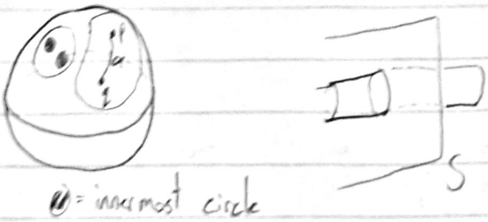
Feb 17



= innermost circle

Perform surgery to M along an annulus with this innermost circle as core curve.
Let M' be the new surface

Then $\partial M' = \partial M = K_1 \# K_2$.
If the circle is non-separating on M, then $g(M') = g(M) - 1$ which contradicts our choice of M (minimal genus)
So the circle must be separating and we can discard the closed component from M'' to get $g(M'') = g(M)$

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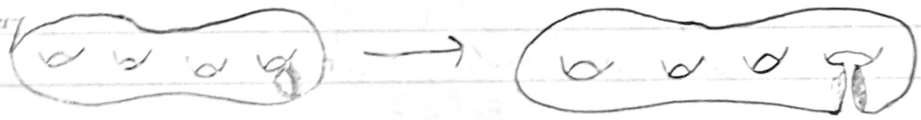
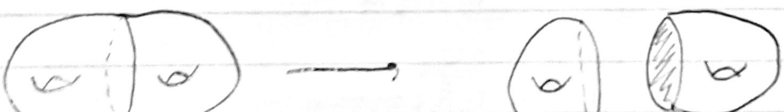
Recall:

Surgery



genus decreases



genus doesn't change

Theorem:
If $K \subset S^3$ is a knot $\exists$ a closed orientable surface F with $\partial F = K$.

Def^n:
The genus of a knot K is the minimal genus of $F \ni F$ is a compact orientable surface with $\partial F = K$. The genus of K is denoted $g(K)$.

Note: The genus of knots is difficult to compute in general.

Theorem:
$g(K_1 \# K_2) = g(K_1) + g(K_2)$ where $K_1 \# K_2$ is the connected sum of K_1 and K_2

Note:
$g(K_1 \# K_2) \le g(K_1) + g(K_2)$ by construction.

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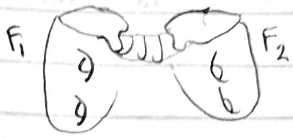
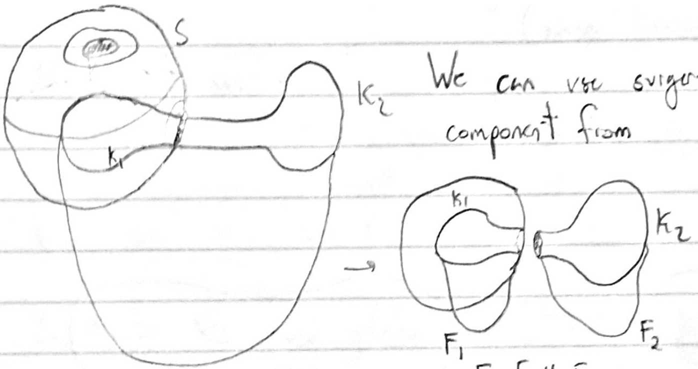
Recall cont:



$$\partial(F_1 \#_b F_2) = K_1 \# K_2$$

$$g(K_1 \# K_2) \leq g(F_1 \#_b F_2) = g(K_1) + g(K_2)$$

The other direction of the inequality was harder. We proved that if F is a compact orientable surface with $\partial F = K_1 \# K_2$ of minimal genus then $g(F) \geq g(K_1) + g(K_2)$



We can use surgery to remove atleast one circle component from $S \cap F$.

$F = F_1 \#_b F_2$

$$\Rightarrow g(K_1 \# K_2) = g(F) = g(F_1) + g(F_2) \geq g(K_1) + g(K_2)$$

$$\Rightarrow \quad g(K_1) + g(K_2) \leq g(K_1 \# K_2) \leq g(K_1) + g(K_2)$$

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Applications of Additivity of Knot genus, Prime Knots

Theorem:

If K_1 and K_2 are both non-trivial knots then $K_1 \# K_2$ is also non-trivial

Proof:

J is trivial $\iff g(J)=0$

$\mathcal{V} = \partial F$ where F is a disk. Any knot which bounds a disk is equivalent to $\mathcal{V}$

This is like saying there are no zero divisors in knot theory.

Defn:

A knot K is prime if for any decomposition into $K = K_1 \# K_2$ one of K_1 or K_2 is the unknot.

Remark:

Any knot with $g(K)=1$ is prime. This follows from the additivity of the genus.

Theorem: [Prime decomposition theorem for knots]

Any knot K can be written as a connected sum of prime knots.

ie. There exists prime knots $K_1, \dots, K_n \ni K = K_1 \# K_2 \# \dots \# K_n$

Further the prime factors of K, $K_1, \dots, K_n$, are unique up to reordering.

Namely if $K = J_1 \# J_2 \# \dots \# J_m$ for some $J_1, \dots, J_m$ are prime knots then $n=m$ and the prime factors where $K_i = J_{\sigma(i)}$ $\forall i = 1, \dots, n$ where σ is a permutation of $\{1, \dots, n\}$

Proof of Existence:

This follows by induction on the genus.

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1. Existence of Prime decomposition.

Proof of Existence:

We start with $g(K)=1$. If K is a knot with $g(K)=1$ then it is prime so it is equal to its own decomposition. So there is nothing to prove.

Now assume existence has been established for all knots with $g(J)<g$ and $g>1$. Consider a knot K with $g(K)=g$.

If K is prime then it is equal to its own decomposition.
Otherwise $K=K'\#K''$ where K' and K'' are nontrivial knots. So $0<g(K'), g(K'')$, and $g(K)=g(K')+g(K'')$.

Hence $g(K')<g$ and $g(K'')<g$. So each of K' and K'' have prime decompositions.

So $K=K'\#K''=(K_1'\#K_2'\#\cdots\#K_n')\#(K_1''\#\cdots\#K_m'')$ where K_i' and K_j'' are all prime.
So K has a prime decomposition. ∎

The proof of uniqueness is a more sophisticated argument using surgery techniques (Thm 2.10 Lickorish)

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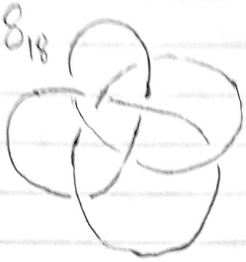
Example Problems

Ex: 8_{18} and 9_{24} both have the same Alexander polynomial

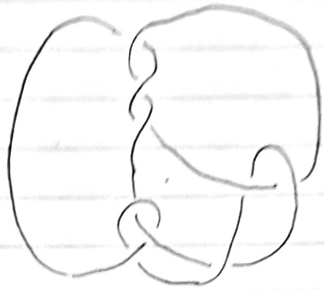
$$\Delta_K(t) = t^6 - 5t^5 + 10t^4 - 13t^3 + 10t^2 - 5t + 1$$

How could we distinguish them? ie show $8_{18} \ncong 9_{24}$

8_{18}



9_{24}



We can use the jones polynomial to distinguish them.

$V_{8_{18}}(t) \neq V_{9_{24}}(t)$ since span $V_{8_{18}}(t) = 8$ and span $V_{9_{24}}(t) = 9$ since they are both reduced alternating diagrams.

What about knot determinant or tri-colourability or mod p-labelability?

Recall: $\det(K) = |\Delta_K(-1)|$, so since $\Delta_{8_{18}}(t) = \Delta_{9_{24}}(t)$, $\det(8_{18}) = \det(9_{24})$

Both 8_{18} and 9_{24} are mod 3-labelable but they have a different number of mod 3 labels

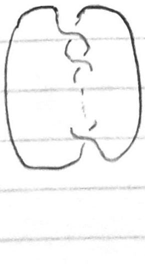
Exercises

1) Count the number of distinct 3-colorings of 8_{18}, 9_{24}

2) Which of the $(2,n)$ torus knots and the n-twist knots are 3-colourable?

(2, n) Torus knot

n odd
n twists



n-twist Knot

n twists