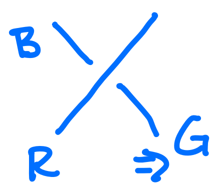
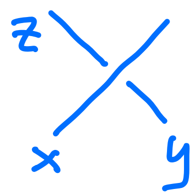
# Lecture 5

Last week we discussed tricolorability and mod $p$ labelings of knot diagrams. These properties are defined in terms of diagrams but are invariants of the knot (or link) type.




tricol.



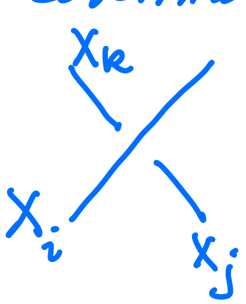

$2x - y - z \equiv 0 \pmod{p}$

mod $p$ labeling

For a knot diagram with $n$ crossings we get an $n \times n$ matrix as follows:

- label the arcs $x_1, x_2, \dots, x_n$
- number the crossing $1, 2, \dots, n$

Construct a matrix $A$ whose rows are indexed by the crossings and columns indexed by the arcs.




$\ell^{\text{th}}$ crossing $\Rightarrow$ Place a $2, -1, -1$ into the $i^{\text{th}}, j^{\text{th}}, k^{\text{th}}$ columns of the $\ell^{\text{th}}$ row of $A$.

(Degenerate cases when $i=j$ or $i=k$. In those cases, place $1, -1$ into two columns.)


The result will be an $n \times n$ matrix $A$, whose rows sum to $0$.

Let $\hat{A}$ be the $(n-1)\times(n-1)$ matrix from $A$ by removing last row and column.

<u>Defn</u> If $D$ is a knot diagram, then set $\det(D) = |\det \hat{A}|$

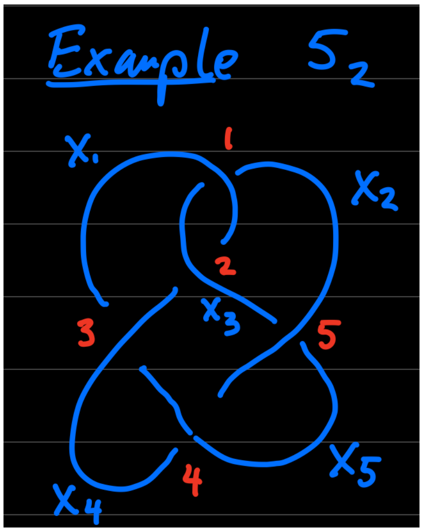
mod $p$ rank $(D)$ = nullity of $\hat{A}$ over $\mathbb{F}_p$

(here $p \neq 2$ is prime)

$(\hat{A} : \mathbb{F}_p^{n-1} \longrightarrow \mathbb{F}_p^{n-1})$

<u>Thm</u> $\det(D)$ and mod $p$ rank $(D)$ are independent of the choice of variables $x_1, \dots, x_n$ for arcs, numbering of the crossings. Moreover, these quantities are unchanged by Reidemeister moves.

$\Rightarrow$ $\det(K)$, mod $p$ rank $(K)$ are invariants of the underlying knot type.




$$\hat{A} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\det \hat{A} = (-1)\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= (-1)[8-1-2] + (-1)[2-0] = -5-2$$

$$= -7$$

$$\det(K = 5_2) = 7$$

<u>Remark</u> Notice that $K$ has a mod $p$ labeling iff $p$ divides $\det(K)$.

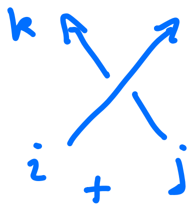
<u>Conclusion</u> $5_2$ admits a mod 7 labeling.

Q: How many mod 7 labelings does this knot admit?

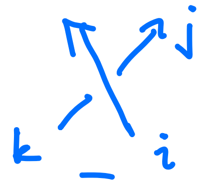
## <u>Alexander Polynomial</u>

We can use a similar approach to define the Alexander polynomial.

Orient the diagram $D$, and number arcs and crossings as before. Each crossing is either positive (RH) or negative (LH)




| $i$ | $j$ | $k$ |
|---|---|---|
| $1-t$ | $-1$ | $t$ |



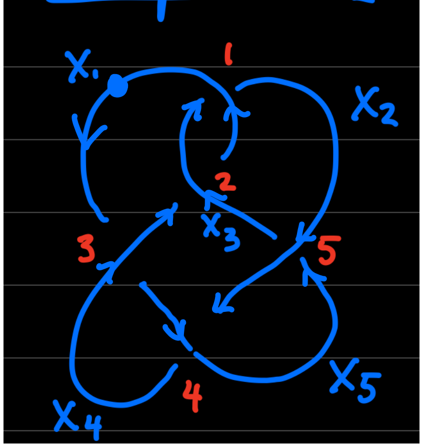

We use the above table to build an $n \times n$ matrix with entries in $\mathbb{Z}[t]$

(Degenerate case: $i=j$, add the $i^{\text{th}}$ + $j^{\text{th}}$ column entries

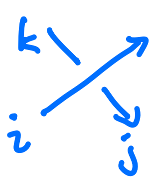
| $i=j$ | $k$ |
|---|---|
| $-t$ | $t$ |

)

<u>Example</u> $5_2$




All left handed crossings!




$$\left\{\begin{array}{lll} i & 1-t & 2 \\ j & -1 & -1 \quad (t=-1) \\ k & t & -1 \end{array}\right.$$

First crossing: $i = 1$

$j = 2$

$k = 3$

$$A = \begin{bmatrix} 1-t & -1 & t & 0 & 0 \\ -1 & 0 & 1-t & t & 0 \\ t & 0 & 0 & 1-t & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & t & 0 & -1 & 1-t \\ 0 & 1-t & -1 & 0 & t \end{bmatrix}$$

Let $\hat{A}$ be the $(n-1)\times(n-1)$ matrix obtained from $A$ by removing last row & column. Then set $\Delta_K(t) = \det(\hat{A})$ – Alexander poly.

For $K = 5_2$, $\Delta_K(t) = \det \begin{bmatrix} 1-t & -1 & t & 0 \\ -1 & 0 & 1-t & t \\ t & 0 & 0 & 1-t \\ 0 & t & 0 & -1 \end{bmatrix}$

$$= \cdots = 2t^3 - 3t^2 + 2t$$

$$= t(2t^2 - 3t + 2)$$

$$|\Delta_K(-1)| = 7 = \det(K)$$

<u>Defn</u> The Alexander matrix is the $(n-1)\times(n-1)$ matrix obtained by removal of last row and column. The Alexander polynomial is the determinant of the Alexander matrix.

It is invariant under the Reidemeister moves up to multiplication by $\pm t^m$ for $m \in \mathbb{Z}$.

<u>Remark</u> If we set $t = -1$ in the Alex. poly, we get $\det(K)$.

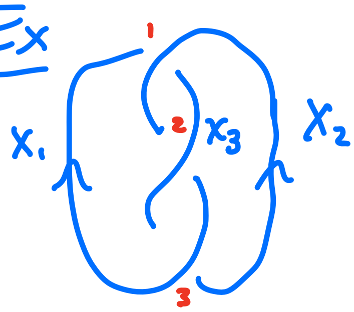
<u>Notation</u> • We write $f(t) \doteq g(t)$ if $f(t) = \pm t^k g(t)$ for some $k \in \mathbb{Z}$.

This applies to $f, g \in \mathbb{Z}[t]$ and to $f, g \in \mathbb{Z}[t, t^{-1}]$

• We write $\Delta_K(t)$ for Alexander poly of $K$. By convention, we write it as $\Delta_K(t) = a_n t^n + \dots + a_1 t + a_0$, where $a_0 \neq 0$.

Thus, $\Delta_K(t)$ is well-defined up to $\pm 1$.

<u>Ex</u>



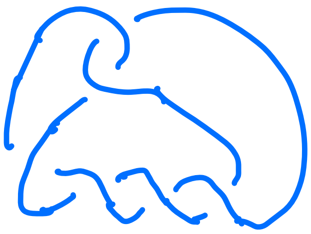

All crossings pos.

$$A = \begin{bmatrix} -1 & 1-t & t \\ t & -1 & 1-t \\ 1-t & t & -1 \end{bmatrix}$$

$$\Delta_K(t) = \det \begin{bmatrix} -1 & 1-t \\ t & -1 \end{bmatrix} = 1 - t(1-t)$$
$$= 1 - t + t^2$$

<u>Remark</u> The Alexander polynomial is remarkably effective at distinguishing low crossing knots.

For example, it separates knots up to 8 crossings, and the first duplicate pair of knots is $6_1$ and $9_{46}$ which both have $\Delta_K(t) = 2t^2 - 5t + 2$

$6_1 =$ 

$9_{46}$

- The Alexander polynomial does not detect chirality.

If $D$ is a knot diagram, the mirror knot $D^*$ is obtained by switching all the crossings of $D$.

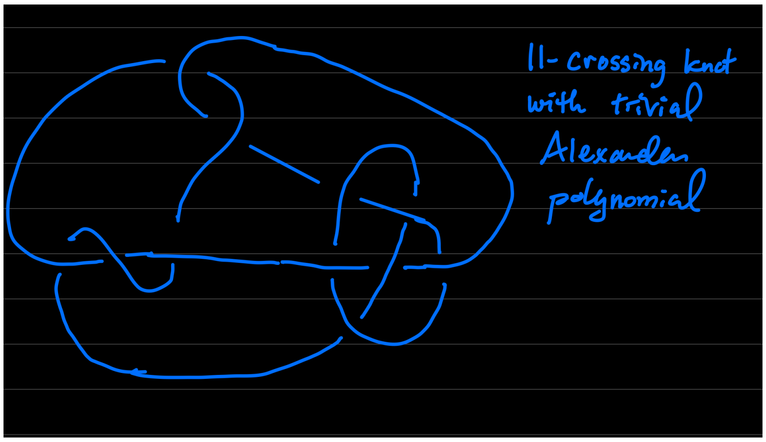
$K$ $\qquad\qquad$ $K^*$

$$\Delta_K(t) \doteq \Delta_{K^*}(t)$$

- It also does not detect the unknot.

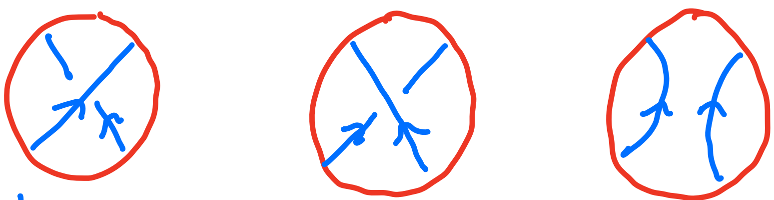
$$\Delta_U(t) = 1$$

Below is an example of a knot with 11 crossings and $\Delta_K(t) = 1$.




To verify that, we will use skein theory to compute $\Delta_K(t)$.

## Conway's polynomial



$L_+$ $\qquad$ $L_-$ $\qquad$ $L_0$

Imagine $L_+$, $L_-$, $L_0$ are 3 nearly identical link diagrams, they are equal except in the neighborhood of a crossing where they appear as above.

The Conway polynomial is written $\nabla_K(z)$ is the unique polynomial satisfying:

(1) $\nabla_U(z) \overset{!}{=} 1$ for $U$ unknot

(2) $\nabla_{L_+}(z) - \nabla_{L_-}(z) + z\,\nabla_{L_0}(z) = 0$