ADEQUATE LINKS IN THICKENED SURFACES AND THE GENERALIZED TAIT CONJECTURES

HANS U. BODEN, HOMAYUN KARIMI, ADAM S. SIKORA

Abstract. In this paper, we apply Kauffman bracket skein algebras to develop a theory of skein adequate links in thickened surfaces. We show that any alternating link diagram on a surface is skein adequate. We apply our theory to establish the first and second Tait conjectures for adequate links in thickened surfaces. Our notion of skein adequacy is broader and more powerful than the corresponding notions of adequacy previously considered for link diagrams in surfaces.

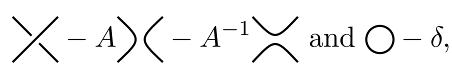
For a link diagram D on a surface Σ of minimal genus $g(\Sigma)$, we show that

$$\operatorname{span}([D]_\Sigma) \leq 4c(D) + 4|D| - 4g(\Sigma),$$

where $[D]$ is its skein bracket, $|D|$ is the number of connected components of D, and $c(D)$ is the number of crossings. This extends a classical result of Kauffman, Murasugi, and Thistlethwaite. We further show that the above inequality is an equality if and only if D is weakly alternating. This is a generalization of a well-known result for classical links due to Thistlethwaite. Thus the skein bracket detects the crossing number for weakly alternating links. As an application, we show that the crossing number is additive under connected sum for adequate links in thickened surfaces.

1. Introduction

The Kauffman bracket is a $\mathbb{Z}[A^{\pm 1}]$-valued invariant of framed links in $\mathbb{R}^3$ determined by the skein relations:

(1) 

where $\delta = -A^2 - A^{-2}$.

It naturally extends to an invariant of framed links in an arbitrary oriented 3-manifold M (possibly with boundary), via the skein module construction: let $\mathscr{L}(M)$ be the set of all unoriented, framed links in M, including the empty link $\varnothing$. The **skein module** $\mathscr{S}(M)$ of M is the quotient of the free $\mathbb{Z}[A^{\pm 1}]$-module spanned by $\mathscr{L}(M)$ by the submodule generated by the Kauffman bracket skein relations (1), cf. [Prz99], [Tur88, Tur91].

By this construction, the bracket

$$[\,\cdot\,] \colon \mathscr{L}(M) \to \mathscr{S}(M),$$

sending framed links to their equivalence classes in $\mathscr{S}(M)$, called the **skein bracket**, is the universal invariant of framed links in M satisfying (1).

Independently of this initial motivation, skein modules quickly began to play a much broader role in the development of quantum topology, for example in connection with

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$SL(2,\mathbb{C})$ character varieties [Bul97, PS00, FKBL17, Tur91, BFKB99], topological quantum field theory, [BHMV95, Tur94], (quantum) Teichmüller spaces and (quantum) cluster algebras [BW11, CL19, FG06, FST08, Mul16], the AJ conjecture [FGL02, Lê06], and many more.

In this paper we develop a general theory of skein adequacy (called adequacy, for short) for links in thickened surfaces with the aid of skein modules.

Let Σ be an oriented surface and $I = [0,1]$ be the unit interval. The skein module of the thickened surface $\Sigma \times I$ comes naturally equipped with a product structure given by stacking, i.e., the product $L_1 \cdot L_2$ is defined by placing L_1 on top of L_2 in $\Sigma \times I$. With this product structure, the skein module $\mathscr{S}(\Sigma \times I)$ becomes an algebra over $\mathbb{Z}[A^{\pm 1}]$.

Let $\mathscr{C}(\Sigma)$ denote the set of all non-trivial unoriented simple loops on Σ up to isotopy and $\mathscr{MC}(\Sigma)$ denote the set of all non-trivial unoriented multi-loops on Σ, i.e., collections of disjoint simple non-contractible loops, including $\varnothing$, up to isotopy. Then by [Prz99] (cf., [SW07]), the skein module $\mathscr{S}(\Sigma \times I)$ is a free $\mathbb{Z}[A^{\pm 1}]$-module with basis $\mathscr{MC}(\Sigma)$. Consequently, via this identification, the skein bracket gives a map

$$[\,\cdot\,]_\Sigma \colon \mathscr{L}(\Sigma \times I) \to \mathscr{S}(\Sigma \times I) = \mathbb{Z}[A^{\pm 1}]\mathscr{MC}(\Sigma). \tag{2}$$

We use the association (2) to develop a theory of skein adequacy for links in $\Sigma \times I$ which extends that for classical links. This theory is broader and more powerful than the corresponding notions of simple adequacy [LT88] and homological adequacy [BK19]. For example, we will see that every weakly alternating link in $\Sigma \times I$ without removable nugatory crossings is skein adequate.

We will apply the skein bracket to establish the first and the second Tait conjecture for skein adequate link diagrams on surfaces. The first one says that skein adequate diagrams have minimal crossing number, and the second one says that two skein adequate diagrams for the same oriented link have the same writhe. (The writhe of a link diagram D is denoted by $w(D)$ and is defined to be the sum of its crossing signs.) These results strengthen the earlier work of Adams et al., [AFLT02], who showed the minimal crossing number result for reduced alternating knot diagrams in surfaces. We also strengthen the minimality result of [BK19], for homologically adequate link diagrams in surfaces, and further show that any connected sum of two skein adequate link diagrams on surfaces is again skein adequate. This implies that the crossing number and writhe are essentially additive under connected sum of skein adequate links in thickened surfaces.

For any link diagram D on a surface Σ of minimal genus, we prove that

$$\operatorname{span}([D]_\Sigma) \leq 4c(D) + 4|D| - 4g(\Sigma),$$

where $|D|$ is the number of connected components of D, $c(D)$ is the number of crossings, and $g(\Sigma)$ is the genus of Σ. This inequality generalizes a result proved by Kauffman, Murasugi, and Thistlethwaite for link diagrams on $\mathbb{R}^2$ [Kau87, Mur87, Thi87], extending their nice geometric application of the Kauffman bracket. It also extends and strengthens an analogous recent result proved in [BK19] using the homological Kauffman bracket.

Additionally, we prove that the above inequality is an equality if and only if D is weakly alternating. Therefore, the skein bracket, together with the crossing number, distinguishes weakly alternating links. That generalizes the analogous result of Thistlethwaite for classical links.

Broader context and motivation. While the results presented here are new only for links in non-contractible surfaces, generalized link theory is of growing interest and has many potential connections to classical links and 3-dimensional geometry. We take a moment to discuss some of them.

One motivation for our results is their connection to the theory of virtual knots and links, which can be viewed as links in thickened surfaces, considered up to homeomorphisms and stabilization [CKS02]. By Kuperberg's theorem, minimal genus realizations of virtual links are unique up to homeomorphism [Kup03]. Our theory of adequate and alternating links in thickened surfaces is invariant under surface homeomorphisms and, therefore, many of the results given here can be restated in the language of virtual links.

A second motivation involves potentially novel applications to classical link theory. The Turaev surface construction associates to any classical link diagram an alternating link in a thickened surface [Tur87, DFK$^+$08, CK14]. Menasco famously proved hyperbolicity for alternating (non-torus) links in S^3 [Men84], and his result has been extended to prime alternating links $L \subset \Sigma \times I$ in [AARH$^+$19]. This result opens the door to using the hyperbolic geometry of alternating links in higher genus surfaces to profitably study non-alternating classical links, e.g., see [AEG$^+$19] and the many other papers cited below.

In [DL07], Dasbach and Lin proved a remarkable result giving a bound on the volume of alternating link complements in terms of the second and penultimate coefficients of the Jones polynomial. In [Lac04], Lackenby established an equally remarkable bound on the volume of alternating link complements in terms of the diagrammatic *twist number* For alternating hyperbolic links in S^3, the results of [DL07] imply that the twist number is essentially an isotopy invariant of L, but this is not true in general.

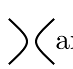
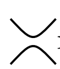
These methods have been generalized to non-alternating hyperbolic links in S^3 [Bla09, BAR19] and to hyperbolic links in arbitrary compact oriented 3-manifolds [HP20]. In general, there is a notion of weakly generalized alternating link diagrams on surfaces due to Howie [How15], extended to links in compact oriented 3-manifolds via "generalized projection surfaces" by Howie and Purcell [HP20].

The volume bounds have been extended to alternating links in thickened surfaces by Bavier and Kalfagianni [?] and Will [Wil20] and also to virtual alternating links by Champanerkar and Kofman [CK20]. In [CK20] and [Wil20], the volume bounds are expressed in terms of the Jones-Krushkal polynomial [Kru11, BK19], and in [?] they are expressed in terms of a skein invariant derived from fully contractible smoothings. In [?, Corollary 1.3], they deduce that, for alternating links in thickened surfaces, the twist number is an isotopy invariant. Interestingly, this result is consistent with the generalized Tait flyping conjecture.

2. State sum formula and the generalized Jones polynomial

We will assume throughout this paper that Σ is an oriented surface with one or more connected components, which may also have boundary. Links in $\Sigma \times I$ will be represented as diagrams on Σ up to Reidemeister moves.

Every framed link in $\Sigma \times I$ can also be represented by a link diagram with framing given by the blackboard framing. Equivalence of framed links is given by regular isotopy, which includes the second and third Reidemeister moves and the modified first Reidemeister move.

Let D be a link diagram on a surface Σ. Given a crossing  of D, we consider its A-type  and B-type  resolution, as in the Kauffman bracket construction. A choice of resolution for each crossing of D is called a **state**. Let $\mathfrak{S}(D)$ denote the set of all states of D. Thus $|\mathfrak{S}(D)| = 2^{c(D)}$, where $c(D)$ is the crossing number of D.

For $S \in \mathfrak{S}(D)$, let $t(S)$ denote the number of contractible loops in S and let $\widehat{S}$ denote S with contractible loops removed. Hence, $\widehat{S} \in \mathscr{MC}(\Sigma)$. The following state sum formula is an immediate consequence of the definition and it generalizes the usual formula for the classical Kauffman bracket:

$$[D]_\Sigma = \sum_{S \in \mathfrak{S}(D)} A^{a(S)-b(S)} \delta^{t(S)} \widehat{S} \in \mathbb{Z}[A^{\pm 1}]\mathscr{MC}(\Sigma), \tag{3}$$

where $a(S), b(S)$ are the numbers of A- and B-smoothings in S and $\delta = -A^2 - A^{-2}$ as before. A similar formula appears in the paper of Dye and Kauffman on the surface bracket polynomial [DK05].

Any invariant of framed links in $\Sigma \times I$ satisfying (1) can be normalized to obtain a Jones-type polynomial invariant of oriented links. In the case of the skein bracket (2), one obtains the **generalized Jones polynomial**, an invariant for oriented links in $\Sigma \times I$ given by

$$J_\Sigma(D) = (-1)^{w(D)} t^{3w(D)/4} ([D]_\Sigma)_{A=t^{-1/4}}. \tag{4}$$

3. Adequate link diagrams in surfaces

Given a link diagram D, let S_A be the pure A state and let S_B be the pure B state. Then S_A and S_B are the states which theoretically give rise to the terms of maximal and minimal degree in (3). The notion of adequacy of a link diagram is designed to guarantee that the terms from S_A and S_B survive in the state sum formula. Therefore, when D is a skein adequate diagram, its skein bracket $[D]_\Sigma$ has maximal possible span.

Two states S, S' are said to be **adjacent** if their resolutions differ at exactly one crossing.

Definition 1. *A link diagram D on a surface Σ is said to be A-adequate if $t(S) \leq t(S_A)$ or $\widehat{S} \neq \widehat{S}_A$ in $\mathscr{MC}(\Sigma)$ for any state S adjacent to S_A. It is said to be B-adequate if $t(S) \leq t(S_B)$ or $\widehat{S} \neq \widehat{S}_B$ for any state S adjacent to S_B. The diagram D is called skein adequate if it is both A- and B-adequate.*

The notions of A- and B-adequacy are modeled on the notions of plus- and minus-adequacy for classical links [Lic97]. Recall that a classical link diagram is said to be **plus-adequate** if $|S| = |S_A| - 1$ for any state S adjacent to S_A, and it is **minus-adequate** if $|S| = |S_B| - 1$ for any state S adjacent to S_B. This simpler notion of adequacy extends verbatim to link diagrams on surfaces. For link diagrams on surfaces, plus- and minus-adequacy is a special case of the notion of homological adequacy, which was introduced in [BK19] and will be reviewed in Section 4. We will see that adequacy as defined above is more general than simple or homological adequacy.

The following provides an alternative definition of adequacy:

Proposition 2. *(1) A link diagram D on Σ is A-adequate if and only if $t(S) \leq t(S_A)$ or $|\widehat{S}| \neq |\widehat{S}_A|$ for any state S adjacent to S_A.*
(2) A link diagram D on Σ is B-adequate if and only if $t(S) \leq t(S_B)$ or $|\widehat{S}| \neq |\widehat{S}_B|$ for any state S adjacent to S_B.

Proof. We begin with some general comments. Given a link diagram D and two adjacent states S, S', the transition from S to S' is one of the following types:

(i) $|S'| = |S| + 1$, i.e., one cycle of S splits into two cycles of S'.
(ii) $|S'| = |S| - 1$, i.e., two cycles of S merge into one cycle of S'.
(iii) $|S'| = |S|$, i.e., one cycle C of S rearranges itself into a new cycle C' of S'.

In cases (ii) and (iii), either $t(S') \leq t(S)$ or $\widehat{S'} \neq \widehat{S}$. Specifically, in case (ii), $t(S') > t(S)$ only when two non-trivial parallel cycles in S merge to form one trivial cycle in S', which implies that $\widehat{S} \neq \widehat{S'}$. Likewise, in case (iii), we claim that neither C nor C' is trivial and, consequently, $t(S') = t(S)$. To see that, note that if S' is obtained from S by a smoothing change of a crossing x then there are two simple closed loops $\alpha, \beta \subset \Sigma$ intersecting at x only and such that the two different smoothings of x yield C and C'. Assigning some orientations to α and β, we see that C and C' with some orientations equal $\pm(\alpha+\beta)$ and $\pm(\alpha-\beta)$ in $H_1(\Sigma)$. Since the algebraic intersection number of α and β is 1, we know that $\alpha \neq \pm\beta$ and, consequently, neither C nor C' is trivial.

Therefore, to verify that a given diagram is A- or B-adequate, it is enough to check that the conditions of Definition 1 hold in case (i).

We will now prove part (1). Suppose S is a state adjacent to S_A with $t(S) = t(S_A)+1$. Then the transition from S_A to S must either be case (i) or (ii).

If it is case (i), then $|S| = |S_A| + 1$ and $t(S) = t(S_A) + 1$, therefore $\widehat{S} = \widehat{S}_A$. Thus D is not A-adequate and $|\widehat{S}| = |\widehat{S}_A|$. If it is case (ii), then $|S| = |S_A| - 1$, and two nontrivial cycles of S_A must merge into a trivial cycle of S. In this case, D is A-adequate and $|\widehat{S}| \neq |\widehat{S}_A|$.

The proof of part (2) is similar and is left to the reader. □

For any diagram D, its bracket has a unique presentation

$$[D]_\Sigma = \sum_\mu p_\mu(D)\mu \in \mathscr{S}(\Sigma \times I),$$

where the sum is over all multi-loops μ in Σ. Denote the maximal and minimal degrees (in the variable A) of the non-zero polynomials $p_\mu(D)$ in this expression by $d_{max}([D]_\Sigma)$ and $d_{min}([D]_\Sigma)$.

Proposition 3. *For any link diagram D on Σ,*
(1) $d_{max}([D]_\Sigma) \leq c(D) + 2t(S_A)$, with equality if D is A-adequate.
(2) $d_{min}([D]_\Sigma) \geq -c(D) - 2t(S_B)$, with equality if D is B-adequate.

Proof of (1). By (3), $[D]_\Sigma$ is given by a state sum with the term $A^{c(D)+2t(S_A)}\widehat{S}_A$ for the state S_A. Now the inequality of (1) follows from the fact that every change of a smoothing in S_A decreases $a(S) - b(S)$ by two and increases $t(S)$ by at most one.

The proof of equality in (1) when D is A-adequate follows immediately from part (1) of the lemma below.

The proof of (2) is analogous, and the proof of equality in (2) when D is B-adequate follows from part (2) of the lemma below. □

Lemma 4. *(1) If D is A-adequate and S is a state with at least one B-smoothing, then either*

$$a(S) - b(S) + 2t(S) < c(D) + 2t(S_A) \quad \text{or} \quad \widehat{S} \neq \widehat{S}_A.$$

(2) If D is B-adequate and S is a state with at least one A-smoothing, then either

$$a(S) - b(S) + 2t(S) > -c(D) - 2t(S_B) \quad \text{or} \quad \widehat{S} \neq \widehat{S}_B.$$

Proof. We prove (1) by contradiction: Suppose to the contrary that S is a state with at least one B-smoothing such that $\widehat{S} = \widehat{S}_A$ and

$$a(S) - b(S) + 2t(S) = c(D) + 2t(S_A).$$

Clearly, S can be obtained from S_A by a sequence of smoothing changes from A to B, $S_A = S_0 \to S_1 \to \cdots \to S_k = S$. Further, each smoothing change must increase $t(\cdot)$ by one, i.e., $t(S_{i+1}) = t(S_i) + 1$, for $i = 0, \dots, k-1$. Since each smoothing change increases the number of cycles in a state by at most one, none of these smoothing changes can add a new cycle to $\widehat{S}_i$, $i = 0, \dots, k$. Therefore $|\widehat{S}_i| \leq |\widehat{S}_{i-1}|$ for $i = 0, \dots, k-1$. However, since $\widehat{S} = \widehat{S}_A$, none of the smoothing changes can decrease $|\widehat{S}_i|$ either. It follows that $\widehat{S}_{i+1} = \widehat{S}_i$ for $i = 0, \dots, k-1$. Thus $|\widehat{S}_{i+1}| = |\widehat{S}_i|$ and

$$|S_{i+1}| = t(S_{i+1}) + |\widehat{S}_{i+1}| = t(S_i) + 1 + |\widehat{S}_i| = |S_i| + 1,$$

for $i = 0, \dots, k-1$. In particular, each transition $S_i \to S_{i+1}$ is of type (i) as discussed in the proof of Proposition 2, i.e., one where a cycle of S_i splits into two cycles of S_{i+1}.

However, since D is A-adequate, the first smoothing change $S_A = S_0 \to S_1$ has either $t(S_1) \leq t(S_A)$ or $\widehat{S}_1 \neq \widehat{S}_A$, which is a contradiction.

This completes the proof of the first statement. The proof of the second one is similar and is left to the reader. □

The next result is an immediate consequence of Proposition 3.

Corollary 5. *If D is a link diagram on Σ, then*

$$\operatorname{span}([D]_\Sigma) \leq 2c(D) + 2t(S_A) + 2t(S_B),$$

with equality if D is skein adequate.

The map $\Psi\colon \mathscr{MC}(\Sigma) \to \mathbb{Z}[z]$ sending S to $z^{|S|}$ extends linearly to the skein module,

$$\Psi\colon \mathscr{S}(\Sigma \times I) = \mathbb{Z}[A^{\pm 1}]\mathscr{MC}(\Sigma) \longrightarrow \mathbb{Z}[A^{\pm 1}, z].$$

The composition $\Psi([D]_\Sigma)$ is called the **reduced homotopy Kauffman bracket** Obviously,

$$\operatorname{span}(\Psi([D]_\Sigma) \leq \operatorname{span}([D]_\Sigma).$$

Proposition 6. *If D is a skein adequate link diagram on Σ, then*

$$\operatorname{span}(\Psi([D]_\Sigma)) = \operatorname{span}([D]_\Sigma).$$

Proof. Let S be a state with at least one B-smoothing such that $|\widehat{S}| = |\widehat{S}_A|$ and $a(S) - b(S) + 2t(S) = c(D) + 2t(S_A)$. As before, S can be obtained from S_A by a sequence of smoothing changes from A to B, each smoothing change can increase $t(\cdot)$ by at most one, i.e., $S_A = S_0 \to S_1 \to \cdots \to S_k = S$. As in the proof of Lemma 4, we must have $t(S_{i+1}) = t(S_i) + 1$. Further, since a smoothing change can increase the number of cycles in S_i by at most one, we have $|\widehat{S}_{i+1}| \leq |\widehat{S}_i|$ for $i = 0, \dots, k-1$. Now the assumption that $|\widehat{S}| = |\widehat{S}_A|$ then implies that $|\widehat{S}_{i+1}| = |\widehat{S}_i|$ for $i = 0, \dots, k-1$. However, since D is adequate, for the first transition $S_A = S_0 \to S_1$, either $t(S_1) \neq t(S_0) + 1$ or $\widehat{S}_1 \neq \widehat{S}_0$. But $t(S_1) = t(S_0) + 1$ and $|\widehat{S}_1| = |\widehat{S}_0|$ imply that $\widehat{S}_1 = \widehat{S}_0$, which gives a contradiction.

Therefore, the term with maximum A-degree in $\Psi([D]_\Sigma)$ must survive. A similar argument applies to show that the term with minimum A-degree survives. It follows that

$$\operatorname{span}(\Psi([D]_\Sigma)) = 2c(D) + 2t(S_A) + 2t(S_B) = \operatorname{span}([D]_\Sigma).$$

□

The next proposition shows that skein adequacy is inherited under passing to subsurfaces $\Sigma' \subset \Sigma$.

Proposition 7. *If a link diagram D on a subsurface Σ' of Σ is A- or B-adequate in Σ then it is A- or B-adequate (respectively) in Σ'.*

Proof. Suppose D is not A-adequate in Σ'. By Proposition 2, there exists a state S adjacent to S_A with $t(S,\Sigma') = t(S_A,\Sigma') + 1$ and $|\widehat{S}| = |\widehat{S}_A|$ in Σ'. In particular, $|S| = |S_A| + 1$, and the transition from S_A to S must involve one cycle C of S_A splitting into two cycles C_1 and C_2 of S. At least one of C_1, C_2 must be trivial in Σ', for otherwise $t(S,\Sigma') = t(S_A,\Sigma')$. If say C_1 is trivial in Σ', then it must also be trivial in Σ, because $\Sigma' \subset \Sigma$ is a subsurface.

As a cycle in Σ, C is either trivial or nontrivial. If it is trivial, then C_2 must also be trivial in Σ, and so in fact all three of C, C_1, C_2 are trivial. This implies that $t(S,\Sigma) = t(S_A,\Sigma) + 1$ and $|\widehat{S}| = |\widehat{S}_A|$ in Σ, contradicting the assumption of A-adequacy of D.

If, on the other hand, C is nontrivial in Σ, then C_2 must also be nontrivial in Σ. This again implies that $t(S,\Sigma) = t(S_A,\Sigma) + 1$ and $|\widehat{S}| = |\widehat{S}_A|$ in Σ, leading to the same contradiction. Therefore, D must be A-adequate on Σ'.

The proof of B-adequacy of D is identical. □

4. Skein and homological adequacy

For completeness of discussion, in this section we compare Definition 1 of skein adequacy to two legacy versions, namely simple and homological adequacy. We will see that our notion of adequacy is broader and that the statements of Lemma 4 and Corollary 5 are strictly stronger than the corresponding statements for simple and homological adequacy. Henceforth, we will say a link diagram on a surface is adequate if it is skein adequate.

For any state $S \subset \Sigma$, let us denote the ranks of the kernel and the image of

$$i_*\colon H_1(S;\mathbb{Z}/2) \to H_1(\Sigma;\mathbb{Z}/2),$$

by $k(S)$ and $r(S)$, respectively.

The **homological Kauffman bracket**,

$$\langle D\rangle_\Sigma = \sum_{S\in\mathfrak{S}(D)} A^{a(S)-b(S)}\delta^{k(S)}z^{r(S)},$$

was introduced by Krushkal [Kru11] and studied in [BK19].

Based on this invariant, [BK19] introduced the notion of homological adequacy for link diagrams in surfaces. A diagram D on Σ is **homologically A-adequate** if $k(S) \leq k(S_A)$ for any state S adjacent to S_A, and it is **homologically B-adequate** if $k(S) \leq k(S_B)$ for any state S adjacent to S_B. A diagram D is **homologically adequate** if it is both homologically A- and B-adequate.

It is not difficult to show that a diagram that is plus-adequate is homologically A-adequate, and one that is minus-adequate is homologically B-adequate. (For further details, see §2.2 of [BK19].)

Proposition 8. *Every homologically A-adequate link diagram is A-adequate and every homologically B-adequate link diagram is B-adequate.*

Proof. Recall from the discussion at the beginning of the proof of Proposition 2 that there are the three possible cases, and to verify that a given diagram is A- or B-adequate, it is enough to check that the conditions of Definition 1 hold in case (i). Hence, it is enough to focus on states S adjacent to S_A or S_B with $|S| = |S_A|+1$ or $|S| = |S_B|+1$, respectively.

If D is not A-adequate, then there exists a state S adjacent to S_A with $|S| = |S_A|+1$, $t(S) = t(S_A)+1$, and $\widehat{S} = \widehat{S}_A$. (Notice that if $|S| = |S_A|+1$ and $t(S) = t(S_A)+1$, then $\widehat{S} = \widehat{S}_A$ automatically holds.) In this case, we have $k(S) = k(S_A)+1$, and it follows that D is not homologically A-adequate. This proves the first statement in the proposition, and the proof of the second statement on B-adequacy is similar. □

In summary then, for a link diagram D on a surface Σ, it follows that

$$\text{plus-adequacy} \Longrightarrow \text{homological } A\text{-adequacy} \Longrightarrow A\text{-adequacy}, \tag{5}$$

with similar statements relating minus-adequacy, homological B-adequacy, and B-adequacy.

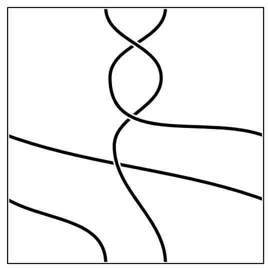


FIGURE 1. An alternating diagram on the torus.

In Example 20, we will see a knot diagram in a genus two surface which is adequate but not homologically adequate. On the other hand, it is easy to construct examples which are homologically adequate but not simply adequate. For instance, consider the alternating diagram D with three crossings on the torus in Figure 1. A straightforward calculation shows that it is homologically adequate but not simply adequate. These examples show that none of the reverse implications in (5) hold, therefore the notion of adequacy in Definition 1 is strictly more general than either homological or simple adequacy.

In general, notice that

$$\operatorname{span}(\langle D\rangle_\Sigma) \le \operatorname{span}([D]_\Sigma) \le 2c(D) + 2t(S_A) + 2t(S_B) \le 2c(D) + 2k(S_A) + 2k(S_B).$$

Therefore, Corollary 5 immediately implies an analogous inequality holds for homological adequacy, cf., [BK19, Corollary 2.7].

5. Alternating links and the Tait conjectures

When tabulating knots, Tait formulated three conjectures on alternating links. The first one states that any reduced alternating diagram of a classical link has minimal crossing number. The second one asserts that any two such diagrams representing the same link have the same writhe. The third one states that any two reduced alternating diagrams of the same link are related by flype moves. The first two conjectures were resolved almost 100 years later, independently by Kauffman, Murasugi, and Thistlethwaite, using the newly discovered Jones polynomial, [Kau87, Mur87, Thi87]. The third conjecture was established shortly after by Menasco and Thistlethwaite [MT93]. The first two Tait conjectures actually hold more generally for adequate links [LT88], and their proofs have been generalized to homologically adequate links in thickened surfaces in [BK19]. Here, we generalize these results even further to adequate links in thickened surfaces.

Henceforth, all links in thickened surfaces will be unframed, unless stated otherwise. Given an oriented link diagram D, let $c_+(D)$ and $c_-(D)$ be the numbers of positive and negative crossings, respectively. The proof of the following theorem can be found in Section 7.1.

Theorem 9. *Let D and E be oriented link diagrams on Σ representing the same oriented unframed link in $\Sigma \times I$.*

(i) *If D is A-adequate, then $c_-(D) \leq c_-(E)$.*
(ii) *If D is B-adequate, then $c_+(D) \leq c_+(E)$.*

The **crossing number** of a link $L \subset \Sigma \times I$, $c(L)$, is defined as the minimal crossing number among all diagram representatives of L. A link $L \subset \Sigma \times I$ is said to be **adequate** if it admits an adequate diagram on Σ.

Using Theorem 9, one can deduce the first and second Tait conjectures for adequate links in surfaces.

Corollary 10. *(i) Any adequate diagram of a link in $\Sigma \times I$ has $c(L)$ crossings.*
(ii) Any two adequate diagrams of the same oriented link in $\Sigma \times I$ have the same writhe.

Proof. Statements (i) and (ii) are immediate consequences of Theorem 9. In the case of (ii), if adequate diagrams D and E represent the same oriented link, then $c_+(D) = c_+(E)$ and $c_-(D) = c_-(E)$ by the above theorem and, hence,

$$w(D) = c_+(D) - c_-(D) = c_+(E) - c_-(E) = w(E). \quad \square$$

Corollary 10 implies that for an adequate link $L \subset \Sigma \times I$, the writhe is a well-defined invariant of its oriented link type.

Let $g(\Sigma)$ be the sum of the genera of the connected components of Σ. A link diagram D on Σ is **minimally embedded** if it does not lie on a subsurface of Σ of smaller genus. In other words, the complement of D on Σ has no non-separating loops. Let N_D be a neighborhood of D in Σ small enough so that it is a ribbon surface retractible onto D. A diagram D is minimally embedded if and only if $g(N_D) = g(\Sigma)$.

Furthermore, note that if D is connected and Σ is closed, then D is minimally embedded if and only if $\Sigma \smallsetminus D$ is composed of disks. In that case, we say that D is **cellularly embedded**.

A link diagram D on a closed surface Σ is said to have **minimal genus** if it is minimally embedded within its isotopy class.

In [Man13], it is proved that any cellularly embedded knot diagram with minimal crossing number has minimal genus. This result was recently extended to link diagrams, and the following is a restatement of Theorem 1 of [BR20].

Theorem 11. *Any cellularly embedded link diagram with minimal crossing number has minimal genus.*

A link diagram D on Σ is **alternating** if, when traveling along any of its components, its crossings alternate between over and under. A link $L \subset \Sigma \times I$ is **alternating** if it can be represented by an alternating link diagram.

A crossing x of D is **nugatory** if there is a simple loop in Σ which separates Σ and intersects D only at x.

As observed in [BK19], although nugatory crossings in diagrams in $\Sigma = \mathbb{R}^2$ can always be removed by rotating one side of the diagram 180° relative to the other, that is not always true for diagrams in non-contractible surfaces Σ, see Figure 2. A nugatory crossing is said to be **removable** if the simple loop can be chosen to bound a disk, otherwise it is called **essential**. A link diagram is **reduced** if it does not contain any removable nugatory crossings. For example, the knot in Figure 6 contains an essential nugatory crossing.

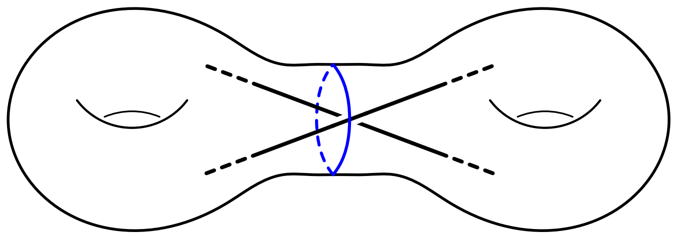


FIGURE 2. An essential nugatory crossing.

The following strengthens Proposition 2.8 of [BK19]. Its proof is given in Section 7.2.

Theorem 12. *Any reduced alternating diagram is adequate.*

Note that unlike Proposition 2.8 of [BK19], we do not assume here that D is cellularly embedded, checkerboard colorable, nor that D has no nugatory crossings.

A link diagram on Σ is said to be **weakly alternating** if it is a connected sum $D_0 \# D_1 \# \cdots \# D_k$ of an alternating diagram D_0 in Σ and with alternating diagrams $D_1, \ldots, D_k$ in S^2 (cf., Lemma 16). Theorem 12 can be generalized to show that weakly alternating diagrams are adequate. In fact, in the next section we will prove Proposition 17, showing that any diagram on a surface obtained as the connected sum of two adequate link diagrams is itself adequate.

Let us return to Tait conjectures now. By Corollary 10, any reduced alternating diagram D has the minimal crossing number for all diagrams representing the same unframed link L in $\Sigma \times I$. Furthermore, all such oriented diagrams representing the same link L have the same writhe.

The results of Kauffman, Murasugi, Thistlethwaite [Kau87, Mur87, Thi87] imply that the span of the Kauffman bracket of any diagram $D \subset S^2$ satisfies

$$\operatorname{span}([D]_{S^2}) \leq 4c(D) + 4,$$

or equivalently for the Jones polynomial, that $\operatorname{span}(V_D(t)) \leq c(D)$, with equality if D is alternating. Furthermore, in [Thi87] Thistlethwaite proved that if $D \subset S^2$ is prime and non-alternating, then

$$\operatorname{span}([D]_{S^2}) < 4c(D) + 4.$$

In [Tur87], it is observed that the above results hold if $D \subset S^2$ is weakly alternating, namely if D is a connected sum of alternating diagrams. Thus the Kauffman bracket $[D]_{S^2}$, together with $c(D)$, *detect* weakly alternating classical links.

The homological Kauffman bracket of [BK19] is not sufficiently strong to prove an analogous statement for links in thickened surfaces. Consider the two knots in the genus two surface in Figure 3. These knots have the same homological Kauffman bracket, namely

$$\langle D_1 \rangle_\Sigma = \langle D_2 \rangle_\Sigma = 3\delta z^2 - 4\delta^2 z + (A^4 + 3 + A^{-4})\delta,$$

but one of them is alternating and the other is not. Consequently, the homological Kauffman bracket does not detect alternating knots in thickened surfaces.

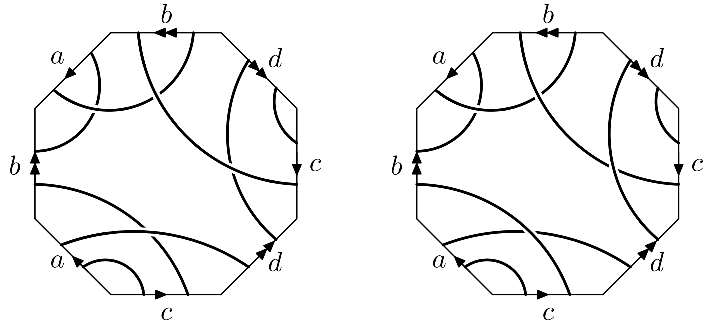


FIGURE 3. Two knots in a genus two surface with the same homological Kauffman bracket.

However, we are going to show that Kauffman, Murasugi, Thistlethwaite statements hold for the Kauffman bracket $[\cdot]_\Sigma$ of diagrams in closed surfaces Σ after replacing 4 by $4|D| - 4g(\Sigma)$ on the right.

Let $|D|$ denote the number of connected components of D (which may be smaller than the number of connected components of the link in $\Sigma \times I$ represented by D). The proof of the next result is given in Section 7.4.

Theorem 13. *(i) If D is minimally embedded in Σ then*

$$\operatorname{span}([D]_\Sigma) \leq 4c(D) + 4|D| - 4g(\Sigma).$$

(ii) If D is cellularly embedded, reduced, and weakly alternating, then

$$\operatorname{span}([D]_\Sigma) = 4c(D) + 4|D| - 4g(\Sigma).$$

(iii) If D is minimally embedded and not weakly alternating then

$$\operatorname{span}([D]_\Sigma) < 4c(D) + 4|D| - 4g(\Sigma).$$

The assumptions of Theorem 13 are necessary:
(i) If D is not required to be minimally embedded, then by adding handles to Σ, one can make the right hand side of the above inequality a negative number of an arbitrarily large magnitude.
(ii) If D has a removable nugatory crossings, then eliminating it decreases the right hand side of the above equality but not the left hand side. Therefore, (ii) does hold for diagrams with removable crossings.

It also fails unless D is cellularly embedded. For example, consider the alternating link in Figure 4. It has $t(S_A) = 4$ and $t(S_B) = 2$, and so $\operatorname{span}([D]_\Sigma) \leq 16 + 12 = 28$ whereas $4c(D) + 4|D| - 4g(\Sigma) = 35$. Note that this diagram is minimally embedded but not cellularly embedded.

Although (ii) holds for weakly alternating diagrams, in the next section we will see that it does not hold generally for connected sums of alternating diagrams in arbitrary surfaces (see Example 19).

Corollary 14. *Let L be a link in $\Sigma \times I$ with a reduced, weakly alternating diagram D which is cellularly embedded. Then any other cellularly embedded diagram E for L satisfies $c(D) \leq c(E)$. If E is not weakly alternating, then $c(D) < c(E)$.*

Proof. The first part of the proof is a direct consequence of Tait conjecture (Corollary 10). Let us prove the full statement now: Any cellularly embedded link diagram on

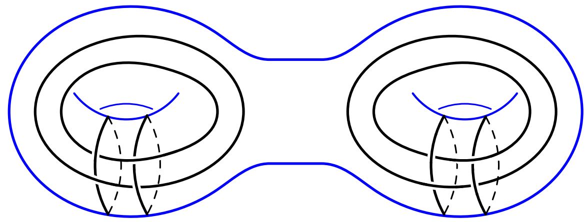


FIGURE 4. Minimally embedded alternating diagram for which the equality of Theorem 13 (ii) does not hold.

a connected surface is itself connected. Therefore, it is enough to prove the statement under the assumption that Σ and D are both connected. Theorem 13 (ii) then implies that $c(D) = \operatorname{span}([D]_\Sigma)/4 + g(\Sigma) - 1$. If E is a second link diagram for L on Σ, then since E is cellularly embedded, it must also be connected. Theorem 13 (i) implies that

$$c(D) = \operatorname{span}([D]_\Sigma)/4 + g(\Sigma) - 1 = \operatorname{span}([E]_\Sigma)/4 + g(\Sigma) - 1 \leq c(E).$$

If E is not weakly alternating, then Theorem 13 (iii) shows the last inequality is strict, therefore it follows that $c(D) < c(E)$. □

Remark 15. The corollary gives an alternate proof of Theorem 11 for non-split alternating links as follows. Let L be a non-split alternating link in $\Sigma \times I$, where Σ is closed oriented surface, and let $D \subset \Sigma$ a minimal crossing cellularly embedded diagram for L. Then Corollary 14 implies that D is an alternating diagram. The argument is completed by appealing to Proposition 6 of [BK20b], which shows that alternating link diagrams have minimal genus.

6. Crossing number and connected sums

In this section, we will study the behavior of the crossing number under connected sum of links in thickened surfaces. This problem is closely related to an old and famous conjecture for classical links, which asserts that, for any two links L_1, L_2,

$$c(L_1 \# L_2) = c(L_1) + c(L_2). \tag{6}$$

This conjecture has been verified for a wide class of links, including alternating links, adequate links, and torus links [Dia04]. Clearly, $c(L_1 \# L_2) \leq c(L_1) + c(L_2)$. In addition, in [Lac09], Lackenby has proved that, in general, one has a lower bound of the form:

$$c(L_1 \# L_2) \geq \tfrac{1}{152} \left(c(L_1) + c(L_2) \right).$$

The operation of connected sum is not so well-behaved for arbitrary links in thickened surfaces.

Just as for classical links, it depends on the choice of components which are joined as well as their orientations. However, unless one of the links is in $S^2 \times I$, it also depends on the diagram representatives as well as the choice of basepoints $x_i \in D_i$ where the link components are joined. The issue is the fact that a Reidemeister move applied to either of the link diagrams may change the link type of their connected sum. We take a moment to quickly review its construction.

Suppose Σ_1 and Σ_2 are oriented surfaces and let $\Sigma_1 \# \Sigma_2$ denote their connected sum. It is obtained from the union $(\Sigma_1 \smallsetminus \operatorname{int} B_1) \cup (\Sigma_2 \smallsetminus \operatorname{int} B_2)$ by gluing $\partial B_1 \subset \Sigma_1$ to $\partial B_2 \subset \Sigma_2$ by an orientation reversing homeomorphism $g \colon \partial B_1 \to \partial B_2$. For connected surfaces, $\Sigma_1 \# \Sigma_2$ is independent of the choice of disks $B_i \subset \Sigma_i$ and gluing map.

If $D_1 \subset \Sigma_1$ and $D_2 \subset \Sigma_2$ are link diagrams, we can choose cutting points $x_i \in D_i$ and disk neighborhoods B_i from Σ_i such that $B_i \cap D_i$ is an interval for $i = 1, 2$. Then the surface $\Sigma_1 \# \Sigma_2$ can be formed in such a way that $D = (D_1 \smallsetminus \operatorname{int} B_1) \cup (D_2 \smallsetminus \operatorname{int} B_2)$ is a link diagram in $\Sigma_1 \# \Sigma_2$. If D_1, D_2 are oriented link diagrams, then we require the gluing to respect the orientations of the arcs. The resulting diagram is called a **connected sum** of D_1 and D_2. In general, it depends on the choice of link diagrams D_1, D_2, components being joined, and the points $x_i \in D_i$. However, it is independent of the choice of disk neighborhoods B_i containing x_i.

The next result shows that when one of the diagrams lies in $S^2 \times I$, the operation of connected sum is well-behaved.

Lemma 16. *Let $D_1 \subset \Sigma \times I$ and $D_2 \subset S^2 \times I$ be oriented diagrams, where Σ is an arbitrary surface. Then the connected sum of D_1 and D_2 is independent of the choice of the cutting points x_1, x_2 on the selected components of D_1 and of D_2.*

We will denote the connected sum in this case by $D_1 \# D_2$. The oriented link type of $D_1 \# D_2$ depends only on the link types of D_1 and D_2 and a choice of which components are joined.

Proof. One can shrink the image of D_2 in the connected sum so that all its crossings lie in a small 3-ball B^3 in $\Sigma \times I$. By an isotopy, we can move the ball along arcs of D_1 representing the component to which D_1 is joined, and moving over or under the other arcs at any crossing that we encounter.

This shows that the connected sum is independent of the choice of the cut point x_1 on D_1. The independence on the cut point x_2 on D_2 follows from the well-known fact that all long knots, or rather $(1, 1)$ tangles, obtained by cutting D_2 at different points x_2 of its specified component are isotopic (as $(1, 1)$ tangles). Shrinking D_2 into a small 3-ball also allows one to translate any Reidemeister move of D_1 or D_2 into a Reidemeister move on the connected sum $D_1 \# D_2$. This proves the last statement. □

Proposition 17. *Any connected sum of two A- or B-adequate diagrams is itself A- or B-adequate (respectively).*

Proof. Let D be a link diagram in $\Sigma_1 \# \Sigma_2$ obtained as the connected sum of A-adequate diagrams $D_1 \subset \Sigma_1$ and $D_2 \subset \Sigma_2$, and suppose to the contrary that D is not A-adequate. By Proposition 2, there is a state S for D adjacent to S_A with $t(S, \Sigma_1 \# \Sigma_2) = t(S_A, \Sigma_1 \# \Sigma_2) + 1$ and $|\widehat{S}| = |\widehat{S}_A|$ in $\Sigma_1 \# \Sigma_2$. In particular, $|S| = |S_A| + 1$, and the transition from S_A to S involves one cycle of S_A splitting into two cycles.

Each cycle of D is a connected sum of a cycle of D_1 and a cycle of D_2. Let x be the crossing of D where the smoothing is changed in the transition from S_A to S. We can assume, without loss of generality, that x is a crossing from D_1. Let $C_1 \# C_2$ be the cycle of S_A that splits into two cycles, $C_1' \# C_2$ and $C_1'' \# C_2$ under this transition. Since $t(S, \Sigma_1 \# \Sigma_2) = t(S_A, \Sigma_1 \# \Sigma_2) + 1$, one of the cycles $C_1' \# C_2$ and $C_1'' \# C_2$, say $C_1' \# C_2$, must be trivial. This implies that C_1' is trivial in Σ_1 and C_2 is trivial in Σ_2.

If $C_1 \# C_2$ is trivial, then $C_1'' \# C_2$ must also be trivial. That would imply that all three of C_1, C_1', C_1'' are trivial in Σ_1. This contradicts the assumption that D_1 is A-adequate, and we take a moment to explain this point.

Let $S(D_1)$ be the corresponding state for D_1. It is obtained from $S_A(D_1)$ by switching the smoothing at x. The transition from $S_A(D_1)$ to $S(D_1)$ involves C_1 splitting into C_1' and C_1''. Since all three of C_1, C_1', C_1'' are trivial in Σ_1, we have $t(S(D_1)) = t(S_A(D_1)) + 1$ and $|\widehat{S}(D_1)| = \widehat{S}_A(D_1)$ in Σ_1, which contradicts the assumption of A-adequacy of D_1.

The other possibility is that $C_1\#C_2$ is non-trivial. Since C_2 is trivial in Σ_2, the cycles C_1 and C_1'' must both be nontrivial in Σ_1. The transition from $S_A(D_1)$ to $S(D_1)$ still involves C_1 splitting into C_1' and C_1'', only now C_1, C_1'' are nontrivial and C_1' is trivial in Σ_1. Thus $t(S(D_1)) = t(S_A(D_1))+1$ and $|\widehat{S}(D_1)| = \widehat{S}_A(D_1)|$ in Σ_1, which again contradicts the assumption of A-adequacy of D_1. Therefore, $D = D_1\#D_2$ must be A-adequate.

The proof of B-adequacy of D is similar. □

Corollary 18. *Suppose $L_1 \subset \Sigma_1 \times I$ and $L_2 \subset \Sigma_2 \times I$ are links represented by adequate diagrams $D_1 \subset \Sigma_1$ and $D_2 \subset \Sigma_2$. Then any link L in $(\Sigma_1\#\Sigma_2)\times I$ admitting a diagram which is a connected sum of D_1 and D_2 is itself adequate. Further, the crossing number and writhe satisfy $c(L) = c(L_1) + c(L_2)$ and $w(L) = w(L_1) + w(L_2)$.*

Proof. Suppose L is represented by $D = D_1\#D_2 \subset \Sigma_1\#\Sigma_2$. Then D is adequate by Proposition 17. Further, by parts (i) and (ii) of Corollary 10, we see that:

$$\begin{aligned} c(L) &= c(D) = c(D_1) + c(D_2) = c(L_1) + c(L_2) \quad \text{and} \\ w(L) &= w(D) = w(D_1) + w(D_2) = w(L_1) + w(L_2). \quad \square \end{aligned}$$

Example 19. Figure 5 shows a knot diagram D in the genus two surface obtained as the connected sum of two alternating diagrams of the same knot in the torus. One can easily verify that D is reduced and cellularly embedded, but not alternating. Further, Proposition 17 implies that this diagram is adequate, and therefore a minimal crossing diagram for the knot type. Direct calculation reveals that $t(S_A) = 2, t(S_B) = 0$, and $|\hat{S}_A| = |\hat{S}_B| = 1$. Therefore $\text{span}([D]_\Sigma) = 16$. On the other hand, since $4(c(D) + |D| - g(\Sigma)) = 20$, by Theorem 13 (ii), it follows that D is not weakly alternating, and in fact not equivalent to any weakly alternating knot in $\Sigma \times I$.

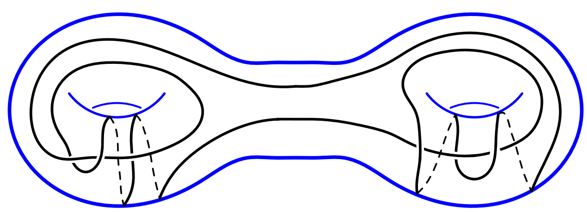


FIGURE 5. A connected sum of alternating diagrams.

Example 20. Figure 6 shows a knot in a genus two surface with an essential nugatory crossing. Since it is reduced and alternating, Theorem 12 shows that it is adequate. Note that this diagram is not homologically adequate. In fact, if S is the state with a B-smoothing at the nugatory crossing and A-smoothings at all the other crossings, then one can show that $|S| = |S_A| + 1$ and $k(S) > k(S_A)$.

Notice that this knot can also be obtained as the connected sum of two alternating knots K_1, K_2 in $T^2 \times I$, with $c(K_i) = 3$ but after performing a Reidemeister one move on one of them to obtain a diagram with four crossings. In particular, this example shows that a connected sum of two diagrams $D_1 \subset \Sigma_1$ and $D_2 \subset \Sigma_2$ can be adequate even when one of them is not adequate.

Suppose $L_1 \subset \Sigma_1 \times I$ and $L_2 \subset \Sigma_2 \times I$ are two alternating links in thickened surfaces with $g(\Sigma_i) > 0$ for $i = 1, 2$. Suppose further that D_i is a link diagram on Σ_i representing L_i for $i = 1, 2$, and that D_1, D_2 are both reduced and alternating.

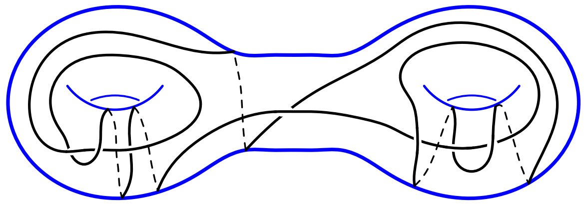


FIGURE 6. An alternating diagram with an essential nugatory crossing.

Instead of forming the connected sum of D_1 and D_2, take one of the diagrams and insert an arbitrary number (say n) of twists before forming the connected sum. See Figure 7 for an illustration.

The result will be a diagram D, which is similar to a connected sum of D_1 and D_2, but with n essential nugatory crossings in between. This construction can be carried out so that D is reduced and alternating. In particular, it will have crossing number $c(D) = c(D_1) + c(D_2) + n$. If L denotes the link type of D, and since D_1 and D_2 are alternating and have minimal crossing number, this shows that the analogue of (6) can fail arbitrarily badly for links in thickened surfaces other than $S^2 \times I$.

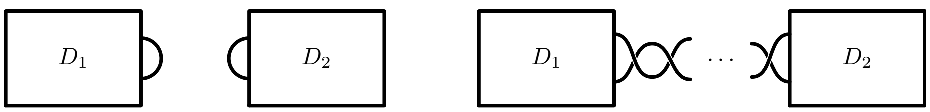


FIGURE 7. Adding twists to a connected sum to create essential nugatory crossings.

The reason (6) fails in general for connected sums of links in thickened surfaces is due to the use of non-minimal diagrams in forming the connected sum. However, if one restricts the connected sum operation to minimal crossing diagrams, then one gets a plausible generalization:

Conjecture 21. *Suppose $L_1 \subset \Sigma_1 \times I$ and $L_2 \subset \Sigma_2 \times I$ are links in thickened surfaces with minimal crossing representatives D_1, D_2, respectively. Then any link L in the thickening of $\Sigma_1 \# \Sigma_2$ arising as a connected sum of D_1 and D_2 satisfies*

$$c(L) = c(L_1) + c(L_2).$$

Note that the assumption that D_1, D_2 are minimal crossing representative implies immediately that

$$c(L) \leq c(L_1) + c(L_2).$$

In fact, the inequality may fail without that assumption. This is related to the fact that crossing number is not additive under connected sum for virtual knots. For example, the Kishino knot is the connected sum of two virtual unknots. As evidence, notice that Corollary 18 confirms that the conjecture is true if L_1 and L_2 are adequate links in thickened surfaces. In particular, it holds for alternating and weakly alternating links.

7. PROOFS OF THEOREMS 9, 12, AND 13

7.1. **Proof of Theorem 9.** Given a link diagram D on Σ and positive integer r, the **r-th parallel** of D is the link diagram D^r on Σ in which each link component of D is replaced by r parallel copies, with each one repeating the same "over" and "under" behavior of the original component.

Lemma 22. *If D is A-adequate, then D^r is also A-adequate. If D is B-adequate, then D^r is also B-adequate.*

Proof. Let $S_A(D)$ and $S_A(D^r)$ be the pure A-smoothings of D and the pure A-smoothings of D^r, respectively. It is straightforward to check that $S_A(D^r)$ is the r-parallel of $S_A(D)$.

Suppose D^r is not A-adequate. Then there is a state S' obtained by switching one A-smoothing in $S_A(D^r)$ to a B-smoothing, such that $t(S_A(D^r)) < t(S')$, and $\widehat{S}_A(D^r) = \widehat{S}'$. In the terminology of the proof of Proposition 2, that can only happen for a smoothing change of type (i). More specifically, when the smoothing change involves one of the innermost cycles in $S_A(D^r)$ which is trivial and self-abutting. Notice that it is only possible if there is a trivial self-abutting cycle in $S_A(D)$. Since D is A-adequate, this cannot happen.

An analogous argument proves the statement for B-adequate diagrams. □

Proof of Theorem 9. (i) Since

$$c(D) - w(D) = c_+(D) + c_-(D) - (c_+(D) - c_-(D)) = 2c_-(D),$$

we will prove that

$$c(D) - w(D) \leq c(E) - w(E).$$

Our argument is an adaptation of Stong's proof [Sto94] (cf., Theorem 5.13 [Lic97]).

Let $L_1, \ldots, L_m$ be the components of L and let D_i and E_i be the subdiagrams of D and E corresponding to L_i. For each $i = 1, \ldots, m$, choose non-negative integers μ_i and ν_i such that $w(D_i) + \mu_i = w(E_i) + \nu_i$. Let D' be composed of components $D'_1, \ldots, D'_m$, where each D'_i is obtained from D_i by adding μ_i positive kinks to it. (These kinks do not cross with other components). Similarly, let E' be composed of components $E'_1, \ldots, E'_m$, where each E'_i is obtained from E_i by adding ν_i positive kinks to it. Notice that D' is still A-adequate.

The writhes of the individual components satisfy:

$$w(D'_i) = w(D_i) + \mu_i = w(E_i) + \nu_i = w(E'_i).$$

Since, furthermore, there is a 1-1 correspondence between the crossings $D'_i \cap D'_j$ and $E'_i \cap E'_j$ and the crossing signs coincide, $w(D') = w(E')$.

For any r, consider the r-th parallels $(D')^r$ and $(E')^r$ now. Then $w((D')^r) = r^2 w(D')$, because each crossing of D' corresponds to r^2 crossings in $(D')^r$ of the same sign. The diagrams $(D')^r$ and $(E')^r$, are equivalent and have the same writhe, thus their Kauffman brackets must be equal. In particular, we have $d_{\max}([(D')^r]_\Sigma) = d_{\max}([(E')^r]_\Sigma)$. Proposition 3 implies now that

$$\begin{aligned} d_{\max}([(D')^r]_\Sigma) &= \left(c(D) + \sum_{i=1}^m \mu_i\right) r^2 + 2\left(t(S_A(D)) + \sum_{i=1}^m \mu_i\right) r, \\ d_{\max}([(E')^r]_\Sigma) &\leq \left(c(E) + \sum_{i=1}^m \nu_i\right) r^2 + 2\left(t(S_A(E)) + \sum_{i=1}^m \nu_i\right) r. \end{aligned}$$

Since this is true for all r, by comparing coefficients of the r^2 terms, we find that:

$$c(D) + \sum_{i=1}^m \mu_i \leq c(E) + \sum_{i=1}^m \nu_i. \tag{7}$$

Subtracting $\sum_{i=1}^{m}(\mu_i + w(D_i)) = \sum_{i=1}^{m}(\nu_i + w(E_i))$ from both sides of (7), we get that

$$c(D) - \sum_{i=1}^{m} w(D_i) \leq c(E) - \sum_{i=1}^{m} w(E_i). \tag{8}$$

Subtracting the total linking number of L from both sides of (8) gives the desired inequality.

The proof of (ii) is analogous. One adds negative kinks to D and E in this case. □

7.2. **Proof of Theorem 12.** A link diagram D on Σ is **alternable** if it can be made alternating by inverting some of its crossings. Every classical link diagram is alternable, but the same is not true for link diagrams in arbitrary surfaces. For example, the knot diagram in the torus in Figure 8 is not alternable.

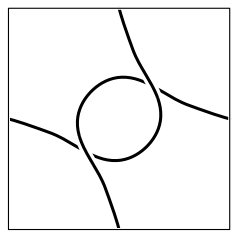


FIGURE 8. A knot diagram in the torus which is not alternable.

A link diagram D on Σ is **checkerboard colorable** if the components of $\Sigma \setminus D$ can be colored by two colors such that any two components of $\Sigma \setminus D$ that share an edge have opposite colors.

Proposition 23. *Any minimal embedding D on Σ is alternable if and only if it is checkerboard colorable.*

Proof. Filling the boundaries of Σ with disks does not affect alternability nor checkerboard colorability. This has two consequences:
(a) it is enough to prove this statement for surfaces Σ with all boundary components capped, i.e. for closed surfaces.
(b) Since Kamada proved that if a diagram D is a deformation retract of Σ then it is alternable if and only if it is checkerboard colorable, [Kam02, Lemma 7], our statement holds for cellularly embedded diagrams.

Our strategy is to reduce the proof to this case of cellular embeddings. Suppose that C is a non-disk component of $\Sigma \setminus D$. Then it contains a non-contractible simple closed loop α. Let Σ' be obtained by cutting Σ along α and by capping the boundary components. The loop α must be separating Σ, since otherwise $D \hookrightarrow \Sigma'$ would be a lower genus embedding of D. Observe now that since Σ is a connected sum of two surfaces $\Sigma_1 \# \Sigma_2$, where $\Sigma_1 \cup \Sigma_2 = \Sigma'$ and D is a disjoint union of $D \cap \Sigma_1$ and of $D \cap \Sigma_2$, it is enough to prove that $D \subset \Sigma_i$ is checkerboard colored for $i = 1, 2$.

By repeating this process as long as possible we reduce the statement to cellularly embedded diagrams, which is covered by (b) above. □

Lemma 24. *Any alternable diagram can be extended by disjoint simple closed loops to a checkerboard colorable one.*

Proof. The surface $N_D \subset \Sigma$, being a regular neighborhood of D, is checkerboard colorable by the earlier mentioned result of Kamada, [Kam02, Lemma 7]. The only reason that coloring does not extend to $D \subset \Sigma$ is that some connected components C of $\Sigma \setminus \text{int}\, N_D$ may have multiple connected components of their boundary whose neighborhoods are colored differently. However, that issue can be resolved by adding simple closed loops around those boundary components of C which are white. □

Proof of Theorem 12: Let D be alternating diagram without removable crossings. By Proposition 23, by adding disjoint simple closed loops to D we obtain a diagram D' which is alternating and checkerboard colorable. Hence, it is enough to prove that D' is adequate. Let us assume for simplicity of notation that D is checkerboard colorable.

We will prove the A-adequacy of D only, as the proof of B-adequacy is identical. Let S be a state with all A-smoothings except for a B-smoothing at a crossing x of D. We will prove that D is A-adequate "at x," meaning that $t(S) \leq t(S_A)$ or $\widehat{S} \neq \widehat{S}_A$ in $\mathscr{S}(\Sigma \times I)$.

As in the proof of Proposition 2, there are three cases, and to check adequacy, it is enough to check that the conditions of Definition 1 hold in the first case, namely when $|S| = |S_A| + 1$. Therefore, S_A must contain a self-approaching cycle C, and in the transition from S_A to S, the cycle C splits into two cycles C_1, C_2 of S. Since D is alternating and checkerboard colorable, S_A bounds a subsurface Σ' of Σ of a certain color, say white, which contains no crossings of D.

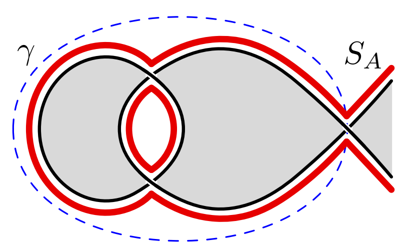


Figure 9.

We claim that neither C_1 nor C_2 is trivial. Indeed, if say C_1 were trivial, then there would be a loop γ parallel to C_1 totally inside Σ' except for a little neighborhood of x, in which it would cross x. Such a curve would imply that the crossing x is removable, (see for example Figure 9), which is a contradiction. Therefore neither C_1 nor C_2 is trivial, and it follows that $t(S) = t(S_A)$. Therefore, D is A-adequate at x, and this completes the proof of the theorem. □

7.3. **Link diagrams and shadows.** A **link shadow** in Σ is a 4-valent graph in Σ, possibly with loop components. In other words, a shadow is a link diagram with crossing types ignored. For that reason we refer to shadow vertices as crossings and the components of any link realization of a shadow as its components. (Not to be confused with connected components of a shadow.)

Some properties of link diagrams are entirely determined by its link shadow. For example, we will say that a link shadow D on Σ is **checkerboard colorable** if the components of $\Sigma \setminus D$ can be colored by two colors such that any two components of $\Sigma \setminus D$ that share an edge have opposite colors. Clearly, a link diagram is checkerboard colorable if and only if its link shadow is. Similarly, a link shadow is **minimally embedded** if it does not lie in a subsurface of Σ of smaller genus, and it is immediate that a link diagram on Σ is minimally embedded if and only if its link shadow is.

Each shadow crossing has two smoothings, which cannot be differentiated as A- and B-type, as in the case of link diagrams. For that reason, for shadow links it is customary to place markers at the crossings indicating the smoothing as in Figure 10.

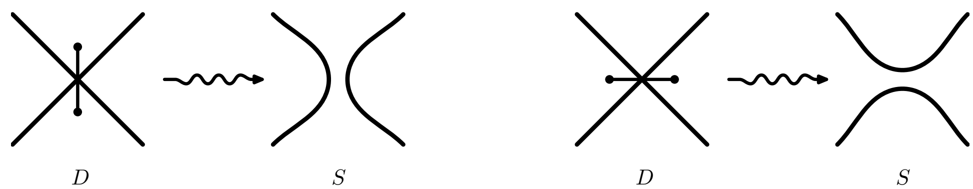


FIGURE 10. Two types of markers for a state of a link shadow.

Two consecutive crossings can have identical or opposite smoothings, see Figure 11. An **alternating state** of a shadow is one with alternating crossing smoothings along all of its components. In other words, a state is alternating if the smoothings at every pair of consecutive crossings are opposite. Not all shadow links admit alternating smoothings, for example the shadow of the non-alternable knot in the torus in Figure 8. On the other hand, any shadow link of an alternating link diagram admits two alternating smoothings, namely the shadow smoothings coming from S_A and S_B.

Given a state S for a link shadow D, the dual state is denoted $S^\vee$ and has opposite smoothing to S at each crossing of D. Notice that a state S is alternating if and only if its dual state $S^\vee$ is alternating.

We say that a 2-disk D^2 is 2-cutting, or simply, **cutting** a shadow D if its boundary intersects D transversely at two points (which are not crossings) and $D^2 \cap D$ contains some but not all the crossings of D. A connected shadow D is said to be **strongly prime** if it has no cutting disk. More generally, a shadow D is strongly prime if all of its connected components are.

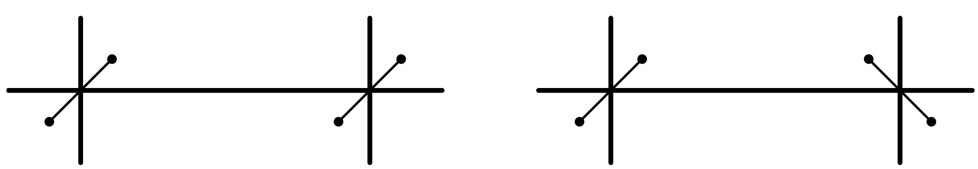


FIGURE 11. Two consecutive crossings with identical markers (left) and opposite markers (right).

Lemma 25. *Every crossing of every strongly prime shadow $D \subset \Sigma$ has at least one smoothing producing a shadow which is again strongly prime.*

For classical links, a proof of this statement can be found in [Lic97]. That proof relies on checkerboard colorability of the diagram, which is of course true for classical links. Below, we give a proof that does not require the shadow to be checkerboard colorable.

Proof. Without loss of generality we can assume that D is connected. Assume now that the smoothings of a crossing v in a strongly prime D produce diagrams D_1, D_2 neither of which is strongly prime. Let B_1, B_2 be cutting disks for D_1 and D_2. Since D is strongly prime, we can assume that $v \in \partial B_i$ for $i = 1, 2$. We can also assume that ∂B_1 and ∂B_2 are in transversal position. Let C be the connected component of $B_1 \cap B_2$ containing v, as in Figure 12 (left). The circles ∂B_1, ∂B_2 are broken because they may intersect each other many times.

By modifying B_1 or B_2 slightly if necessary we can assume that D does not contain the second intersection point, w, of $\partial B_1 \cap \partial B_2$ in C.

Let $\alpha_1 = \text{int}(C \cap \partial B_1)$ and $\alpha_2 = \text{int}(C \cap \partial B_2)$. (Note that $v \notin \alpha_1 \cup \alpha_2$.) Since D intersects $\partial B_i - \{v\}$ twice, for $i = 1, 2$, and D intersects $\alpha_1 \cup \alpha_2$ at an odd number of points, we have the following possibilities:
(1) $|D \cap \alpha_2| = 1$, $D \cap \alpha_1 = \varnothing$
(2) $|D \cap \alpha_2| = 2$, $|D \cap \alpha_1| = 1$
(3) one of the two cases above with α_1 interchanged with α_2. We will ignore it without loss of generality.

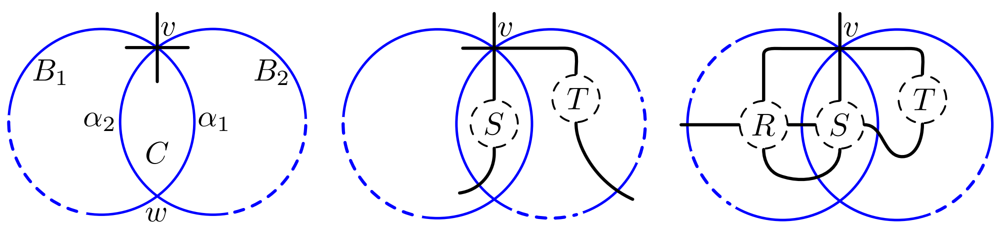


Figure 12.

In the first case, D looks like in Figure 12 (center), where S, T (in dashed circles) are shadow tangles. In that case, since neighborhoods of S, T are not cutting disks for D, the tangles S, T are crossingless. That means that B_2 is not a cutting disk for D_2 – a contradiction.

In the second case, D looks like in Figure 12 (right), where R, S, T are shadow tangles. Note that all crossings of D, other than v, are contained in R, S or T, since otherwise a disk containing v, R, S and T but no other crossings of D would be cutting for D. Note also that, as in the first case, T is crossingless. That means that all crossings of D_1 are in R and S. Hence, B_1 is not cutting for D_1 – a contradiction. □

Proposition 26. *Let D be a link shadow minimally embedded in Σ (not necessarily connected). Then for any state S of D,*
(1)

$$t(S) + t(S^\vee) \leq c(D) + 2|D| - 2g(\Sigma).$$

(2) If D is strongly prime and S is non-alternating, then

$$t(S) + t(S^\vee) < c(D) + 2|D| - 2g(\Sigma).$$

Proof. If D is disconnected then there is a non-trivial simple closed loop $\alpha \subset \Sigma$ in the complement of D. Since D is minimally embedded, that loop is separating. Cutting Σ along α and capping the holes does not affect the right hand side of the above inequalities and it can only increase the left hand side. Therefore, it is enough to prove statements (1) and (2) for connected shadows in closed surfaces. By the discussion at the end of Section 5, when D is connected and Σ is closed, the diagram D is minimally embedded if and only if it is cellularly embedded.

Recall that N_D denotes a neighborhood of D in Σ small enough so that it is a ribbon surface retractible onto D. For the remainder of the proof it will be useful to remember that D is minimally embedded if and only if $g(N_D) = g(\Sigma)$.

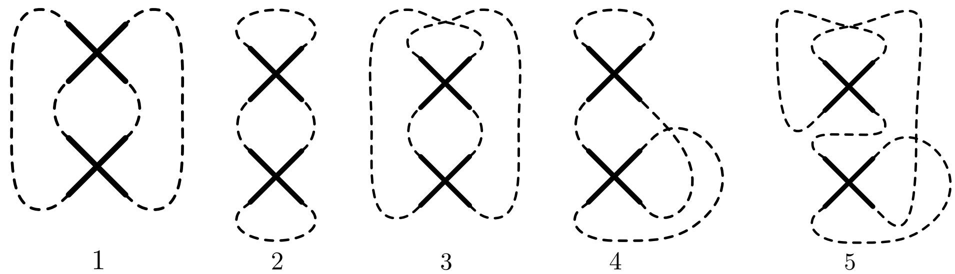


FIGURE 13. Connected shadow diagrams with 2-crossings

If $c(D) = 0$ then D is a loop, $|S| = |S^\vee| = 1$, $g(\Sigma) = 0$ and statements (1) and (2) hold. (Note that D is alternating.)

Assume $c(D) = 1$ now. If the crossing of D is a loop self-intersection then $g(\Sigma) = 0$,

$$t(S) + t(S^\vee) \leq 2 + 1,$$

S is alternating and statement (1) holds and statement (2) is vacuously true. If the crossing of D is between different components (of a link lift of D), then both smoothings of the crossing produce non-trivial loops and, hence, $t(S) = t(S^\vee) = 0$. Since $g(\Sigma) = 1$ statements (1) and (2) hold.

Let us prove part (2) for two crossing shadows. All possible abstract connected 2-crossing shadows D are shown in Figure 13.

For a connected two crossing shadow diagram D of the first kind, the cellularity assumption implies that $\Sigma \smallsetminus D$ consists of disks only. Consequently, D must lie in $\Sigma = S^2$. The non-alternating states S of D of the first kind are shown in Figure 14.

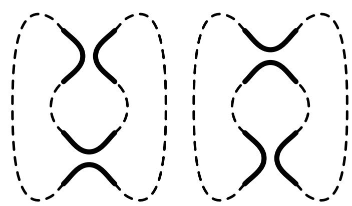


FIGURE 14. Non-alternating states for the first kind of a two crossing connected shadow diagram

Since $t(S) = t(S^\vee) = 1$, statement (2) holds in this case.

In case D is a diagram of the second kind, the cellularity assumption again implies that D must lie in $\Sigma = S^2$. However, such a diagram cannot be strongly prime, so there is nothing to prove for part (2) in this case.

Note that a cellularly embedded shadow of the third, fourth, and fifth kind must lie in a torus and that $t(S) = t(S^\vee) = 0$ for its non-alternating states in these cases. Consequently, statement (2) holds.

Now we continue the proof by induction on the crossing number $c(D)$. The base case is $c(D) = 1$ for part (1) and $c(D) = 2$ for part (2).

Inductive step: Assume the statement holds for all shadows with at most $c - 1$ crossings. Let S be a state of a cellularly embedded, c-crossing shadow D and x a crossing in D.

For part (2) we assume additionally that D is strongly prime and S non-alternating. Then S has two consecutive smoothings that are identical, and we require that x is a third crossing of D. Let D' be obtained from D by a smoothing of x which is arbitrary for part (1) but strongly prime for part (2). Such a smoothing exists by Lemma 25.

The chosen smoothing coincides either with the smoothing of x in S or in $S^\vee$ and, without loss of generality, we can assume that it coincides with the smoothing of x in S. Then S, but not $S^\vee$, is a state of D'. Denote the dual state to S in D' by $S^{\vee\prime}$. (It differs from $S^\vee$ at x only.) Then D' is a $(c-1)$–crossing shadow and S is a state for D', non-alternating in case of part (2).

If D' is cellularly embedded (and, hence, connected), then

$$t(S) + t(S^{\vee\prime}) \leq c(D') + 2 - 2g(\Sigma),$$

by the inductive assumption, with the sharp inequality for statement (2). Since

$$t(S^\vee) \leq t(S^{\vee\prime}) + 1 \text{ and } c(D') + 1 = c(D),$$

the statement follows.

Assume now that D' is not cellularly embedded in Σ. Since D is cellularly embedded, that means that there is a component of $\Sigma \smallsetminus D$ (necessarily a disk), which becomes an annulus in $\Sigma \smallsetminus D'$. Then $S^{\vee\prime}$ has two components along the boundary of that annulus, cf. Figure 15. Consequently, D' is cellularly embedded in a new surface Σ' obtained from Σ

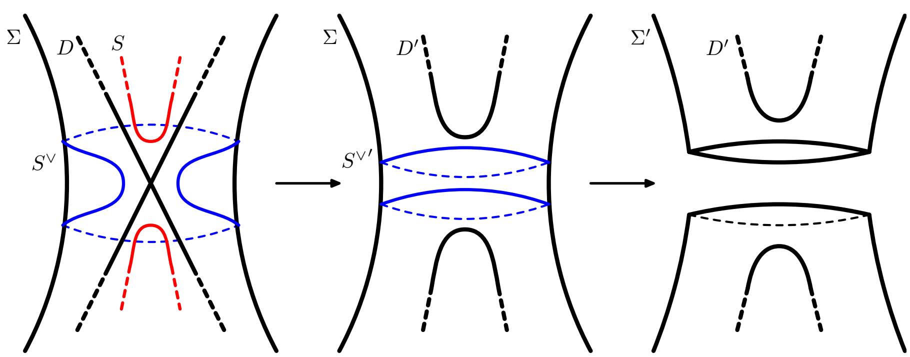


FIGURE 15. Transition from D to D' creating an annulus in $\Sigma \smallsetminus D'$.

by cutting it along the core of the annulus and capping the two boundary components. Now,

$$t(S, \Sigma') + t(S^{\vee\prime}, \Sigma') \leq c(D') + \chi(\Sigma'),$$

by the inductive assumption, where $t(S, \Sigma'), t(S^{\vee\prime}, \Sigma')$ denote the numbers of trivial components in Σ'. That is a sharp inequality for statement (2).

Since the transition from $S^\vee \subset \Sigma$ to $S^{\vee\prime} \subset \Sigma'$ eliminates one trivial cycle and creates two cycles (boundaries of the annulus) which become trivial on Σ',

$$t(S^\vee) + 1 \leq t(S^{\vee\prime}, \Sigma').$$

(Other cycles of $S^{\vee\prime}$ can be trivialized as well in the process of transforming Σ into Σ'.) Furthermore,

$$t(S) \leq t(S, \Sigma'), \text{ and } \chi(\Sigma') = \chi(\Sigma) + 2.$$

Hence,

$$t(S) + t(S^\vee) \leq t(S, \Sigma') + t(S^{\vee\prime}, \Sigma') - 1 \leq c(D') + \chi(\Sigma') - 1 = c(D) + \chi(\Sigma).$$

This inequality is sharp in part (2). Hence, the statement follows. $\square$

7.4. **Proof of Theorem 13.** Part (i) follows immediately by combining Proposition 26 and Corollary 5.

For parts (ii) and (iii), note that if D is a connected sum of $D_0 \subset \Sigma$, and $D_1, \ldots, D_k \subset S^2$ then

$$[D]_\Sigma = \delta^{-k}[D_0]_\Sigma \cdot \prod_{i=1}^{k}[D_i]_{S^2}. \tag{9}$$

Therefore, it is enough to prove parts (ii) and (iii) for *prime* diagrams (alternating for (2) and non-alternating for (3)).

The condition that D is prime implies that it is not a nontrivial connected sum diagram as above. More precisely, a link diagram D on Σ is said to be **prime** if any contractible simple loop γ in Σ that meets D transversely at two points bounds a 2-disk that intersects D in an unknotted arc (possibly with self-crossings).

Proof of (iii): Assume D is prime. If the shadow diagram of D is strongly prime, then the statement follows from Proposition 26 and Corollary 5. If it is not strongly prime then D must contain a self-crossing trivial arc. Let D' be obtained by replacing it by a simple trivial arc. Since D' is minimally embedded and $\text{span}([D]_\Sigma)$ is invariant under Reidemeister moves,

$$\text{span}([D]_\Sigma) = \text{span}([D']_\Sigma) \leq 4c(D') + 4|D| - 4g(\Sigma) < 4c(D) + 4|D| - 4g(\Sigma),$$

by part (i).

Our proof of (ii) follows that of [BK19, Thm. 2.9]. Since both sides of the equality in (ii) are additive under disjoint union of diagrams, it is enough to prove it for connected diagrams.

By Proposition 23, D is checkerboard colorable. Then all regions of one color, say white, are enclosed by the cycles in the state S_A of D, and all regions of the other color, i.e., black, are enclosed by the cycles in the state S_B. Therefore, the numbers of white and black regions are $t(S_A)$ and $t(S_B)$, respectively. Since D defines a cellular decomposition of Σ into $c(D)$ 0-cells, $2c(D)$ 1-cells, and $t(S_A) + t(S_B)$ 2-cells,

$$2 - 2g(\Sigma) = \chi(\Sigma) = c(D) - 2c(D) + t(S_A) + t(S_B),$$

and

$$t(S_A) + t(S_B) = c(D) + 2 - 2g(\Sigma).$$

By Proposition 3,

$$\begin{aligned}\text{span}([D]_\Sigma) &= d_{max}([D]_\Sigma)) - d_{min}([D]_\Sigma)),\\ &= 2c(D) + 2t(S_A) + 2t(S_B),\\ &= 4c(D) + 4 - 4g(\Sigma).\end{aligned}$$

□

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