Recall from last time that two knot diagrams represent equivalent knots iff they are related by Reidemeister moves. One must also consider planar isotopy of the diagrams.

Today we introduce various concepts from geometric topology.

Are you familiar with the notions of

(a) topological space

(b) smooth manifold?

$$S^3 = \{(x,y,z,t) \in \mathbb{R}^4 \mid x^2+y^2+z^2+t^2=1\}$$

smooth submfd of $\mathbb{R}^4$ cut out by one eqn.

Defn If X, Y are topological spaces, then a homeomorphism is a map $f: X \to Y$ that is a continuous bijection whose inverse is also continuous.

- f is one-to-one, f is onto
- f is continuous, $f^{-1}: Y \to X$ is continuous.

Ex A knot K is a subset of $\mathbb{R}^3$ that is homeomorphic to $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

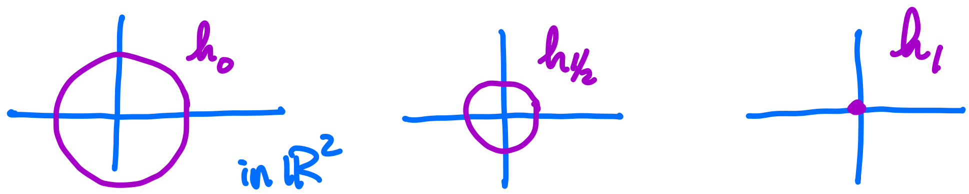
Defn If X, Y are smooth manifolds, a diffeomorphism is a map $f: X \to Y$ that is differentiable, a bijection, and the inverse map is also differentiable.

Defn A homotopy of a space $X \subseteq \mathbb{R}^n$ is a continuous map $h : X \times [0,1] \longrightarrow \mathbb{R}^n$ such that $h_0(x) := h(x,0) = x$, ie $h_0 = id_{(x\in X)}$

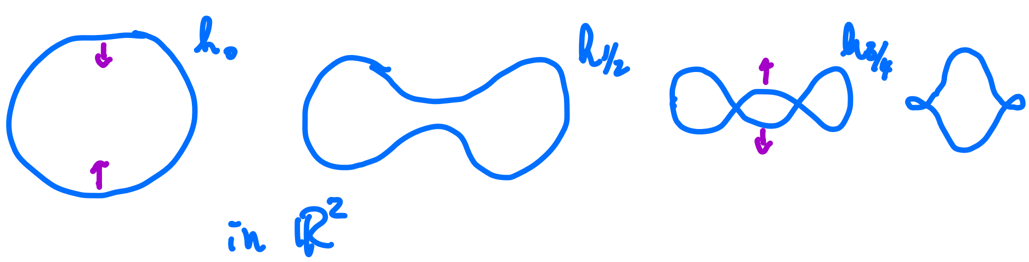
[Let $h_t(x) = h(x,t)$. In that way, we can think of h_t as a 1-parameter family of maps $h_t : X \longrightarrow \mathbb{R}^n$.]

Examples $X = S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

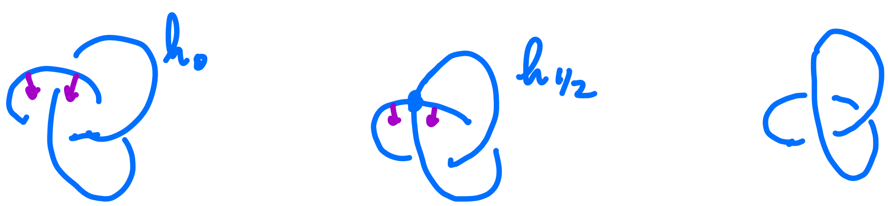
(1) $h(u,t) = (1-t)u$ $\qquad u = (x,y) \in S^1$



(2)



(3)



trefoil $\qquad$ sing. knot $\qquad$ unknot diagram

Observe : The knot type is <u>not</u> preserved under a homotopy.

Reason : The "singular" knot at time $t = 1/2$ allows us to perform a crossing change.

Unknotting number of trefoil is 1, but unknown for some low-crossing knots such as 10_{11} (either 2 or 3), $11n_3$ (either 1 or 2), and $11n_{141}$ (either 1, 2 or 3).
[KNOTINFO !!].

<u>Defn</u> A map $f: X \to Y$ is an <u>embedding</u> if it is one-to-one, continuous, and gives a homeomorphism from X to $f(X) \subseteq Y$.

<u>Ex</u> 1. A knot is an embedding of S^1 into $\mathbb{R}^3$

2. $(0,1) =$  $f: (0,1) \to \mathbb{R}^2$ is not an embedding. 

(f is one-to-one and continuous, but it is not a homeomorphism onto its image.)

<u>Defn</u> An <u>isotopy</u> is a homotopy $h: X \times [0,1] \to \mathbb{R}^n$ such that each $h_t: X \to \mathbb{R}^n$ is an embedding.

<u>Ex</u> $t=0$ $t=1/2$ $t=3/4$ $t=15/16$ $t=1$

A problematic isotopy.

Ambient isotopy

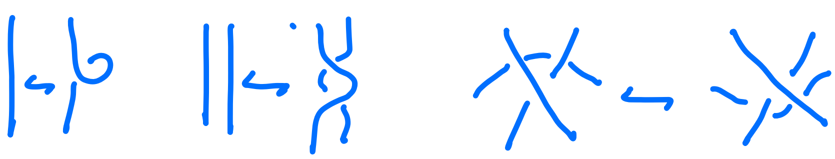
We say K_1 and K_2 are ambient isotopic if there is an isotopy $h: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ with $h_0(K_1) = K_1$

$$h_1(K_1) = K_2$$

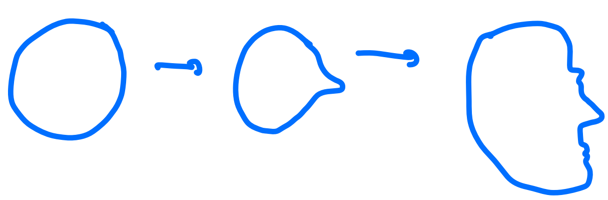
Two knots are equivalent if there is an ambient isotopy of $\mathbb{R}^3$ carrying one to the other.

Theorem Two knots K_1 and K_2 are equivalent if and only if they admit knot diagrams D_1 and D_2 which are related by a sequence of Reidemeister moves and planar isotopies.

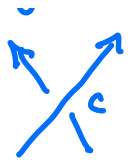
RMs



planar isotopy

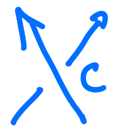


Example Fix a knot and diagram of it and also fix an orientation on the knot. Then every crossing in the diagram is either right-handed or left-handed.



RH crossing

$w(c) = 1$



LH crossing

$w(c) = -1$

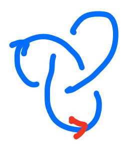
To tell difference:

(a) RH rule

(b) Rotate overcrossing arc counterclockwise, compare direction to undercrossing arc.

Defn The <u>writhe</u> of a knot diagram is the total number of positive (RH) crossings minus the total of negative (LH) crossings.

Example



$w = 3$

$w = -3$

$w = 2 - 2 = 0$

Exercise Show that the writhe is unchanged by Reidemeister moves of type 2 or 3. Under a type 1 RM, the writhe changes by ± 1.

Linking Number

Consider an oriented link $L = K_1 \cup K_2$ with two components. Consider only the crossings that involve both components, we define the linking number

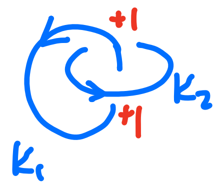
$$lk(K_1, K_2) = \frac{\#(\text{pos. crossings}) - \#(\text{neg. crossings})}{2}$$

Examples

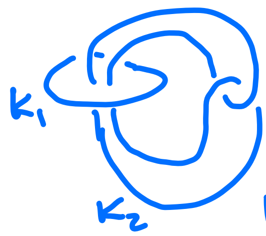
1\. 

Hopf link

$lk(K_1,K_2) = \frac{-2}{2} = -1$

2\. 

$lk(K_1,K_2) = \frac{2}{2} = 1$

3\. 

Whitehead link

$lk(K_1,K_2) = 0$

Observations:

- $lk(K_1,K_2) = lk(K_2,K_1)$
- If $-L$ is the same link but with orientation switched on both components, then the linking number is unchanged.

Exercise: The linking number $lk(K_1,K_2)$ is unchanged by planar isotopy and RMs.

$\Rightarrow$ $lk(K_1,K_2)$ is an invariant of the link.

For example, this tells us the positive Hopf link is not equivalent to the negative Hopf link or the Whitehead link.