>knot</u> is a simple closed polygonal curve in $\mathbb{R}^3$.

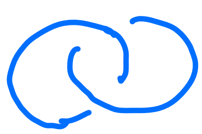
<u>Ex</u> Unknot is (p_1, p_2, p_3) for any 3 non-colinnear points.



<u>Defn</u> A <u>link</u> is a finite union of disjoint knots.

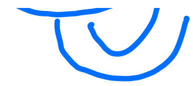
<u>Ex</u> The unlink is a union of unknots that can all be made to lie in the same 2-plane.





Hopf link

whitehead

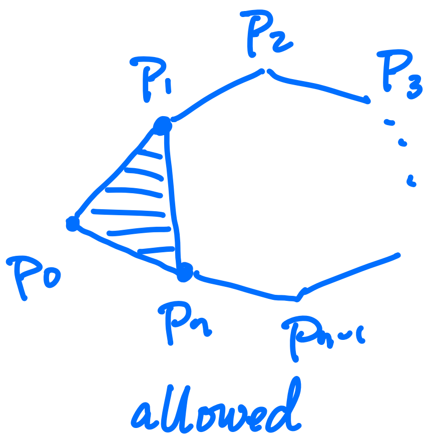


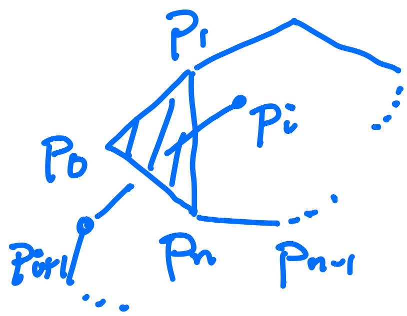
link

Deformation and equivalence of knots

If K is the knot $(p_1, \dots, p_n)$ and $p_0 \in \mathbb{R}^3$ is some other point, when is $(p_0, p_1, \dots, p_n)$ be equivalent to K?

Requirements
- $p_0 \in \overline{p_n p_1}$
- $\Delta(p_0, p_1, p_n)$ intersects K only in edge $\overline{p_n p_1}$





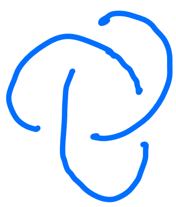
Defn ① If $\Delta(p_0, p_1, p_n)$ intersects K only in $\overline{p_1 p_n}$, then we say that $J = (p_0, p_1, \dots, p_n)$ is an elementary deformation of $K = (p_1, \dots, p_n)$.

② Two knots K and J are equivalent if there is a sequence of elementary deformations

$$K = K_0, K_1, \dots, K_N = J$$

i.e. for each i, K_{i+1} is an elementary deformation of K_i.

Notice that a polygonal curve $(p_1, \ldots, p_n)$ has a built-in direction in which to traverse the edges of the knot. This is an orientation on the knot, and usually it is indicated by placing an arrow on one arc.

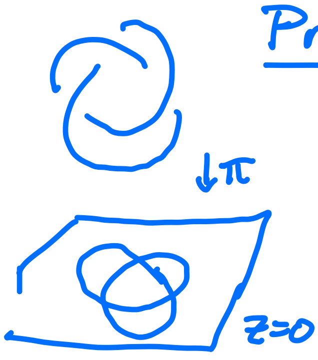


a knot

an oriented knot K

the same knot with opposite orientation

<u>Dfn</u> A knot K is <u>reversible</u> if K^r is equivalent to K.



Projections of knots

If K is a knot in $\mathbb{R}^3$ given by a simple closed polygonal curve $(p_1, \ldots, p_n)$ then the projection of K is the shadow of K in the 2-plane $\{z=0\} \cong \mathbb{R}^2$

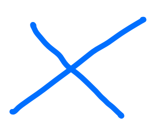
Let $\pi : \mathbb{R}^3 \longrightarrow \{z=0\} \cong \mathbb{R}^2$ be proj. map,

so $\pi(x,y,z) = (x,y,0)$
Notice that $\pi(K)$ is just a polygonal path in the xy-plane given by
$(\pi(p_1), \pi(p_2), \ldots, \pi(p_n))$.

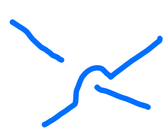
<u>Defn</u> We say that K has a <u>regular projection</u> if for any $q \in \mathbb{R}^2$, the preimage $\pi^{-1}(q) \cap K$ has at most two points, and vertices of K are never "double points".

<u>Exercise</u> Show that K has only finitely many double point if the projection is regular.

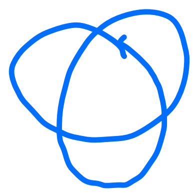
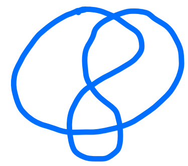
From the projection alone, it is not possible to reconstruct the knot, as too much information has been lost. However, if the projection is regular, then you can recover the knot with additional information of overcrossing/undercrossing arcs at the double points.



$\pi(K)$



<u>Question</u> How many different knots have projection

 or  ?

How many ways are there to resolve the crossings? What are chances of getting a nontrivial knot if the crossings are resolved in a random way?

<u>Defn</u> A knot projection which is regular together with over and under crossing information for each double point is called a <u>knot</u> <u>diagram</u>.

Every knot admits a knot diagram. This is intuitively clear but takes work to prove it rigorously. Read § 2.4

<u>Sketch</u>

(a) From a knot diagram, one can build a nearly flat knot using bridges for the overcrossing arcs. The original knot will be equivalent to this nearly flat knot.

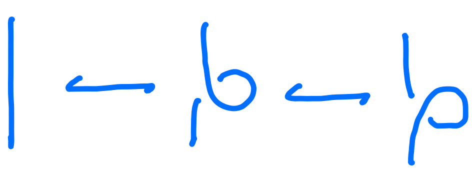
(b) If K has a regular projection, then any other knot J sufficiently close to K

will also have a regular projection.

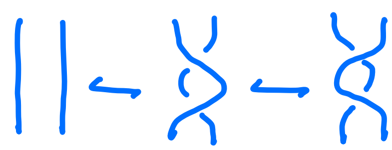
(c) If K does not have a regular projection, then we can find a knot K' arbitrarily close to K such that K' does have a regular projection.

<u>Question</u>: If D_1 and D_2 are two knot diagrams for knots K_1 and K_2, respectively, when are the knots K_1 and K_2 equivalent?

The answer was provided in 1926 by K. Reidemeister, who gave local moves on knot diagrams.



type I



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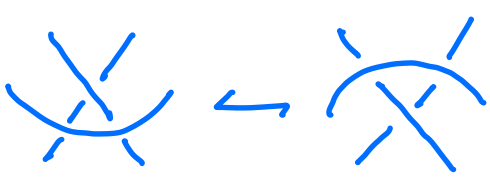
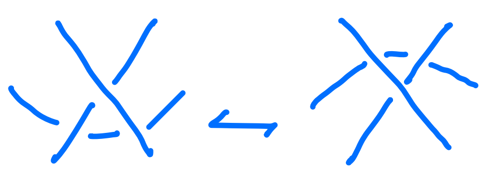


type III



type III

Theorem Let K_1 and K_2 be knots with diagrams D_1 and D_2. Then K_1 is equivalent to K_2 if and only if there is a finite sequence of Reidemeister moves relating D_1 to D_2.

Exercise Find a sequence of Reidemeister moves to transform the diagrams into the unknot.

A

B