A CHARACTERIZATION OF ALTERNATING LINKS IN THICKENED SURFACES

HANS U. BODEN AND HOMAYUN KARIMI

ABSTRACT. We use an extension of Gordon-Litherland pairing to thickened surfaces to give a topological characterization of alternating links in thickened surfaces. If Σ is a closed oriented surface and F is a compact unoriented surface in $\Sigma \times I$, then the Gordon-Litherland pairing defines a symmetric bilinear pairing on the first homology of F. A compact surface in $\Sigma \times I$ is called *definite* if its Gordon-Litherland pairing is a definite form. We prove that a non-split link L in a thickened surface is alternating if and only if it bounds two definite surfaces of opposite sign.

§1. **Introduction.** Alternating knots in S^3 can be characterized as precisely those knots which simultaneously bound both positive and negative definite spanning surfaces. This beautiful result was established recently and independently by Greene [Gre17] and Howie [How17]. Indeed, in [Gre17], Greene proves a more general statement characterizing alternating links in $\mathbb{Z}/2$ homology 3-spheres.

In this paper we study links in thickened surfaces and we take up the problem of establishing a topological characterization of alternating links in thickened surfaces.

Let Σ be a compact, connected, oriented surface and $I = [0, 1]$, the unit interval. A link in $\Sigma \times I$ is an embedding $L\colon \bigsqcup_{i=1}^m S^1 \hookrightarrow \Sigma \times I$, considered up to isotopy and orientation preserving homeomorphisms of the pair $(\Sigma \times I, \Sigma \times \{0\})$. A link L in a thickened surface $\Sigma \times I$ can be represented by its link diagram, which is the tetravalent graph D on Σ obtained under projection $p\colon \Sigma \times I \to \Sigma$. The arcs of D are drawn to indicate over and under crossings near vertices in the usual way. The link diagram D on Σ is said to be *alternating* if its crossings alternate from over to under around each component of the link. A link L in $\Sigma \times I$ is alternating if it admits an alternating link diagram on Σ (see Figure 1).

Given a compact unoriented surface F in S^3, Gordon and Litherland defined a symmetric bilinear pairing on $H_1(F)$. In the case that F is a spanning surface for a link $L \subset S^3$, the signature of L can be computed in terms of the signature of the pairing together with a correction term. The Gordon-Litherland pairing is extended to $\mathbb{Z}/2$ homology 3-spheres in [Gre17] and to thickened surfaces $\Sigma \times I$ in [BCK20]. Our characterization of alternating links in $\Sigma \times I$ is phrased in terms of the Gordon-Litherland pairing on spanning surfaces for the link.

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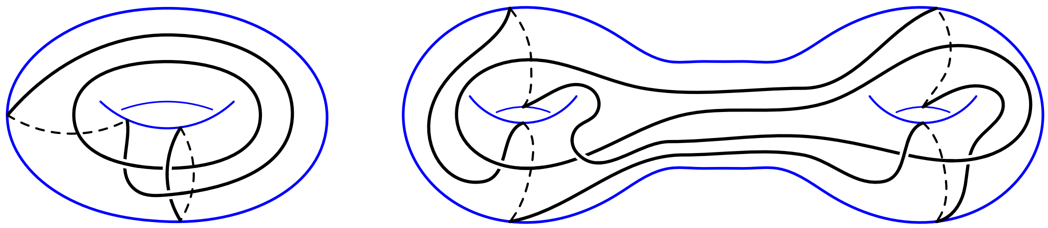

FIGURE 1. An alternating link on the torus and an alternating knot on a genus two surface.

A link L in $\Sigma \times I$ is said to be *split* if it can be represented by a disconnected diagram D on Σ. A link diagram D on Σ is said to be *cellularly embedded* if $\Sigma \smallsetminus D$ is a union of disks.

Let L be a non-split link in $\Sigma \times I$ with alternating diagram D. We further assume that D is cellularly embedded. It follows that the complementary regions of $\Sigma \smallsetminus D$ admit a checkerboard coloring, and that the black and white regions form spanning surfaces for L which we denote B and W. A straighforward argument (see Lemma 5) shows that B is negative definite and W is positive definite with respect to their associated Gordon-Litherland pairings.

Our main result is a converse to this statement given by the following theorem.

Theorem 1. *Let L be a non-split link in $\Sigma \times I$, and assume that L bounds a positive definite spanning surface and a negative definite spanning surface. Then L is an alternating link in $\Sigma \times I$.*

Theorem 1 applies to give a characterization of alternating virtual links.

Virtual links can be defined as virtual link diagrams up to the generalized Reidemeister moves [Kau99]. One can also define them as stable equivalence classes of links in thickened surfaces. Here, two links $L_0 \subset \Sigma_0 \times I$ and $L_1 \subset \Sigma_1 \times I$ are said to be *stably equivalent* if one is obtained from the other by a finite sequence of isotopies, diffeomorphisms, stabilizations, and destabilizations. Stabilization is the operation of adding a handle to Σ to obtain a new surface Σ', and destabilization is the reverse procedure. In [CKS02], Carter, Kamada, and Saito give a one-to-one correspondence between virtual links and stable equivalence classes of links in thickened surfaces.

The virtual genus of a virtual link is the minimum genus over all surfaces Σ such that $\Sigma \times I$ contains a representative for L. In that case, we say that the representative link $L \subset \Sigma \times I$ has *minimal genus*. A necessary but not sufficient condition for a link $L \subset \Sigma \times I$ to have minimal genus is for its diagram $D \subset \Sigma$ to be cellularly embedded. Kuperberg's theorem shows that every virtual link has an irreducible representative which is unique up to diffeomorphism [Kup03]. In particular, it implies that two minimal genus representatives of the same virtual link are equivalent under isotopy and orientation-preserving homeomorphism of the pair $(\Sigma \times I, \Sigma \times \{0\})$.

In Corollary 8, we will show that any non-split alternating link $L \subset \Sigma \times I$ has minimal genus.

Corollary 2. *A non-split virtual link is alternating if and only if it admits a representative L in $\Sigma \times I$ which bounds a positive definite spanning surface and a negative definite spanning surface.*

§**2**. **Gordon-Litherland pairing.** In this section, we review the Gordon-Litherland pairing [GL78] and its extension to links in thickened surfaces [BCK20]. The pairing is defined for any link $L \subset \Sigma \times I$ that admits a *spanning surface*, which is a compact, connected surface F embedded in $\Sigma \times I$ with boundary $\partial F = L$. The surface F may or may not be orientable, and here we consider it as an *unoriented* surface. Not all links $L \subset \Sigma \times I$ admit spanning surfaces, and in fact Proposition 1.7 of [BK19] implies that L admits a spanning surface if and only if $[L]$ is trivial in $H_1(\Sigma; \mathbb{Z}/2)$.

The link diagram D is the decorated graph on Σ obtained as the image of L under the projection $p\colon \Sigma \times I \to \Sigma$. Then D is called *checkerboard colorable* if the complementary regions of $\Sigma \smallsetminus D$ can be colored black and white so that, whenever two regions share an edge, one is white and the other is black. A link L in $\Sigma \times I$ is said to be checkerboard colorable if it admits a diagram which is checkerboard colorable. The black regions of a checkerboard coloring determine an unoriented spanning surface which we call the *checkerboard surface*. A straightforward argument shows that a link in $\Sigma \times I$ is checkerboard colorable if and only if it bounds an unoriented spanning surface (see Proposition 1.7 of [BK19]).

Next, we recall the definition of the asymmetric linking for simple closed curves in a thickened surface. Let Σ be a compact, connected, oriented surface. The asymmetric linking pairing in $\Sigma \times I$ is taken relative to $\Sigma \times \{1\}$ and defined as follows. Given a simple closed curve J in $\Sigma \times I$, then by Proposition 7.1 of [BGH$^+$17], $H_1(\Sigma \times I, \Sigma \times I \smallsetminus J)$ is infinite cyclic and generated by a meridian μ of J. If K is a simple closed curve in $\Sigma \times I$ disjoint from J, then define $\ell k(J, K)$ to be the unique integer m such that $[K] = m\mu$ in $H_1(\Sigma \times I, \Sigma \times I \smallsetminus J)$. Alternatively, if B is a 2-chain in $\Sigma \times I$ such that $\partial B = K - v$, where v is a 1-cycle in $\Sigma \times \{1\}$, then $\ell k(J, K) = J \cdot B$, where $\cdot$ denotes the intersection number.

One can determine the asymmetric linking numbers easily using the following simple diagrammatic description. If J, K are two disjoint simple closed curves in $\Sigma \times I$ represented as diagrams on Σ, then $\ell k(J, K)$ is equal to the number of times J goes above K with sign given by comparing with orientation of Σ, where "above" refers to the positive I direction in $\Sigma \times I$.

Now suppose that F is a compact, connected, unoriented surface embedded in $\Sigma \times I$. Its normal bundle $N(F)$ has boundary a $\{\pm 1\}$-bundle $\widetilde{F} \xrightarrow{\pi} F$, a double cover with $\widetilde{F}$ oriented. (If F is oriented, then $\widetilde{F}$ is the trivial double cover.) For a simple closed curve α on F, set $\tau a = \pi^{-1}(a) \subset \widetilde{F}$. We extend this to a map $\tau\colon H_1(F) \to H_1(\widetilde{F})$ called the transfer map.

The Gordon-Litherland pairing is extended to thickened surfaces $\Sigma \times I$ in [BCK20], and we review its definition. Suppose F is a compact, unoriented surface without closed components in $\Sigma \times I$ and $a, b \in H_1(F)$. Then $\mathcal{G}_F(a, b) = \ell k(\tau a, b) - p_*[a] \cdot p_*[b]$, where $p\colon \Sigma \times I \to \Sigma$ projection and $\cdot$ refers to algebraic intersection number in Σ. By a straightforward argument (e.g., [BCK20, Lemma 2.1]) one can show that this gives a symmetric bilinear pairing

$$\mathcal{G}_F \colon H_1(F) \times H_1(F) \to \mathbb{Z}.$$

For $x \in H_1(F)$, let $\lvert x \rvert_F = \mathcal{G}_F(x,x)$. Clearly, $\lvert -x \rvert_F = \lvert x \rvert_F$.

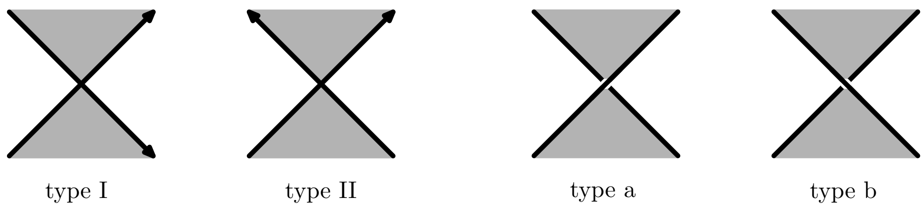

FIGURE 2. Type I and II crossings (left), and incidence numbers (right).

Assume now that $L \subset \Sigma \times I$ is a link with m components and write $L = K_1 \cup \cdots \cup K_m$. Suppose further that $F \subset \Sigma \times I$ is a spanning surface for L. Clearly, each component determines an element $[K_i] \in H_1(F)$, well-defined up to sign, and $\frac{1}{2}\lvert [K_i] \rvert_F$ is equal to the framing that the surface F induces on K_i. Set $e(F) = -\frac{1}{2}\sum_{i=1}^m \lvert [K_i] \rvert_F$, the Euler number of F. Further, set $e(F,L) = -\frac{1}{2}\lvert [L] \rvert_F$. If $L' = K_1' \cup \cdots \cup K_m'$ denotes the push-off of L that misses F, then it follows that

$$\begin{aligned} e(F) &= -\sum_{i=1}^m \ell k(K_i, K_i'), \\ e(F,L) &= -\ell k(L, L') = -\sum_{i,j=1}^m \ell k(K_i, K_j'). \end{aligned}$$

Note that $e(F)$ is independent of the choice of orientation on L whereas $e(F,L)$ is not. To see this, notice that in the formula for $e(F)$, each component K_i can be oriented arbitrarily provided its push-off K_i' is oriented in a compatible manner. The two quantities are related by the formula

$$e(F,L) = e(F) - \lambda(L), \tag{1}$$

where $\lambda(L) = \sum_{i \neq j} \ell k(K_i, K_j)$ denotes the total linking number of L.

Two spanning surfaces are said to be S^*-*equivalent* if one can be obtained from the other by ambient isotopy, attachment or removal of 1-handles, and attachment or removal of a small half-twisted band, as depicted in Figure 3.

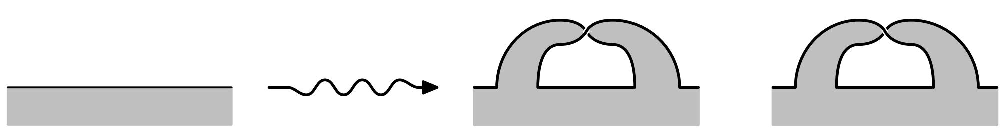

FIGURE 3. Attaching a small half-twisted band.

The signature of L, F is defined by setting $\sigma_F(L) = \mathrm{sig}(\mathcal{G}_F) + \frac{1}{2}e(F,L)$. The following lemma shows that $\sigma_F(L)$ gives a well-defined invariant of the pair (F,L) depending only on the S^*-equivalence class of F. For a proof, see [BCK20, Lemma 2.6].

Lemma 3. *If F and F' are S^*-equivalent with $\partial F = L = \partial F'$, then $\sigma_F(L) = \sigma_{F'}(L)$.*

Further, unless Σ is the 2-sphere, two spanning surfaces F and F' are S^*-equivalent if and only if $[F] = [F']$ as elements in $H_1(\Sigma \times I, L; \mathbb{Z}/2)$ (see [BCK20, Lemma 2.4]).

In the case that F is a checkerboard surface for L, the correction term $\frac{1}{2}e(F,L)$ is given by a sum of incidence numbers of crossings of type I or II (see Figure 2). The incidence number of a crossing x is denoted η_x and is defined by setting

$$\eta_x = \begin{cases} 1 & \text{if } x \text{ is type } a, \\ -1 & \text{if } x \text{ is type } b. \end{cases}$$

Specifically, if B and W denote the black and white surfaces of the checkerboard coloring, set

$$\mu_W(D) = \sum_{x \text{ type I}} -\eta_x \qquad \text{and} \qquad \mu_B(D) = \sum_{x \text{ type II}} \eta_x.$$

By Lemma 2.7 [BCK20], we see that $\mu_W(D) = -\frac{1}{2}e(W,L)$ and $\mu_B(D) = -\frac{1}{2}e(B,L)$.

§3. **Definite surfaces.** In this section, we show that a connected checkerboard colorable link diagram D on Σ is alternating if and only if its checkerboard surfaces are definite and of opposite sign.

Definition 4. A compact, connected surface F in $\Sigma \times I$ is positive (or negative) definite if its Gordon-Litherland pairing $\mathcal{G}_F$ is.

Suppose D is a connected link diagram on Σ such that

(i) $\Sigma \smallsetminus D$ is a union of disks,
(ii) D is checkerboard colorable.

Choose a checkerboard coloring of D and let B, W be the black and white surfaces. Then B, W have first Betti numbers

$$\begin{aligned} b_1(W) &= 2g + \beta - 1, \\ b_1(B) &= 2g + \alpha - 1, \end{aligned} \tag{2}$$

where $g = \text{genus}(\Sigma)$ is the genus of Σ, α is the number of white disks and β is the number of black disks.

The Euler characteristic of Σ satisfies $\chi(\Sigma) = 2 - 2g = c(D) - 2c(D) + (\alpha + \beta)$, where $c(D)$ denotes the number of crossings of D. Thus we have

$$\alpha + \beta = 2 - 2g + c(D). \tag{3}$$

In particular, combining equations (2) and (3) gives that

$$b_1(W) + b_1(B) = 2g + c(D). \tag{4}$$

Lemma 5. *If D is a link diagram on Σ which is cellularly embedded and alternating, then the black and white surfaces are definite and of opposite sign.*

Proof. Choose the coloring such that every crossing of D has type b. Let Γ_B be the Tait graph for the black surface, it is the graph in Σ with one vertex for each black disk and one edge for each crossing. The edge associated with a crossing x of D is labeled with η_x, and since every crossing has type b, each edge of Γ_B is labeled -1, see Figure 2. The black surface B admits a deformation retraction onto Γ_B, hence $H_1(B) = H_1(\Gamma_B)$. Moreover, for any nontrivial cycle $\gamma = \sum_{i=1}^k e_i$ of length k in the Tait graph Γ_B, it follows that $\mathcal{G}_B(\gamma,\gamma) = |\gamma|_B = -k < 0$. This shows that $\mathcal{G}_B$ is negative definite. Similarly, for the edge associated to a crossing x, its edge label in the Tait graph Γ_W of the white surface is given by $-\eta_x$. Since all the crossings have type b, each edge of Γ_W is labeled $+1$. Thus, if γ is a nontrivial cycle of length k in the Tait graph Γ_W, then $\mathcal{G}_W(\gamma,\gamma) > 0$. It follows that $\mathcal{G}_W$ is positive definite. □

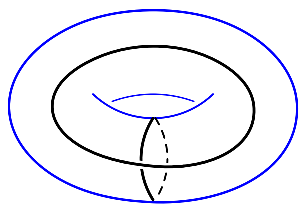

FIGURE 4. A non-checkerboard colorable link.

Remark 6. (i) For classical links, every link admits a spanning surface, and any two are S^*-equivalent [GL78, Theorem 11]. For links in thickened surfaces of genus $g(\Sigma) > 0$, neither statement is true. A link $L \subset \Sigma \times I$ admits a spanning surfaces if and only if it is checkerboard colorable, but not all links in $\Sigma \times I$ are checkerboard colorable (see Figure 4). For links $L \subset \Sigma \times I$ which are checkerboard colorable, not all spanning surfaces are S^*-equivalent to one another. For instance, if L is non-split, then there are exactly two S^*-equivalence classes of spanning surfaces for L. Indeed, the black and white surfaces represent the two S^*-equivalence classes, and any other spanning surface for L is S^*-equivalent to either the black or the white surface.
(ii) More generally, given a spanning surface F for a link $L \subset \Sigma \times I$ in a connected thickened surface of positive genus, we can construct a new spanning surface by connecting F to a parallel copy of Σ near $\Sigma \times \{0\}$ by a small thin tube τ. Let $F' = F \#_\tau \Sigma$ denote the new spanning surface. Then it is not difficult to see that F and F' represent the two S^*-equivalence classes of spanning surfaces for L.
(iii) Given a connected spanning surface F for L, if the Gordon-Litherland pairing $\mathcal{G}_F$ is non-singular, then it will be non-singular for any connected surface S^*-equivalent to F, see §2.3 and §2.4 of [BCK20].

Theorem 7. *Suppose $L \subset \Sigma \times I$ is a non-split link represented by a diagram $D \subset \Sigma$, which is assumed to be checkerboard colored. Let B, W be the two spanning surfaces for L associated to the black and white regions, respectively. If the Gordon-Litherland pairings $\mathcal{G}_B$ and $\mathcal{G}_W$ are both nonsingular, then $L \subset \Sigma \times I$ has minimal genus. In particular, this implies that D is cellularly embedded.*

Proof. Assume to the contrary that L does not have minimal genus. Then there is a link $L' \subset \Sigma \times I$ which is isotopic to L whose diagram D' is not cellularly embedded.

Since L is non-split, D' is connected. In addition, since D is checkerboard colorable, the same is true of D'. Since D' is not cellularly embedded, there is a non-contractible simple closed curve γ in Σ which is disjoint from D'. Since $\gamma \cap D' = \varnothing$, it follows that γ is contained entirely in either a black region or a white region in the coloring of D'. We can arrange that γ lies in a black region by switching the coloring if necessary.

Let F be the black surface for D'. We claim that there is a simple closed curve α lying entirely in a black region such that its homology class $[\alpha]$ is nontrivial as an element in $H_1(F)$. Indeed, if γ is a non-separating curve, then we can take $\alpha = \gamma$. Otherwise, if γ is a separating curve, then since D' is connected, it lies in one of the connected components of $\Sigma \smallsetminus \gamma$. Both components have positive genus, and one of them is contained entirely in a black region. Therefore, we can take α to be a simple closed curve in the component disjoint from D', and we can further choose α so that $[\alpha] \neq 0$.

Since α is a simple closed curve, its homology class $[\alpha]$ is primitive as an element in $H_1(F)$. Therefore, we can find a basis U for $H_1(F)$ with $\alpha \in U$. Further, since α lies entirely within the black region, we have $\mathcal{G}_F(\alpha, \alpha) = |\alpha|_F = 0$. In fact, for any other element $\beta \in U$, one can show that $\mathcal{G}_F(\alpha, \beta) = 0$. It follows that the Gordon-Litherland pairing $\mathcal{G}_F$ is singular. However, by remark 6, we see that F must be S^*-equivalent to either B or W, which contradicts the assumption that $\mathcal{G}_B$ and $\mathcal{G}_W$ are both nonsingular. □

Corollary 8. *Any non-split link L in $\Sigma \times I$ represented by a cellularly embedded alternating diagram has minimal genus.*

Proof. Let D be a cellularly embedded alternating diagram for L. Then D is checkerboard colorable, and Lemma 5 implies the black and white surfaces are definite. In particular, their Gordon-Litherland pairings are non-singular. The conclusion now follows from Theorem 7. □

By convention, given an alternating diagram for a link L in $\Sigma \times I$, we will choose the coloring in which every crossing has type b. With this choice, the white surface becomes positive definite and the black surface becomes negative definite.

Lemma 9. *Suppose D is a connected alternating diagram for a link L in $\Sigma \times I$ with checkerboard coloring such that every crossing has type b. Then*

$$\sigma_W(L) - \sigma_B(L) = 2g(\Sigma).$$

Proof. In general we have

$$\begin{aligned} \sigma_W(L) &= \operatorname{sig}(\mathcal{G}_W) - \mu_W(D), \\ \sigma_B(L) &= \operatorname{sig}(\mathcal{G}_B) - \mu_B(D). \end{aligned}$$

Since all crossings have type b and referring to Figure 5, we see that

$$\begin{aligned} \mu_W(D) &= \sum_{x \text{ type I}} -\eta_x = \sum_{x \text{ type I}} \varepsilon_x = c_+(D), \\ \mu_B(D) &= \sum_{x \text{ type II}} \eta_x = \sum_{x \text{ type II}} \varepsilon_x = -c_-(D), \end{aligned}$$

where $c_+(D)$ is the number of positive crossings of D and $c_-(D)$ is the number of negative crossings. Hence

$$\mu_W(D) - \mu_B(D) = c_+(D) + c_-(D) = c(D). \tag{5}$$

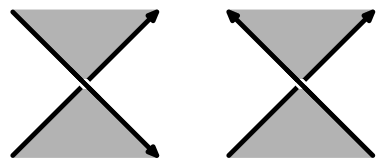

FIGURE 5. Two type b crossings.

Lemma 5 shows that W is positive definite and B is negative definite, hence

$$\begin{aligned}\sigma_W(L) - \sigma_B(L) &= \operatorname{sig}(\mathcal{G}_W) - \mu_W(D) - (\operatorname{sig}(\mathcal{G}_B) - \mu_B(D)),\\ &= b_1(W) + b_1(B) - (\mu_W(D) - \mu_B(D)) = 2g,\end{aligned}$$

where the last step follows from equations (4) and (5). This completes the proof. □

Proposition 10. *Let D be a connected, checkerboard colorable link diagram on $\Sigma \times I$ which is cellularly embedded. Then D is alternating if and only if the black and white surfaces are definite and of opposite sign.*

Proof. If D is alternating, then Lemma 5 gives the desired conclusion.

Conversely, suppose B is negative definite and W is positive definite. Let $a_\pm$ be the number of type a crossings of D with $\varepsilon_x = \pm 1$, and $b_\pm$ be the number of type b crossings of D with $\varepsilon_x = \pm 1$. Then

$$\mu_W(D) = a_- - b_+ \qquad \text{and} \qquad \mu_B(D) = -a_+ + b_-.$$

It follows that

$$\begin{aligned}\mu_W(D) - \mu_B(D) &= a_- - b_+ - (-a_+ + b_-),\\ &= a(D) - b(D),\end{aligned}$$

where $a(D) = a_+ + a_-$ is the total number of type a crossings and $b(D) = b_+ + b_-$ is the total number of type b crossings.

Therefore,

$$|\mu_W(D) - \mu_B(D)| \leq c(D), \tag{6}$$

with equality if and only if $a(D) = 0$ or $b(D) = 0$. In the first case, all crossings have type b, and in the second, they all have type a. In either case, we see that D is alternating.

Given a spanning surface F for L, by Remark 6 (ii), we obtain a new surface $F \#_\tau \Sigma$ by connecting it to a parallel copy of Σ by a thin tube τ. Unless Σ is the 2-sphere, the surfaces F and $F \#_\tau \Sigma$ are not S^*-equivalent.

Since B and W are chromatic duals of one another, it follows that B is S^*-equivalent to $W\#_\tau\Sigma$. Hence $|\sigma_W(L)-\sigma_B(L)|\leq 2g$. This observation, combined with equations (2) and (3) and the inequality (6), shows that:

$$\begin{aligned} 2g &\geq |\sigma_W(L)-\sigma_B(L)|,\\ &= |\operatorname{sig}(\mathcal{G}_W)-\mu_W(D)-(\operatorname{sig}(\mathcal{G}_B)-\mu_B(D))|,\\ &\geq |\operatorname{sig}(\mathcal{G}_W)-\operatorname{sig}(\mathcal{G}_B)|-|\mu_W(D)-\mu_B(D)|,\\ &= b_1(W)+b_1(B)-|\mu_W(D)-\mu_B(D)|,\\ &= (2g+\beta-1)+(2g+\alpha-1)-|\mu_W(D)-\mu_B(D)|,\\ &= 2g+c(D)-|\mu_W(D)-\mu_B(D)|\geq 2g. \end{aligned}$$

Therefore we must have equality throughout, and it follows that D is alternating. □

The next result relates the signature of $\mathcal{G}_{F'}$ to that of $\mathcal{G}_F$ restricted to $\mathcal{K}$, where $F'=F\#_\tau\Sigma$ and $\mathcal{K}=\operatorname{Ker}(H_1(F)\to H_1(\Sigma\times I))$. For a proof, we refer to [BCK20].

Theorem 11 (Theorem 4.1, [BCK20]). *Let $F\subset\Sigma\times I$ be a spanning surface such that $H_1(F)\to H_1(\Sigma\times I)$ is surjective and set $F'=F\#_\tau\Sigma$. Then $\operatorname{sig}(\mathcal{G}_{F'})=\operatorname{sig}(\mathcal{G}_F|_{\mathcal{K}})$, the restriction of $\mathcal{G}_F$ to $\mathcal{K}=\operatorname{Ker}(H_1(F)\to H_1(\Sigma\times I))$.*

§4. **Characterization of alternating links in thickened surfaces.** In this section, we establish Theorem 17, which gives the converse to Proposition 10. This result is a restatement of Theorem 1 and is our main result.

If L is a link in the thickened surface $\Sigma\times I$, let $\nu(L)$ be a tubular neighborhood of L and let $X_L=\Sigma\times I\smallsetminus\operatorname{int}(\nu(L))$ denote the exterior of L. The next result is a restatement of Proposition 6.3 from [CSW14]. Recall that a link L in a 3-manifold M is said to be *local* if it is contained in an embedded 3-ball B in M.

Proposition 12 (Carter-Silver-Williams). *If Σ is a surface of genus $g\geq 1$ and L is a non-split and non-local link in $\Sigma\times I$, then the exterior X_L is irreducible.*

Proof. A detailed proof can be found in [CSW14], so we only sketch the argument. Since Σ has genus $g\geq 1$, by [Hat07, Proposition 1.6] it follows that $\Sigma\times I$ is irreducible. For any non-split, non-local link L in an orientable 3-manifold M, irreducibility of M implies irreducibility of the link complement $M\smallsetminus\operatorname{int}(\nu(L))$. Applying this to links in $\Sigma\times I$ completes the argument. □

The next result is an analogue of Lemma 3.1 from [Gre17].

Lemma 13 (Greene). *If $F\subset\Sigma\times I$ is a definite surface with $\partial F=L$, then $b_1(F)$ is minimal over all spanning surfaces for L which are S^*-equivalent to F and have the same Euler number as F. If F' is another such surface with $b_1(F')=b_1(F)$, then F' is definite and of the same sign as F.*

Proof. If F' is S^*-equivalent to F, then Lemma 3 implies that $\sigma_F(L)=\sigma_{F'}(L)$. If, in addition, $e(F)=e(F')$, then it follows that $\operatorname{sig}(\mathcal{G}_F)=\operatorname{sig}(\mathcal{G}'_F)$.

Now suppose F is definite. Then we have

$$b_1(F)=|\operatorname{sig}(\mathcal{G}_F)|=|\operatorname{sig}(\mathcal{G}_{F'})|\leq b_1(F'),$$

which shows the first claim.

If, in addition, $b_1(F') = b_1(F)$, then we have $b_1(F') = |\operatorname{sig}(\mathcal{G}_{F'})|$, hence F' must also be definite. Since $\operatorname{sig}(\mathcal{G}_F) = \operatorname{sig}(\mathcal{G}_{F'})$, it follows that F and F' must have the same sign. □

Corollary 14. *If $F \subset \Sigma \times I$ is definite, then it is incompressible.*

Proof. Suppose to the contrary that F is compressible. Let F' be the surface obtained from F by a compression. Then F' is S^*-equivalent to F and $b_1(F') < b_1(F)$. Further, $e(F') = e(F)$. However, this is impossible, for it would contradict Lemma 13 if F is definite. □

The next result is Lemma 3.3 from [Gre17]. The proof is the same as in [Gre17] so we will not repeat it here.

Lemma 15 (Greene)**.** *If S is definite and $S' \subset S$ is a compact subsurface with connected boundary, then S' is definite.*

We will use Lemma 15 to prove the following analogue of Lemma 3.4 of [Gre17] for links in thickened surfaces.

Lemma 16. *Let L be a non-split, non-local link in $\Sigma \times I$, where Σ has genus $g \geq 1$. Suppose further that L is checkerboard colorable and that $\Sigma \smallsetminus p(L)$ is a union of disks, where $p\colon \Sigma \times I \to \Sigma$ is projection. Let P and N be two spanning surfaces for L, with P positive definite and N negative definite. If P and N are in minimal position, then $P \cap N \cap X_L$ does not contain a simple closed curve of intersection.*

Proof. Suppose to the contrary that γ is a simple closed curve contained in $P \cap N \cap X_L$. Let $\nu(\gamma)$ be a small regular neighborhood of γ in $\Sigma \times I$, which contains no other intersection of P and N. Then $\nu(\gamma) \approx S^1 \times D^2$, and $\partial \nu(\gamma) = T^2$. Let $\gamma^P = P \cap \partial \nu(\gamma)$ and $\gamma^N = N \cap \partial \nu(\gamma)$. If $\widetilde{P} \xrightarrow{\pi_P} P$ and $\widetilde{N} \xrightarrow{\pi_N} N$ are the double covers, then γ^P is isotopic to $\pi_N^{-1}(\gamma)$ and γ^N is isotopic to $\pi_P^{-1}(\gamma)$. Notice that γ^P and γ^N are isotopic in $X_L \smallsetminus \nu(\gamma)$. Hence $\ell k(\pi_P^{-1}(\gamma), \gamma) = \ell k(\pi_N^{-1}(\gamma), \gamma)$. Therefore

$$0 \leq \| \gamma \|_P = \mathcal{G}_P(\gamma, \gamma) = \mathcal{G}_N(\gamma, \gamma) = \| \gamma \|_N \leq 0.$$

Thus, $\| \gamma \|_P = \| \gamma \|_N = 0$, and it follows that γ is null-homologous in both P and N. Let P' and N' be orientable subsurfaces of P and N, respectively, with $\partial P' = \partial N' = \gamma$. By Lemma 15, P' is positive definite and N' is negative definite.

Since P' and N' are both orientable, we see that $\frac{1}{2} e(P', \gamma) = 0 = \frac{1}{2} e(N', \gamma)$. Therefore $\sigma_{P'}(\gamma) = \operatorname{sig}(\mathcal{G}_{P'})$ and $\sigma_{N'}(\gamma) = \operatorname{sig}(\mathcal{G}_{N'})$.

Suppose firstly that P' and N' are S^*-equivalent. Then

$$0 \leq b_1(P') = \operatorname{sig}(\mathcal{G}_{P'}) = \sigma_{P'}(\gamma) = \sigma_{N'}(\gamma) = \operatorname{sig}(\mathcal{G}_{N'}) = -b_1(N') \leq 0.$$

Therefore, P' and N' are disks in this case.

Suppose now that P' and N' are not S^*-equivalent. Since $\mathcal{G}_{P'}$ and $\mathcal{G}_{N'}$ are definite, they are non-singular. By Theorem 7, it follows that, as a knot in the thickened surface $\Sigma \times I$, γ has minimal genus. The inclusions $P' \subset \Sigma \times I$ and $N' \subset \Sigma \times I$ induce maps $H_1(P') \to H_1(\Sigma \times I)$ and $H_1(N') \to H_1(\Sigma \times I)$, both of which can be seen to be surjective by the proof of Theorem 7. Theorem 11 applies to show that $\operatorname{sig}(\mathcal{G}_{P'})$

is equal to the signature of $\mathcal{G}_{N'}$ restricted to $\mathcal{K}' = \ker(H_1(N') \to H_1(\Sigma))$. (Notice, since P' and N' are not S^*-equivalent, then by Remark 6, $N'\#_\tau\Sigma$ is S^*-equivalent to P', and $\operatorname{sig}(\mathcal{G}_{P'}) = \operatorname{sig}(\mathcal{G}_{N''})$, where $N'' = N'\#_\tau\Sigma$.) However, $\mathcal{G}_{N'}$ is negative definite, and so is its restriction to any subspace of $H_1(N')$. In particular, its signature is non-positive, whereas the signature of $\mathcal{G}_{P'}$ is non-negative. The only way the signatures of $\mathcal{G}_{P'}$ and $\mathcal{G}_{N''}$ can be equal is for $\operatorname{sig}(\mathcal{G}_{P'}) = 0 = \operatorname{sig}(\mathcal{G}_{N''})$. Repeating the argument with the roles of P' and N' reversed shows that $\operatorname{sig}(\mathcal{G}_{N'}) = 0$. Therefore, P' and N' are disks in this case too.

By passing to an innermost disk, we can assume that P' and N' have disjoint interiors, so their union is a sphere. Proposition 12 implies that X_L is irreducible, therefore the sphere $P' \cup N'$ bounds a ball in X_L. Using the ball, we can construct an isotopy that reduces the number of components of $P \cap N \cap X_L$, which contradicts the assumption that P and N are in minimal position. □

Theorem 17. *Suppose $L \subset \Sigma \times I$ is a non-split link admitting a cellularly embedded diagram. Then L is alternating if and only if there exist positive and negative definite spanning surfaces for L.*

Proof. If L is non-split and D is an alternating diagram for L, then Proposition 10 implies that the checkerboard surfaces W and B are positive and negative definite, respectively. This proves one direction, and it remains to prove the converse.

Suppose then that P and N are two definite spanning surfaces for L, with P positive definite and N negative definite. Notice that Corollary 8 applies to show that $L \subset \Sigma \times I$ has minimal genus.

Let $X_L = \Sigma \times I \smallsetminus \operatorname{int}(\nu(L))$ be the exterior of L. We write $\partial X_L = \partial_1 X_L \cup \cdots \cup \partial_m X_L$ according to the components of the link $L = K_1 \cup \cdots \cup K_m$. Clearly, each $\partial_i X_L$ is a torus.

Put $P \cap X_L$ and $N \cap X_L$ into minimal position, and for $i = 1, \ldots, m$ set $\lambda_i^P = P \cap \partial_i X_L$ and $\lambda_i^N = N \cap \partial_i X_L$. Thus λ_i^P and λ_i^N intersect transversely in $\partial_i X_L$. We further set $\lambda^P = \bigcup_i \lambda_i^P$ and $\lambda^N = \bigcup_i \lambda_i^N$.

By Lemma 16, $P \cap N \cap X_L$ does not contain any closed components. Thus $P \cap N \cap X_L$ is a union of arc components which we call *double arcs*. Each double arc connects a pair of distinct points in $\lambda^P \cap \lambda^N$. Thus $\lambda^P \cap \lambda^N$ consists of an even number of points, equal to twice the number of double arcs. Since P is positive definite and N is negative definite, the number of points in $\lambda_i^P \cap \lambda_i^N$ is equal to the difference in framings $\frac{1}{2}\big|[K_i]\big|_P - \frac{1}{2}\big|[K_i]\big|_N$. Summing over the components, we get that

$$\sum_{i=1}^{m} \left(\tfrac{1}{2}\big|[K_i]\big|_P - \tfrac{1}{2}\big|[K_i]\big|_N\right) = e(N) - e(P).$$

The number of arc components in $P \cap N \cap \partial X_L$ is therefore equal to $\frac{1}{2}(e(N) - e(P))$.

An orientation on X_L induces one on ∂X_L, and an orientation on each K_i induces ones on λ_i^P and λ_i^N. This defines a sign $\varepsilon_x \in \{\pm 1\}$ for each point $x \in \lambda_i^P \cap \lambda_i^N$ which is positive if the orientations of λ_i^P and λ_i^N at x agree with the orientation of $\partial_i X_L$, and negative if they do not agree. A key point is that every point in $\lambda_i^P \cap \lambda_i^N$ has the

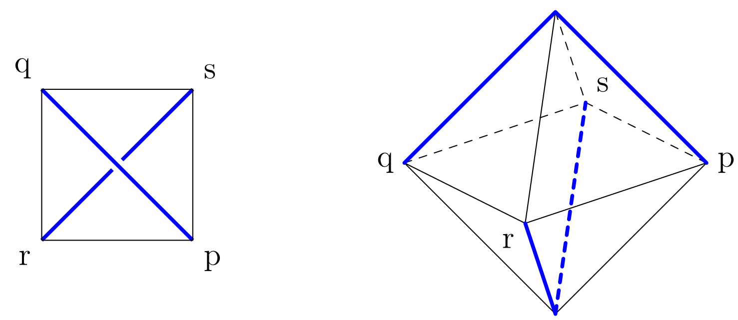

FIGURE 6. A standard crossing (left) placed on an octahedron (right).

same sign; this follows from the fact that

$$\#(\lambda_i^P \cap \lambda_i^N) = \tfrac{1}{2}\big|[K_i]\big|_P - \tfrac{1}{2}\big|[K_i]\big|_N = \Big|\sum \varepsilon_x\Big|,$$

where the sum on the right is taken over all $x \in \lambda_i^P \cap \lambda_i^N$.

As explained in §2 of [How17], there are two kinds of double arcs; one is called a *standard* double arc and the other is called a *parallel* double arc. Lemma 2.1 of [How17] applies to show that every double arc of $P \cap N \cap X_L$ is standard. Each double arc extends to give an arc a in $P \cap N$ with $\partial a = a \cap L$. Since the double arc is standard, there is a neighborhood V of a modelled on a standard neighborhood of a crossing in a link diagram. See Figure 6.

We can choose local coordinates near the crossings so that the arcs of the link lie in the xy plane except at the crossing. At each crossing we place an octahedron which intersects the xy plane in the square with vertices $p = (1, -1)$, $q = (-1, 1)$, $r = (-1, -1)$, and $s = (1, 1)$. We assume that the over-crossing arc connecting p to q is given by

$$\beta(t) = (1 - 2t, -1 + 2t, \min(2t, 2 - 2t)) \text{ for } 0 \le t \le 1,$$

and the under-crossing arc connecting r to s is given by

$$\gamma(t) = (-1 + 2t, -1 + 2t, \max(-2t, -2 + 2t)) \text{ for } 0 \le t \le 1.$$

In the standard crossing, the black surface is parametrized by

$$B(s, t) = \beta(t)(1 - s) + s \cdot \gamma(t) \text{ for } 0 \le s, t \le 1,$$

and the white surface is parametrized by

$$W(s, t) = \beta(t)(1 - s) + s \cdot \gamma(1 - t) \text{ for } 0 \le s, t \le 1.$$

Notice that, at the crossing, the black surface contains a left half-twist, whereas the white surface contains a right half-twist. The black and white surfaces intersect in the vertical arc $(0, 0, 1 - 2t)$, for $0 \le t \le 1$, which connects $(0, 0, 1)$ to $(0, 0, -1)$.

Thus, any standard arc a has a neighborhood $V \subset \Sigma \times I$ such that a is *vertical* and the projection $p \colon \Sigma \times I \to \Sigma$ maps $(P \cup N \smallsetminus a) \cap V$ homeomorphically onto a once-punctured disk in Σ.

Let A denote the union of all the double arcs in $P \cap N \cap X_L$. Lemma 16 implies that $P \cap N \cap X_L$ does not contain any simple closed curves. Thus it follows that $P \cup N \smallsetminus A$ is a two-dimensional manifold. Furthermore, collapsing the standard models of each of the double arcs a in A down, we see that $\nu(P \cup N)$ is homeomorphic to $\nu(S)$ for some connected surface S embedded in $\Sigma \times I$. We can identify $\nu(S) \approx S \times I$ in such a way that the double arcs are all mapped to distinct points under projection $S \times I \to S$.

Set $c = \frac{1}{2}(e(N) - e(P))$, which is equal to the number of arc components in $P \cap N \cap \partial X_L$. Therefore $P \cap N = L \cup A$ and has Euler characteristic $-c$.

Claim: $\chi(S) = \chi(\Sigma)$.

Proof of the Claim. Since $\nu(S) = \nu(P \cup N)$, we have that

$$\begin{aligned} \chi(S) &= \chi(\nu(S)) = \chi(\nu(P \cup N)) = \chi(P \cup N), \\ &= \chi(P) + \chi(N) - \chi(P \cap N), \\ &= (1 - b_1(P)) + (1 - b_1(N)) + c. \end{aligned} \tag{7}$$

On the other hand, computing the signature of L using P and N, we see that

$$\begin{aligned} \sigma_P(L) &= \operatorname{sig}(\mathcal{G}_P) + \tfrac{1}{2}e(P, L), \\ \sigma_N(L) &= \operatorname{sig}(\mathcal{G}_N) + \tfrac{1}{2}e(N, L). \end{aligned}$$

Thus, by equation (1), we have

$$\begin{aligned} \sigma_P(L) - \sigma_N(L) &= \operatorname{sig}(\mathcal{G}_P) - \operatorname{sig}(\mathcal{G}_N) + \tfrac{1}{2}(e(P, L) - e(N, L)), \\ &= \operatorname{sig}(\mathcal{G}_P) - \operatorname{sig}(\mathcal{G}_N) + \tfrac{1}{2}(e(P) - e(N)), \\ &= b_1(P) + b_1(N) - c. \end{aligned}$$

Substituting this into equation (7), it follows that

$$\begin{aligned} \chi(S) &= 2 - (b_1(P) + b_1(N) - c), \\ &= 2 - (\sigma_P(L) - \sigma_N(L)), \\ &= 2 - 2g(\Sigma) = \chi(\Sigma). \end{aligned}$$

Note that we know that $\sigma_P(L) - \sigma_N(L) = 2g(\Sigma)$, using S^*-equivalence between N and $P \#_\tau \Sigma$, and observing that $\sigma_P(L) - \sigma_{P\#_\tau\Sigma}(L) = 2g(\Sigma)$, the latter following since P is positive definite.

Projecting L along $\nu(S) \approx S \times I \to S$, gives a diagram D for L which by the claim has genus $g(S) = g(\Sigma)$. Furthermore, the checkerboard surfaces of D on S are evidently equal to P and N. It follows that D is alternating. Notice that the link diagram on S has the same Gauss code as D, and since both are minimal genus representatives of L, Kuperberg's theorem [Kup03] guarantees that they represent the same virtual link. □

In Theorem 1.2 of [Gre17], Greene uses his characterization to deduce that any two connected, reduced, alternating diagrams of the same classical link have the same crossing number and writhe. A key result is Theorem 5.5 of [Gre17], which shows that two connected bridgeless planar graphs with isometric flow lattices have the same number of edges. In this way, Greene gave a new geometric approach to establishing the first two Tait conjectures. Building on this approach, Kindred recently gave a geometric proof of the Tait flype conjecture [Kin20]. The first two Tait conjectures have been extended to alternating links in thickened surfaces and alternating virtual links in [BK19, BKS20]. In [BK19], the results are deduced using the homological Jones polynomial [Kru11]. In [BKS20], stronger statements are obtained using adequacy of the Kauffman skein bracket. It is an interesting question whether Greene and Kindred's methods can be extended to links in thickened surfaces and used to give alternative, geometric proofs of all three Tait conjectures in the generalized setting.

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Mathematics & Statistics, McMaster University, Hamilton, Ontario
E-mail address: boden@mcmaster.ca

Mathematics & Statistics, McMaster University, Hamilton, Ontario
E-mail address: karimih@math.mcmaster.ca