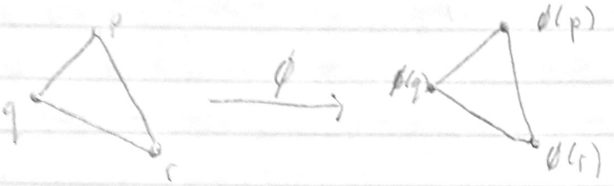
Feb 15

Classification of Compact Surfaces, Euler Characteristic

Recall:

Two polyhedral surfaces M, N are homeomorphic if after subdivision there is a bijection from the vertices of M to those of N such that any three vertices in M which bound a face, the corresponding vertices in N also bound a face.



Question:

Given two polyhedral surfaces how do we tell if they're homeomorphic?

Answer:

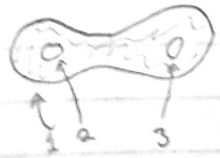
Classification of compact surfaces.

Remark:

1) If M is a compact connected surface with boundary, then each component of the boundary of M, ∂M, is a circle. If N is homeomorphic to M then it will have the same number of components in ∂N.

2) If N and M are homeomorphic then M is orientable iff N is orientable

eg:

M= 

flat surface

∂M has 3 components

Defn:

If M is polyhedral with triangulation T having V vertices, E edges, F faces then the Euler characteristic $\chi(M) = V-E+F$

eg:

V=4

E=6 $\quad \chi(M) = 4-6+4 = 2$

F=4

Feb 15

Genus

Remark:

Notice that under subdivision the Euler characteristic remains unchanged

eg: $M' =$  $V' = V+1$, $E' = E+2$, $F' = F+1$ $\Rightarrow \chi(M') = V' - E' + F' = V+1-E-2+F+1 = V-E+F$

$M =$ 

So $\chi(M)$ is independent of triangulation

Theorem:

If M, N are homeomorphic then $\chi(M) = \chi(N)$

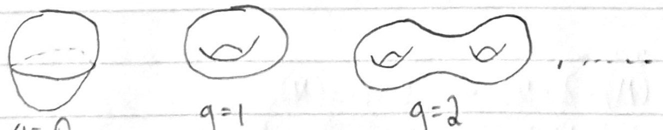
Def^n:

Let M be a connected orientated surface with B boundary components. Then the genus of M is given by

$$g(M) = \frac{2 - \chi(M) - B}{2}$$

Remark:

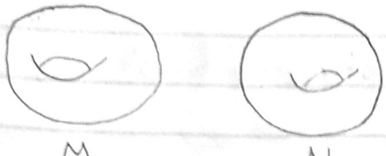
If M is compact and closed (ie $\partial M = \emptyset$) then this is equivalent to saying $\chi(M) = 2 - 2g(M)$.



$g=0$ $\quad$ $g=1$ $\quad$ $g=2$ $\quad$, ...

Feb 15

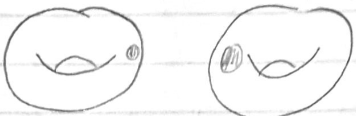
Connected Sum, Closed Surfaces



M N

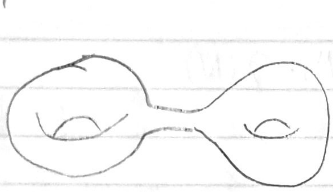
Given two surfaces M, N, we can form the connected sum $M \# N$ as follows

Step 1:
Remove a small 2-disk from M, N



M N

Step 2
Attach a cylinder $S^1 \times I$ so that it connects the two exposed circles in M, N



$M \# N$

Note:
We can use this to recursively construct Σ_g closed surfaces of genus g for all $g \geq 0$

$$\Sigma_{g+1} = \Sigma_g \# T^2$$

($T^2 =$ )

Theorem:
Two compact closed connected orientable surfaces M, N are homeomorphic iff $g(M) = g(N)$

Proof:
If M, N are homeomorphic then $\chi(M) = \chi(N) \Rightarrow g(M) = g(N)$
To prove the converse direction note that a surface M has genus g iff it is homeomorphic to the connected sum $\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g \text{ copies}}$

Remark:
Σ_g can also be constructed as a quotient of the $4g$-gon

Feb 15

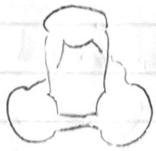
Surfaces with boundaries

The genus is the perfect invariant for closed orientable surfaces. One can use a similar approach to classify surfaces with boundary and non-orientable surfaces.

Theorem

Any compact surface M with boundary can be constructed as a union of disks and bands.

eg:



This surfaces has 2 boundary components

This surface has 1 boundary component

Theorem:

If M is a surface with boundary constructed as a union of disks and bands then $\chi(M) = \#$ of disks $- \#$ of bands.

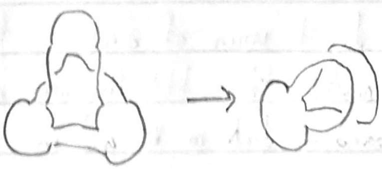
If M is orientable then its genus is, $g(M) = \frac{2-\chi(M)-B}{2} = \frac{2 - \#\text{of disks} + \#\text{of Bands} - \#\text{of boundaries}}{2}$

Theorem

Any compact connected surface with boundary can be constructed as the union of a single disk with bands attached. In this case

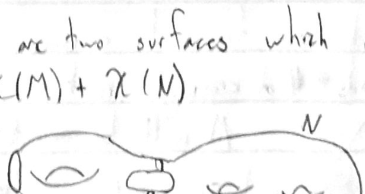
$$g(M) = \frac{1 + \#\text{of bands} - \#\text{of } \partial \text{ components}}{2}$$

eg:



Theorem

If M, N are two surfaces which meet along one or more boundary components then $\chi(M \cup N) = \chi(M) + \chi(N)$.



Feb 15

Classification of Surfaces, Seifert's Algorithm

Theorem

Suppose M and N are two connected surfaces constructed by attaching bands to a single disk. Then M and N are homeomorphic iff the following conditions are satisfied

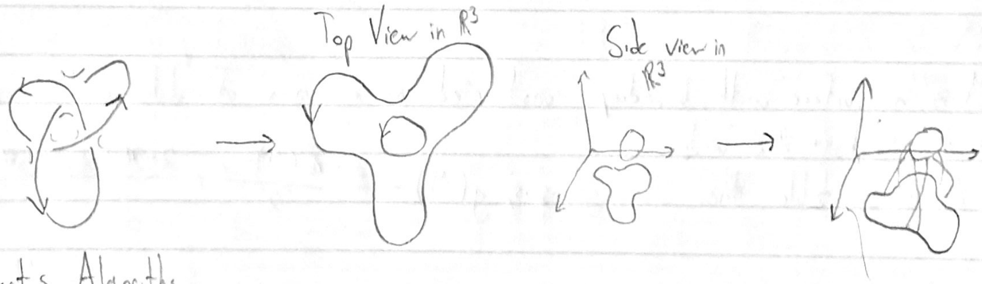
1) M and N have the same number of bands
2) M and N have the same number of boundary components
3) Both M and N are orientable or both are non-orientable

Given any knot or link in S^3 there is a construction of an oriented surface with boundary the given knot or link.

Theorem [Seifert]

Every orientated knot or link is the boundary of a compact connected orientated surface.

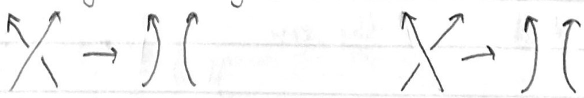
eg:



$S=2$
$C=3$
$\chi(M)=2-3=-1$
$g(M)=\frac{1-(-1)}{2}=1$

Seifert's Algorithm

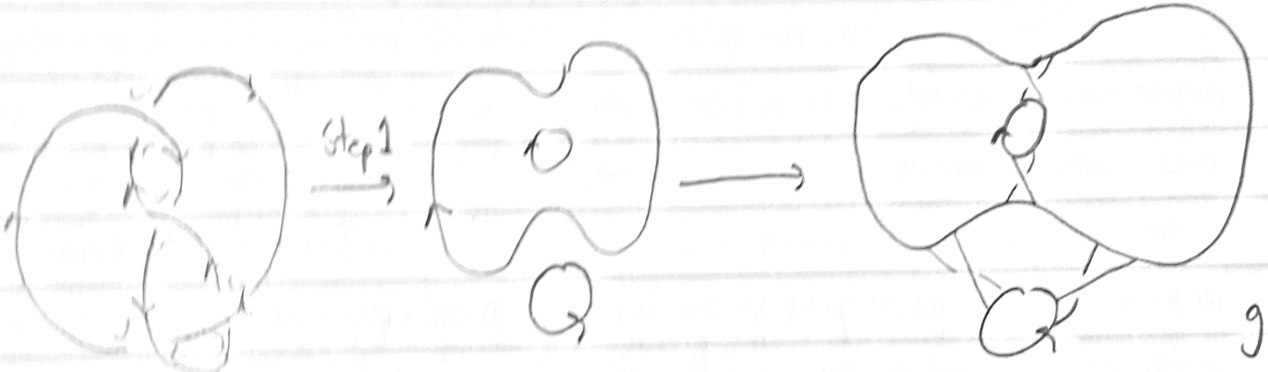
1) Smooth all crossing's according to their orientation



2) After smoothing all crossing you will have a disjoint union of circles in the plane, with the circles possibly being nested. If the circles are nested then lift the inner circles to a higher level in $\mathbb{R}^3$. Note that each circle will bound a disk in $\mathbb{R}^3$ and we can arrange them to be disjoint

3) Attach half twisted bands to the disks one for each crossing according to the crossing type. Notice that the number of bands = # of crossings in K. The resulting surface M will be orientated with $\chi(M)=S-C$ Where S= # of Seifert circles.

Ex:



$S = 3$

$C = 4$

$\chi(M) = 3 - 4$
$= -1$

$g(M) = \frac{1-(-1)}{2} = 1$