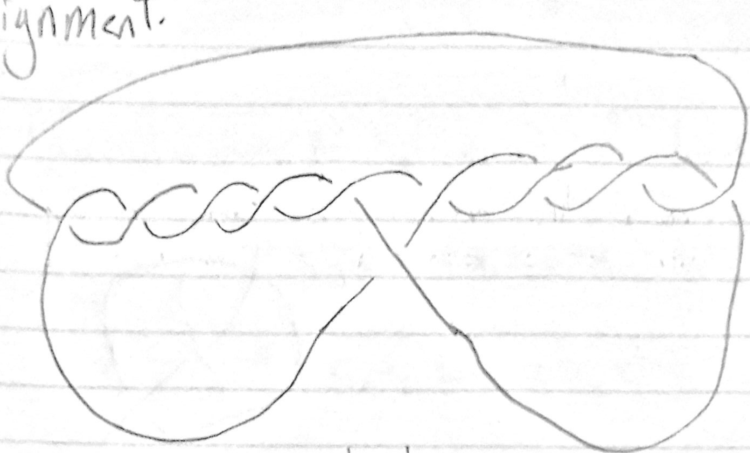
March 24

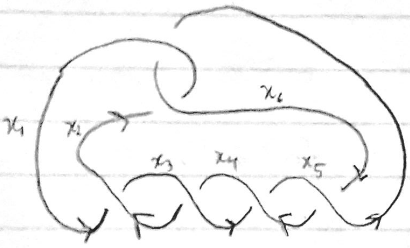
Assignment.



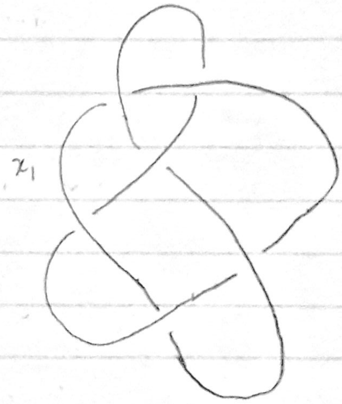
10 crossing knot

Assignment due next tues.

6_1



6_2



1) Write out the Wirtinger presentations for G_K with $K=6_1$ and $K=6_2$ (6 generators, 5 relations)

2) Simplify each of the group presentation to be a 2-generator 1-relator group ie $G_K = \langle x, y \mid r \rangle$

3) Using 2) deduce the Alexander polynomial using Fox derivatives

March 24

Chapter 7, Numerical Invariants

Various ways to associate tell tale integers to knots

There are computable integers such as the genus $g(k)$ of the knot, the determinant $\det(k)$, the signature $\sigma(k)$, and the mod-p rank of k. For each we have an algorithm for computing them from a diagram.

On the other hand we can naturally associated integers to knots, and its not entirely clear how to compute them. Examples include the crossing index $c(k)$ of k, the unknotting number $u(k)$, the bridge number, and braid index. (Last two to be defined)

We will summarize what is known about the various quantities, how they are related, and the many unknown/open problems.

1) genus, crossing index, and the degree of the Alexander polynomial

$$\Delta_k(t) = \det(V - tV^T) \Rightarrow \deg(\Delta_k(t)) \leq 2n \quad \text{since } V \text{ is a } 2n \times 2n \text{ matrix}$$

$$g(k) := \min\{g(F) \mid F \text{ is a Seifert surface of } K\}$$

If F is a disk-band surface with 1 disk then it has an even number of bands $2n$, and $g(k) = n$. Since $\chi(F) = \chi(\ \ ) = 1-2n = (2-2g)-1 \Rightarrow n = g$

$\deg(\Delta_k(t)) \leq 2n = 2g(F)$.

There is also an inequality relating to the crossing number $c(D)$ of a diagram D for K. Seiferts algorithm, let $S(D) = \#$ of Seifert circles adding $c(D)$ half-twisted bands, $\Rightarrow \chi(F) = S(D) - c(D) = 1 - 2g(F)$

$$\Rightarrow 2g(F) = 1 + c(D) - S(D) \leq c(D)$$

$$\Rightarrow 2g(K) \leq c(K)$$

← True for any diagram, and since each diagram gives a surface, we can take minimum on both sides to show its true for the knot.

$$\boxed{\Rightarrow \deg(\Delta_k(t)) \leq 2g(K) \leq c(k)}$$

Unknotting Number

Remarks:

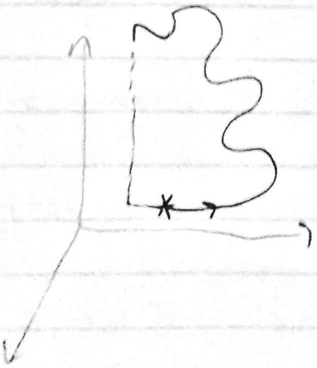
1) $g(K)$ is additive under connected sum
2) $c(K)$ is conjecturally additive under connected sum (big open problem)
   it is known to be additive for alternating knots. (uses jones polynomial)
3) One can use non-orientable surfaces and define an analouge of $g(K)$ called crosscap number.

Unknotting Number.

The unknotting number of a knot diagram is the minimum number of crossings that when changed, result in a diagram of the unknot.

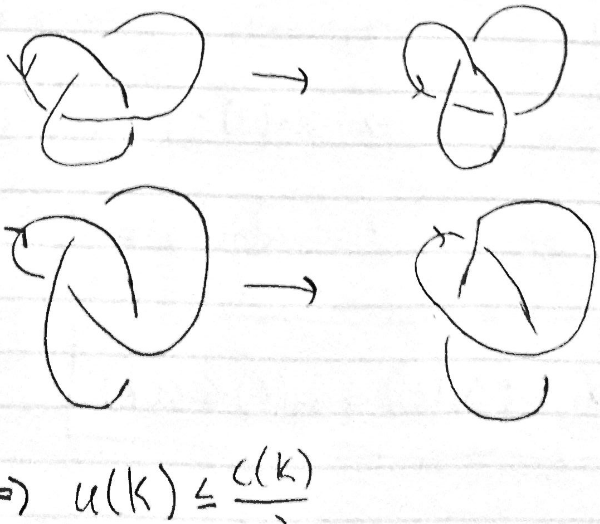
Notice that for any $* \in D$ and either direction ($\rightarrow\!*\!\rightarrow$) we can always change crossings so the resulting diagram is ascending.

An orientated knot diagram with base point is ascending if each crossing is first traversed as an undercrossing and then as an over crossing.



Traversing the knot in the opposite direction will require crossing changes on the complementary set of crossings to make it ascending.

Ex:



$$\Rightarrow u(D) \leq \frac{c(D)}{2} \qquad \Rightarrow u(K) \leq \frac{c(K)}{2}$$

March 24

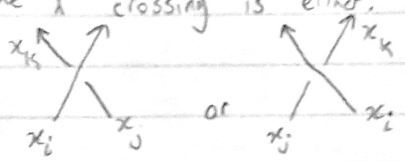
Mod p Rank and the Unknotting number

The twist knots show that $c(K) - 2u(K)$ is unbounded

We have lower bounds for $u(K)$ in terms of the knot signature $\sigma(K)$ and mod p rank of K.

Review:

The mod p rank is determined from the "colouring matrix" defined as follows. Assume D is a diagram with n crossings, label the arcs of D $x_1, \dots, x_n$ and crossings $l = 1, \dots, n$. Construct a matrix $\hat{A}$ by inserting $2, -1, -1$ into the i^{th}, j^{th}, or k^{th} columns if the l^{th} crossing is either. The other entries of $\hat{A}$ are 0



Then obtain A the $(n-1)\times(n-1)$ matrix obtained from removing last row + column from $\hat{A}$. Then the mod p rank of K is the mod p nullity of A

K has a mod p labeling $\iff$ the corresponding set of equations has a solution mod p

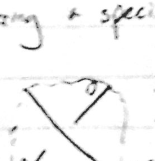
$\iff \det(\hat{A}) \equiv 0 \bmod p$

To determine how many mod p labelings a knot admits, one can diagonalize A over $\mathbb{Z}$ to get a matrix $A' = \begin{bmatrix} t_1 & 0 \\ 0 & t_{n-1} \end{bmatrix}$ where each t_i divides t_{i+1} for $i = 1, \dots, n-1$. These are called the torsion numbers. The mod p rank of K is the number of t_i divisible by p.

Theorem

$\text{mod}\,p\,\text{rank}(K) \leq u(K)$ for each prime p.

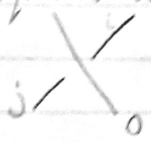
We will demonstrate why this is true by considering a special case $u(K)=1$. Suppose K has $u(K)=1$, and draw the crossing which when changed gives us the unknot



If all arcs in the local picture are labeled the 0 then the same is true for the diagram with the crossing changed

But that is an unknot diagram which is not mod p labelable $\Rightarrow\Leftarrow$

So now suppose modprank$(K) \geq 2$. So we have two linearly independent solutions to the equations over $\mathbb{Z}/p \cong \mathbb{F}_p$



Let i,j be the labels on the incoming under crossing arc, then $i,j \not\equiv 0 \bmod p$

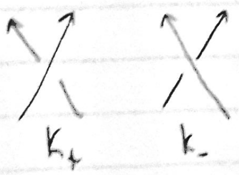
So i,j are both units in $\mathbb{F}_p$. So we can subtract a multiple of the second labeling from the first to get one with a zero on one arc, this is a contradiction.

Theorem:

$$\frac{\sigma(K)}{2} \leq u(K)$$

Remark:

1) The same formula holds for $\sigma_\omega(K)$ provided ω^2 is not a root of $\Delta_K(t)$

2) 

$\sigma(K_+) \leq \sigma(K_-) \leq \sigma(K_+) + 2$