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Signature function (Levine-Tristram Signature)

Recall

A matrix A defined over $\mathbb{C}$ is Hermitian if $A = A^*$ ← Conjugate transpose.

eg: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a,b,c,d \in \mathbb{C}$ $A^* = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix}$

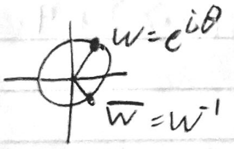
So $A = A^* \iff a,d \in \mathbb{R}$ and $b = \bar{c}$, $c = \bar{b}$

$A = \begin{bmatrix} x & b \\ \bar{b} & y \end{bmatrix}$ $x,y \in \mathbb{R}$

Theorem

Every Hermitian matrix is diagonalizable with real eigenvalues (sign(A) is defined as before)

Let $w \in \mathbb{C}$, $|w| = 1$, then $A = (1-w)V + (1-w^{-1})V^T$ is Hermitian



Def^n:

$\sigma_w(K) = \text{sign}((1-w)V + (1-w^{-1})V^T)$. Then $w \mapsto \sigma_w(K)$ is the signature function $\forall w \in U(1) \subseteq \mathbb{C}$

Main Idea:

$K \rightsquigarrow F \rightsquigarrow V \rightsquigarrow \sigma(K) := \text{sign}(V+V^T) \rightsquigarrow \sigma_w(K) = \text{sign}((1-w)V + (1-w^{-1})V^T)$ $w \in \mathbb{C}$, $|w|=1$

knot, Seifert Surface, Seifert Matrix, Knot Signature, Signature Function

Remark:

There do exist values of $w \in U(1) = \{e^{i\theta} \in \mathbb{C}\}$, the matrix $(1-w)V + (1-w^{-1})V^T$ can be singular. Where as, in general, $V + V^T$ is non-singular with $\sigma(K) \in 2\mathbb{Z}$. If matrix is non-singular then $\sigma_w(K) \in 2\mathbb{Z}$. This will be the case if w^2 is not a root of $\Delta_K(t)$.

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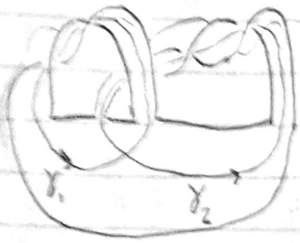
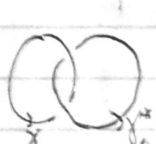
Example of Signatures

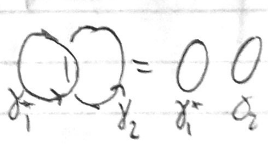
Ex: RH-Trefoil



$V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ $V^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ $(1-w)V + (1-\bar{w})V^T = \begin{bmatrix} w+\bar{w}-2 & 1-w \\ 1-\bar{w} & w+\bar{w}-2 \end{bmatrix} = A_w$

$lk(\gamma_1, \gamma_1^*) = -1$ $lk(\gamma_1, \gamma_2^*) = 1$ $lk(\gamma_2, \gamma_1^*) = 0$ $lk(\gamma_2, \gamma_2^*) = -1$



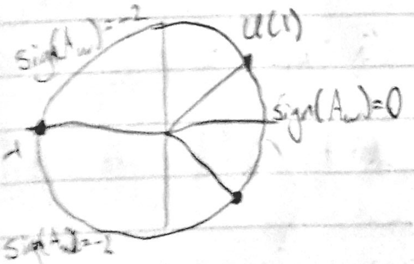


$$\det(A_w) = (w+\bar{w}-2)^2 - (1-w)(1-\bar{w})$$

$$= (w+\bar{w}-2)(w+\bar{w}-2) - (1-\bar{w}-w+1)$$

$$= w^2+\bar{w}^2-3w-3\bar{w}+4 \quad = w^2+\bar{w}^2-3(w+\bar{w})+4$$

Roots are $w = e^{\frac{\pi}{3}i}$ or $e^{-\frac{\pi}{3}i}$



For $w = -1$, $A_w = A_{-1} = 2(A + A^T)$ $\text{sign}(A_{-1}) = \sigma(K) = -2$

March 22

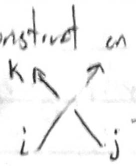
Fox differentiation and the knot group

Review

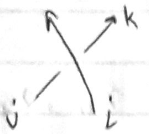
We have seen 3 ways to compute the Alexander polynomial

1) i) Given a knot with n-crossings label the arcs $i=1,\dots,n$.

ii) Construct an $n\times n$ matrix with j^{th} row corresponding to the ℓ^{th} crossing



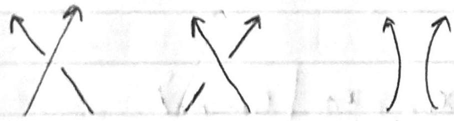
$$\rightsquigarrow \begin{array}{cccccccc} & i & \cdots & j & \cdots & k & \\ 0\cdots0 & 1-t & 0\cdots0 & -1 & 0\cdots0 & t & 0\cdots0 \end{array}$$



$$\rightsquigarrow \begin{array}{ccc} i & j & k \\ 1-t & t & -1 \end{array}$$

This will produce an $n\times n$ matrix A, let $\hat{A}$ be the $(n-1)\times(n-1)$ matrix obtained by removing the last row & column. Then $\det(\hat{A}) = \Delta_K(t)$ (Well defined up to mult by $\pm t^k$)

2) Using the Skein Relation



L_+ L_- L_0

If $\Delta_K(t)$ is normalized $\ni$ $\Delta_K(1)=1$ and $\Delta_K(t^{-1}) = t$

$$\Delta_{L_+} - \Delta_{L_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta_{L_0} \qquad (\text{And } \Delta_O(t)=1, \quad \Delta_{OO}(t)=0)$$

Unknot Unlink

3) Construct a Seifert surface and compute a Seifert matrix V then

$$\Delta_K(t) = \det(V - tV^T)$$

4) Fox differentiation

Wirtinger Presentations

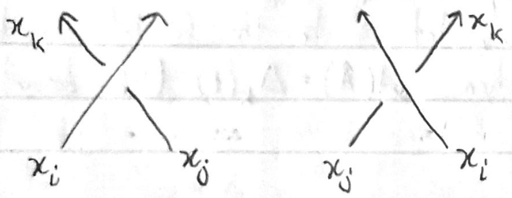
The Alexander polynomial can also be computed from a presentation of the knot group

$$G_K = \pi_1(S^3 \setminus N(K))$$

Recall that G_K can be represented as a finitely presented group (generators/relators) using any knot diagram.

Wirtinger Presentation

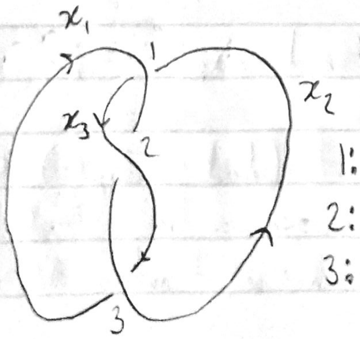
Suppose K is a diagram with n crossings. Label the arcs $x_1, \dots, x_n$. At each crossing we have one relation.



r_λ: $x_k = x_i^{-1} x_j x_i$ $\qquad$ $x_k = x_i x_j x_i^{-1}$

$$G_K = \langle \text{gens} \mid \text{relators} \rangle = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$$

ex:



$$G_K = \langle x_1, x_2, x_3 \mid x_1x_2x_1^{-1}x_3^{-1},\ x_3x_1x_3^{-1}x_2^{-1},\ x_2x_3x_2^{-1}x_1^{-1} \rangle$$

1: $x_1x_2x_1^{-1} = x_3 \Rightarrow x_1x_2x_1^{-1}x_3^{-1} = r_1$

2: $x_3x_1x_3^{-1} = x_2 \Rightarrow x_3x_1x_3^{-1}x_2^{-1} = r_2$

3: $x_2x_3x_2^{-1} = x_1 \Rightarrow x_2x_3x_2^{-1}x_1^{-1} = r_3$

Aside $\quad G_K/[G_K, G_K] \cong \mathbb{Z}$

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Fox Differentiation

$\frac{\partial}{\partial x_i}$ partial differential operator on Free group $\langle x_1, \dots, x_n \rangle$ defined as follows

$$\frac{\partial}{\partial x_i} 1 = 0$$

$$\frac{\partial}{\partial x_i} x_i = 1$$

$$\frac{\partial}{\partial x_i} x_j = 0 \quad \text{if } x_i \neq x_j$$

$$\frac{\partial}{\partial x_i}(w_1 w_2) = \frac{\partial w_1}{\partial x_i} + w_1 \frac{\partial w_2}{\partial x_i} \quad \text{where } w_1, w_2 \text{ are words}$$

Claim:
$$\frac{\partial}{\partial x_i} x_i^{-1} = -x_i^{-1}$$

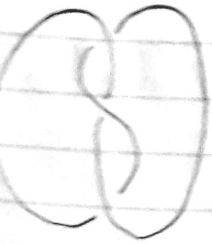
Proof:
$$1 = x_i x_i^{-1}, \quad \frac{\partial}{\partial x_i} 1 = 0 \Rightarrow \frac{\partial x_i}{\partial x_i} + x_i \frac{\partial x_i^{-1}}{\partial x_i} = 0 \Rightarrow 1 + x_i \frac{\partial x_i^{-1}}{\partial x_i} = 0 \Rightarrow \frac{\partial x_i^{-1}}{\partial x_i} = -x_i^{-1} \quad \blacksquare$$

Notice that $\frac{\partial}{\partial x_i}(w)$ is a linear combination of elements in the free group $\langle x_1, \dots, x_n \rangle$ (ie the group algebra $\mathbb{Z}(G)$)

$$\left(\sum_{i=1}^{n} \alpha_i w_i \in \mathbb{Z}(G) \quad \alpha_i \in \mathbb{Z},\ w_i \in G \right)$$

so $\frac{\partial}{\partial x_i}(w) : G \to \mathbb{Z}(G)$

Alexander Polynomial with Fox differentiation



$$\langle x_1, x_2, x_3 \mid x_1 x_2 x_1^{-1} x_3^{-1},\ x_3 x_1 x_3^{-1} x_2^{-1} \rangle$$

$$\cong \langle x_1, x_2 \mid x_1 x_2 x_1^{-1} x_1,\ x_1 x_2^{-1} x_1^{-1} x_2^{-1} \rangle$$

$$\cong \langle x_1, x_2 \mid w = x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1} \rangle$$

$$\frac{\partial}{\partial x_1}(w) = 1 + x_1 x_2 - x_1 x_2 x_1 x_2^{-1} x_1^{-1}$$

$$\frac{\partial w}{\partial x_2} = x_1 - x_1 x_2 x_1 x_2^{-1} - x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1}$$

$$\Delta_K(t) = \left[\frac{\partial w}{\partial x_i}\right]_{x_1, x_2 = t} \begin{array}{ll} = 1 + t^2 - t & \frac{\partial}{\partial x_1} \\ = t - t^2 - 1 & \frac{\partial}{\partial x_2} \end{array}$$

$$1 + t^2 - t \doteq t - t^2 - 1$$
$$\left(= (-1)(1 + t^2 - t)\right)$$

Theorem:

Suppose G_K has a Wirtinger-type presentation, $G_K \cong \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$. Let $A = \left[\frac{\partial r_i}{\partial x_j}\right]_{\substack{i=1,\dots,n-1 \\ j=1,\dots,n}}$ be the matrix of Fox derivatives. Let $\hat{A}$ be the $(n-1)\times(n-1)$ matrix obtained by removing the last column and setting $x_i = t$. Then $\det(\hat{A}) \doteq \Delta_K(t)$

Remark:

This is especially useful for knots K having a Wirtinger presentation of the form $\langle x, y \mid r \rangle$ ie 2 generator, 1-relator groups

$$\Delta_K(t) = \left.\frac{\partial r}{\partial x}\right|_{x, y = t}$$

eg: 2-bridge knots (or rational knots)