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Geometric Techniques in Knot theory

The use of surfaces to study knots.

We will show how to use surfaces to study knots and links in S^3. This will include both orientated and unorientated surfaces, and we will use them to recast the Alexander polynomial and determinant and to derive new invariants such as the knot signature and Arf invariants

The signature invariants are especially interesting and powerful. They are sensitive to questions about amphicheirality, and give a window into 4 dimensional topology.

Although it is not readily apparent, the Jones polynomial and Kauffman bracket are in a sense derived from surface theory. This was hinted at in the proof of the KMT theorem. When we proved equality we relied on the relationship of the checker board colouring and S_A, S_B states. State surfaces contain the knot and have boundary S and $S^\vee$

What is a surface?

A surface is a smooth manifold of dimension 2.

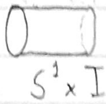
M is a surface if every point $p \in M$ is contained in a neighborhood that is homeomorphic to $D^2 = \{(x,y) \mid x^2+y^2 \le 1\} \subseteq \mathbb{R}^2$. M is a top. space and is usually assumed to be compact

Zorn's Lemma

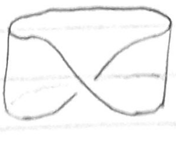
Example:

$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$ 

$T^2 = S^1 \times S^1 = \{(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{C}^2 \mid 0 \le \theta_1, \theta_2 \le 2\pi\}$ 

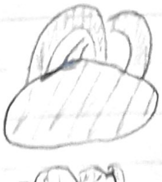
 D^2 A^2  $\simeq$  $S^1 \times I$

The surfaces S^2, T^2 are closed, ie they have empty boundary

 Möbius Strip $= M^2$

All the other surfaces drawn here have non-empty boundary

Disk-Band Surfaces



Going around the boundary gives us a knot, in this case it is the trivial knot.

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Def^n:

We write ∂M for the boundary of M

eg:

$\partial D^2 = S^1$

$\partial A^2 = S^1 \cup S^1$

$\partial M^2 = S^1$

Surfaces can be either compact or not, so far all of these surfaces have been compact. For an example of a non-compact surface consider $S^2 \setminus \{p\}$ for any $p \in S^2$ or $S^2 \setminus \{p,q\}$ where $p,q \in S^2$

Note

$S^2 \setminus \{p\} \cong \mathbb{R}^2$

$S^2 \setminus \{p,q\} \cong S^1 \times \mathbb{R}$

Def^n:

A surface M is disconnected if $\exists$ a proper non-empty subset $M' \subseteq M$ which is both open and closed.

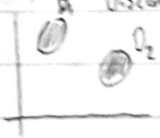
Let $M'' = M \setminus M'$ then $M = M' \cup M''$ is a disjoint union of topological spaces

Both inherit the subspace topology

eg:

In $\mathbb{R}^2$

A Disconnected surface



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Orientation and Polyhedral Surfaces

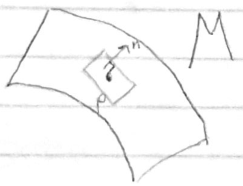
When studying surfaces we are interested in both the intrinsic and extrinsic properties of the surface. (Here imagine $M \subseteq \mathbb{R}^3$, like a ship in a bottle. Intrinsic relate to M and extrinsic relate to how M is put in $\mathbb{R}^3$)

The intrinsic properties reflect the properties that are independent of the embedding of M into $\mathbb{R}^3$. Compactness and connectedness are examples of intrinsic properties but orientability is defined extrinsically but is really an intrinsic property as well

Defn:

A surface $M \subseteq \mathbb{R}^3$ is said to be orientable if it admits a nowhere zero normal vector field.

eg:



Remark: Not all surfaces are orientable

eg: The Mobius strip is not orientable

We can think of orientability as a condition allowing us to paint the surface on both sides with two colours (black and white) in a consistent fashion

Orientability is an intrinsic property for surfaces

We can think of surfaces in terms of polyhedral surfaces in $\mathbb{R}^3$. These are surfaces built as unions of triangles (here a triangle is a convex span of three non-colinear points). The triangles $\{T_j\}$

$\in \mathbb{R}^3$

$$T = \{ap + bq + cr \mid a+b+c=1, \ 0 \le a,b,c\}$$

that comprise the polyhedral surface M satisfy

1) Two triangles T_i, T_j are either disjoint, or they intersect in a single vertex or in an edge.  or 
2) Any given edge is contained in atmost two triangles
3) The edges that sit in only one triangle have union a polygonal curve. This is the boundary of M

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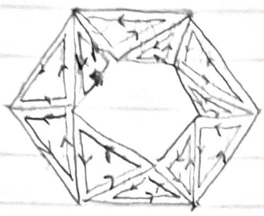
Polyhedral Surfaces

eg: The annulus as a polygonal surface



This polyhedral surface is orientatable

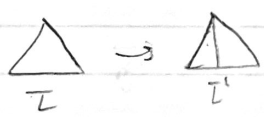
Def^n:

A polyhedral surface M is orientable if each triangle is orientated so that whenever two triangles meet in a common edge, the edge orientations from the two triangles are opposite.

Def^n:

If (M, τ) is a polyhedral surface with triangulation τ, then a refinement or subdivision of τ is a triangulation τ' obtained by subdividing triangles in τ.

eg:



Def^n:

Two polyhedral surfaces (M, τ) and (N, σ) are said to be homeomorphic if after subdividing there is a bijection ϕ from the vertices of (M, τ') to those in (N, σ') so that $p, q, r \in M$ lie on a face of τ' iff $\phi(p), \phi(q), \phi(r) \in N$ also lie on a face of σ'

Question:

When are two polyhedral surfaces going to be homeomorphic?

Answer:

Classification of surfaces.

Question:

When can we deform the surface M into another surface N?