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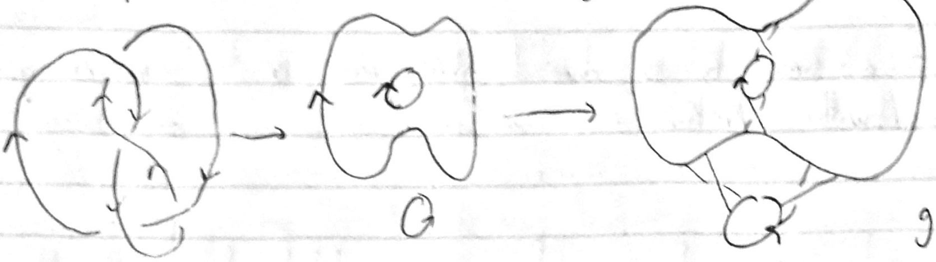
Examples of Seifert's Algorithm

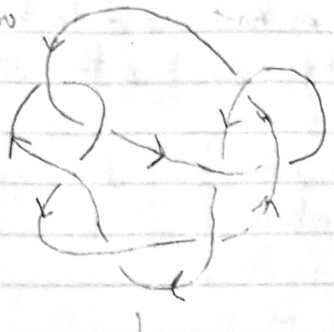
Ex:

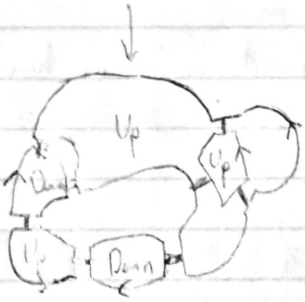


$S = 3$
$c = 4$
$\chi(M) = 3 - 4$
$= -1$

$g(M) = \frac{1-(-1)}{2} = 1$

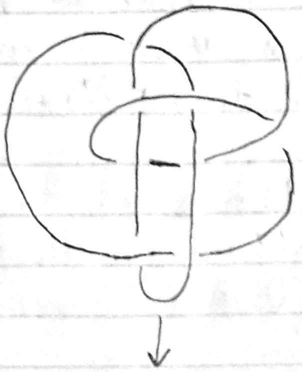
Ex:
Livingston Book

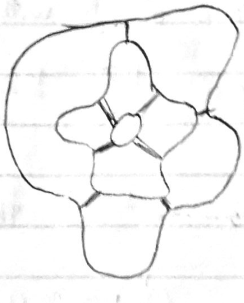




$S = 5 = \#(\text{disks})$
$c = 8 = \#(\text{bands})$
$\chi = 5 - 8 = -3$
$g = \frac{1-\chi}{2} = \frac{4}{2} = 2$

Other book





$S = 3$
$c = 8$
$\chi = 3 - 8 = -5$
$g = \frac{1-\chi}{2} = 3$

▮ = half twisted band

So  K is abstractly our surface.

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Genus of a Knot

Def^n:

The genus of a knot K is denoted $g(K)$ and is the minimum genus over all orientated surfaces M with $\partial M = K$.

Remark:

1) There is a similar notion of genus for an orientated link, with the caveat that it is an invariant of the oriented link and we require the surface M to be connected. For knots, one can assume that M is connected since the boundary is connected.
2) In general it is difficult to compute the genus of a knot (proven to be NP [2016])
3) If $g(K)=0$ then $K = \partial M$ for M a 2-disk. This implies K is the unknot
4) We saw last time that the trefoil and figure eight knots have genus 1 surfaces

   $\Rightarrow$ $g(3_1)=1$ since the trefoil is not the unknot

   $\Rightarrow$ $g(4_1)=1$ " figure-eight knot "

5) For a diagram of K with c crossings and s Seifert circles we have $\chi(M) = s-c$

$$g(K) \leq g(M) = \frac{1-\chi}{2} = \frac{1-s+c}{2}$$

If K is non trivial then $s \geq 2$ so $\frac{1-s+c}{2} \leq \frac{c-1}{2}$

$$\Rightarrow \underbrace{c(K)}_{\text{crossing number}} \geq 2\underbrace{g(K)}_{\text{genus}}+1$$

eg: For 3_1, we have equality

Additivity of the genus of Knots

Theorem:

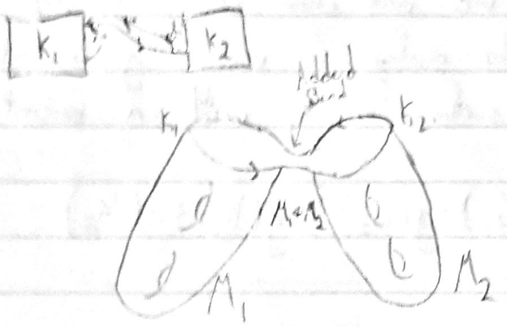
If K_1 and K_2 are two knots then $g(K_1 \# K_2) = g(K_1) + g(K_2)$

Proof:

Uses surgery theory, one direction is easy to see

$$g(K_1 \# K_2) \leq g(K_1) + g(K_2)$$

Let M_1, M_2 be minimal genus surfaces for K_1, K_2



1) $g(M_1 \# M_2) = g(M_1) + g(M_2)$ and 2) $\partial(M_1 \# M_2) = K_1 \# K_2$

Proposition

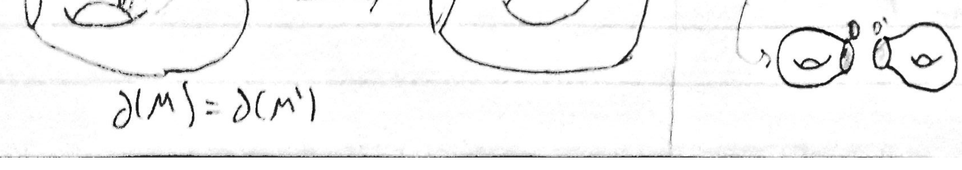
If M_1 and M_2 are two surfaces that intersect in a collection of arcs, then $\chi(M_1 \cup M_2) = \chi(M_1) + \chi(M_2) - \#(\text{arcs})$

1) follows from the above proposition

The hard part is to show $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$, this part will use surgery theory

If γ is a simple closed curve in the interior of a surface M ($\subseteq \mathbb{R}^3$) bounding a disk D with $\text{int}(D)$ disjoint from M then we can remove an annulus from M (with core curve γ) and glue in two disks D, D'.

eg:



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Remark

There are two possibilities

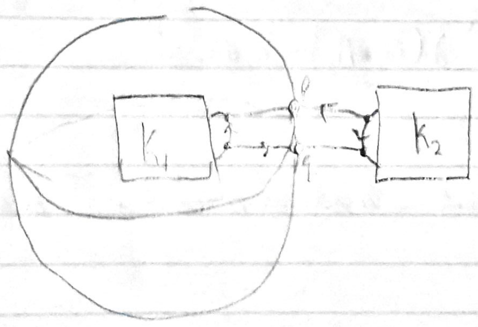
1) If γ is non-seperating on M then the genus of the new surface M' satisfies
$$g(M') = g(M) - 1$$

2) If γ is seperating, and we define the genus for disconnected surfaces as a sum of the genus of the components then $g(M') = g(M)$

Proof cont

To show $g(K_1 \# K_2) = g(K_1) + g(K_2)$ we will show that for any minimal genus seifert surface M for $K_1 \# K_2$ we can find another surface G with $g(G) = g(M)$, $\partial G = \partial M = K_1 \# K_2$ and G is a band sum of a seifert surface for K_1 with a seifert surface for K_2

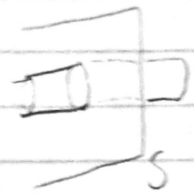
$$G = G_1 \#_b G_2 \quad \overset{g(K_1 \# K_2)}{=} g(G) = g(G_1) + g(G_2) \geq g(K_1) + g(K_2)$$



We can find a sphere S enclosing K_1 and disjoint from $K_1 \cup K_2$. We can form $K_1 \# K_2$ so the arcs pierce S in two points, p, q.

Let M be a minimal genus surface for $K_1 \# K_2$. We can modify M by surgery to obtain G a band connected sum with $g(G) = g(M)$.

By an isotopy of M and S we can arrange that $S \cap M$ consists of a finite union of arcs and circle. In fact there is only one arc α connecting p, q, all other components are circles. We can choose an innermost circle in $S^2 \setminus \alpha$. Such a circle bounds a disk with interior disjoint from M



@: innermost circle.

Perform surgery to M along an annulus with this innermost circle as core curve.
Let M' be the new surface

Then $\partial M' = \partial M = K_1 \# K_2$.

If the circle is non-separating on M, then $g(M') = g(M) - 1$ which contradicts our choice of M (minimal genus)

So the circle must be separating and we can discard the closed component from M' to get $g(M') = g(M)$