UNKNOTTING OPERATIONS

UNKNOTTING OPERATIONS FOR CLASSICAL, VIRTUAL AND WELDED KNOTS

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Chapter 1

Introduction

A central problem in knot theory is that of classifying knots. The goal is to develop methods to tell whether two given knots have the same knot type. Solutions to the truncated version of this problem are given by tabulating all knots up to a given crossing number. If two knots are distinct, it is natural to ask how far apart they are. To answer this, one needs a notion of distance on knots. One way to do this is via an unknotting operation, where distance is measured as the least number of operations needed to convert one knot into another. For instance, crossing change is an unknotting operation (see Figure 1.1), and the unknotting number of a knot is defined to be the least number of crossing changes needed to deform it to the unknot. More generally, the Gordian distance between two knots is defined as the least number of crossing changes needed to deform one knot into the other. As we shall see in Section 5.2, Gordian distance defines a metric on the space of knots.

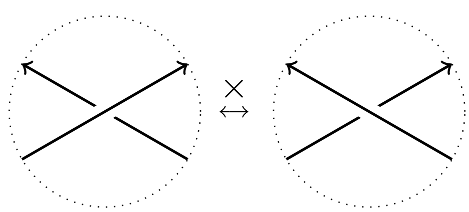


Figure 1.1: The crossing change (or $\times$-move)

The unknotting number is an invariant of central importance in knot theory. Using algebraic methods, one can find lower bounds for the unknotting number of a given knot. For instance, in [Wen37] Wendt showed that the minimum number of generators of the first homology group of the double branched cover branched along a knot gives a lower bound for the unknotting number. This criterion was later subsumed by Nakanishi [Nak81], who gave a similar bound in terms of the rank of the Alexander module (see Definition 2.25). Thus, the rank of the Alexander module is useful in computing the unknotting number of a knot.

The general strategy in working with unknotting operations is to develop algebraic methods to get a lower bound and to use constructive methods to get an upper bound. Every time we find a sequence of crossing changes that unknots a given knot, we obtain an upper bound on its unknotting number. On the other hand, invariants can be used to provide lower bounds. In fortuitous circumstance, these two methods collide to yield bounds that are sharp and can be used to infer the unknotting number.

Some properties of a knot can be deduced from its unknotting number. For example, Scharlemann showed that any knot with unknotting number one is prime. For torus knots, the unknotting number was determined by Kronheimer and Mrowka in [KM93, KM95] using gauge theory. A table of unknotting numbers of classical knots up to 12 crossings can be found in [CL], where 664 entries remain unknown (but only 9 unknown for knots up to 10 crossings).

The tabulation of Gordian distances between knots has been given up to 9 crossings in [Moo10, p.126], but in many cases the information is incomplete. The methods to calculate the Gordian distance are similar to those of unknotting numbers: geometric and algebraic invariants are used to construct lower bounds and constructive methods are used to give upper bounds. It is of great interest to use other unknotting operations to define generalized notions of unknotting number and Gordian distance. In this context, many well-known knot invariants play a key role in providing bounds. Knot signatures, knot determinants, Blanchfield pairing, linking forms, Arf invariants, and polynomial invariants are all useful in determining generalized unknotting numbers and Gordian distances.

The algebraic unknotting number is defined as the number of crossing changes needed to deform a given knot into one with trivial Alexander polynomial. It admits a purely algebraic definition in terms of (S-equivalence classes of) Seifert matrices; see [Mur90, Fog93, Sae99]. Thus, the algebraic unknotting number converts a geometric problem into an algebraic problem, and it has been tabulated for knots up to 12 crossings (see [BF]). More recently, the algebraic unknotting number has been shown to have other, more topological interpretations; see [BF14, BF15]. In [Che19], it is shown that knots with certain Alexander polynomials must all have algebraic unknotting number one.

In this thesis we will consider several other local transformations as potential unknotting operations. We will see that some of them are unknotting operations and some are not. For instance, the Δ-move in Figure 1.2 is known to be an unknotting operation for classical and welded knots. Similarly, the $\sharp$-move and p-move are shown in Figure 1.3. Despite their similar appearance, only one of them (the $\sharp$-move) is an unknotting operation for classical knots.

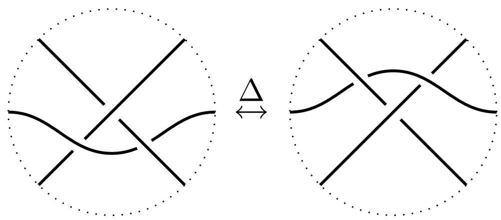


Figure 1.2: The delta move (or Δ-move)

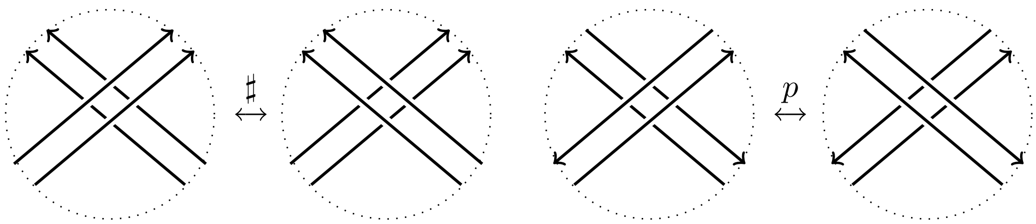


Figure 1.3: The sharp move (or $\sharp$-move) and band-pass move (or p-move)

We discuss these operations and other local transformations in Chapter 4. We are interested in these moves as they reveal some topological properties of knots. For example, Naik and Stanford showed that the Δ^2-move gives a diagrammatic interpretation of S-equivalence, and hence it can be used to relate the unknotting number and the algebraic unknotting number. Local transformations have interest beyond pure mathematics as they also have numerous applications in life sciences and specifically in genetics; see [DEM12, Dar08] and [Ada94, Chap.7]. We are interested not only in calculating generalized unknotting numbers, but also in comparing them among different local transformations and across different knot theories.

This thesis is largely expository, and we provide a survey on unknotting operations for classical, virtual and welded knots, mainly following the papers [Mur85, NNSY18, Sat18], but also drawing on results from a number of other papers, such as [Che19, MN89, Mur70, NS03, Nak81].

In addition, the thesis contains some original work, such as the definition and properties of the algebraic unknotting numbers of virtual and welded knots. Theorems 3.16, 3.17 give an upper bound on the unknotting number. Theorem 4.25 gives a new diagrammatic definition of the algebraic unknotting number. The algebraic

t_n-moves are defined, and we give an algebraic reformulation of t_4-conjecture; see [Prz88, Conjecture 3.6] and [Kir, Problem 1.59]. In Theorem 4.33, we show that the t_4-conjecture holds only if its algebraic version holds. Corollary 4.24 gives examples of many knots which can be unknotted by a sequence of Δ^2-moves and one crossing change. Further, Theorem 5.15 gives a new method to tell if a knot can be turned into a torus knot with one crossing change.

The remainder of this paper is organized as follows. In Chapter 2, we review preliminary material. We define the main objects of study and discuss their basic properties. In Chapter 3, we examine crossing changes for classical, virtual and welded knots and use the crossing number to give an upper bound on the unknotting number of classical and welded knots. We also discuss the sequential and diagrammatic definitions of unknotting numbers and introduce algebraic unknotting numbers for classical, virtual and welded knots. In Chapter 4, we study other local transformations in the context of unknotting operations. For instance, we explain why the $\times$-move, Δ-move, and $\sharp$-move are all unknotting operations for both classical and welded knots. The p-move is an exception. Although it is not an unknotting operation for classical knots, we will see that it is an unknotting operation for welded knots (see Corollary 4.15). In Chapter 5, we discuss generalized Gordian distance and provide some bounds. In Chapter 6 we give a brief summary and outline a few ideas for future research.

Chapter 2

Preliminaries

In this chapter, we start with several topological objects that will be used in later chapters. We also introduce several knot invariants and discuss their properties.

2.1 Knot categories and representations

Definition 2.1. A (classical) *knot* is an embedding of S^1 into S^3. Two knots K_1 and K_2 in S^3 are said to be *equivalent* if there is an ambient isotopy carrying K_1 to K_2.

A (classical n-component) *link* is the disjoint union $\sqcup_n S^1$ embedded in S^3, while the components are allowed to tangle with each other. In this thesis, we mainly discuss properties of knots. It is often to our advantage to represent knots by planar diagrams.

Definition 2.2. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection of a knot K. If $\pi(K)$ has a finite number of singular points, and if they are all transverse double points, then $\pi(K)$ is said to be a *regular projection* of K. The double points in the projection are called *crossings*. A *knot diagram* is a regular projection of a knot with every under-crossing line broken at every double point.

There are some types of classical knots that will be discussed in later chapters.

Definition 2.3. A *torus knot* $T(p, q)$ is a knot that is obtained by taking a line segment of slope p/q in $\mathbb{R}^2$, i.e. $y = (p/q)x$ for $0 \leq x \leq q$, and looking at its image in the quotient of $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$. Note that $T(p, q)$ is a simple closed curve in T^2.

Definition 2.4. A knot is said to be *prime* if it is not a connected sum of two non-trivial knots. A knot is called *composite* if it is not a prime knot.

Definition 2.5. A classical knot is called a 2-bridge knot if it admits an embedding in $\mathbb{R}^3$ such that K has only 2 local maxima.

Equivalently, the definition above says that we can find a plane $\mathbb{R}^2 \subset \mathbb{R}^3$ such that part of K sitting in the plane with no crossing and only two arcs that are not in the plane as they admit the two local maxima. Refer to [BZ03, Chapter 12] for a detailed explanation.

In this thesis, we work with oriented knots, and the orientation is indicated by placing an arrow on the knot diagram. Note that for a same knot embedded in S^3, the diagram is not unique. Therefore, we need to define an equivalence relation between two diagrams.

Definition 2.6. The (classical) *Reidemeister moves* are the three moves shown in Figure 2.1.

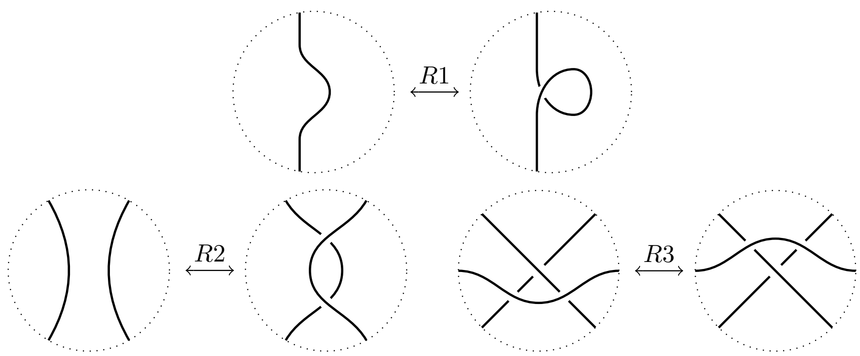


Figure 2.1: The Reidemeister moves

The three moves are called Reidemeister moves as Reidemeister first proved that two classical knots are equivalent if and only if their diagrams are related by a finite sequence of Reidemeister moves and planar isotopies. The three Reidemeister moves and planar isotopies together generate classical isotopies of knot diagrams.

Definition 2.7. A *virtual knot diagram* is a 4-valent planar graph, but each vertex is now allowed to be a classical crossing or *virtual crossing* as in Figure 2.2.

There is an equivalence relation defined on the virtual knot diagram.

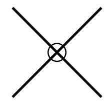


Figure 2.2: The virtual crossing

Definition 2.8. The *virtual Reidemeister moves* are the moves in Figure 2.3. Along with classical isotopies, these virtual Reidemeister moves generate *virtual isotopies* of knots diagrams. Virtual and classical Reidemeister moves are collectively called *generalized Reidemeister moves*.

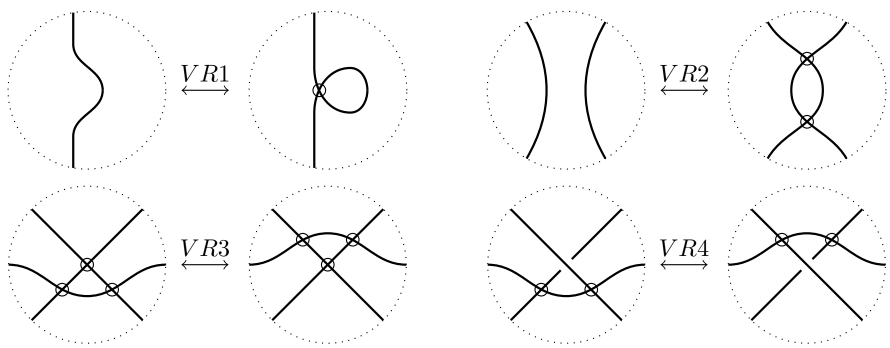


Figure 2.3: The virtual Reidemeister moves

Aside from knot diagram, there are other ways to represent a knot. For instance, one can represent them using Gauss diagrams as follows.

Definition 2.9. For a knot diagram with n crossings, its *Gauss diagram* is a counterclockwise oriented circle with $2n$ points on the circle and n arrows paring the points. Every arrow represents a classical crossing so that arrow head is associated to the over-crossing arc and arrow foot to the under-crossing arc. The arrow head is decorated with a sign $\varepsilon = \pm 1$ according to the writhe of the crossing as in Figure 2.4. The order of the points on the circle tell us adjacency of the crossings in the knot diagram. Note that virtual crossings are not indicated by the Gauss diagram and a Gauss diagram.

The Gauss diagram for the trefoil knot is shown in Figure 2.5 (left). Classical knot diagrams are completely determined by the associated Gauss diagram, but not every Gauss diagram corresponds to a classical knot.

Alternatively, classical isotopies plus detour moves in Figure 2.6 generate virtual knot equivalence. It also explains why virtual crossings can be thought to "not really

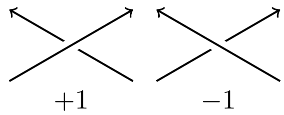


Figure 2.4: The writhe of crossings

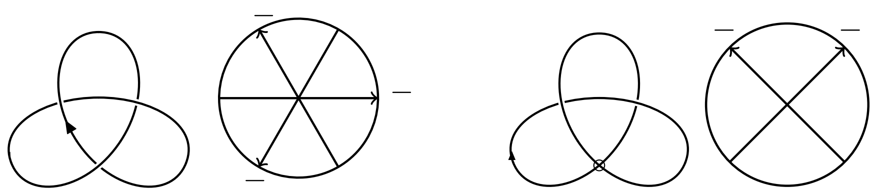


Figure 2.5: Knot diagrams and Gauss diagrams of a classical and virtual knots

exist there". Indeed, virtual crossings are not indicated by the Gauss diagram and a Gauss diagram does not change under a detour move. Hence every Gauss diagram represents one virtual diagram up to detour moves and planar isotopies. The *virtual knot type* is an equivalence class of virtual knot diagrams under virtual isotopy.

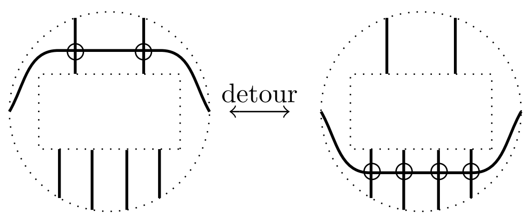


Figure 2.6: The detour move

It is worth mentioning another geometric interpretation of virtual knots. A virtual knot diagram can be realized as a simple closed curve embedded in a thickened surface. Moreover, there is bijection between virtual knots and knots in a thickened surface up to stable equivalence; see [CKS02] for a detailed explanation. Given a virtual knot, the minimal genus over all such surfaces is an invariant of the virtual knot type. Note that for classical knots, this genus is zero. If we regard the virtual crossing as in Figure 2.7 (left), and if every classical crossing is realized by thickening the surface as in Figure 2.7 (right), then every virtual knot can be embedded in a thickened surface. We can attach 2-disks to that surface along its boundaries so that we obtain a closed surface. The resulting surface is called the *Carter surface* of the knot; see [KK00].

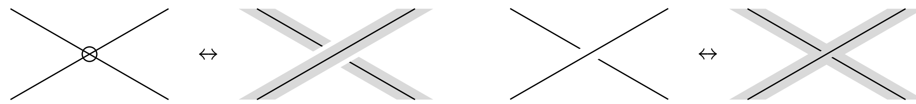


Figure 2.7: The rule to embed a virtual knot as a knot in a thickened surface

The upper and lower forbidden moves (Figure 2.8) are two moves resembling the $R3$ move. Nelson and Kanenobu showed that forbidden moves unknot any virtual knot in [Kan01, Nel01]. The set of virtual knots modulo one of the two forbidden moves gives a nontrivial theory defined as follows ; see [FRR97].

Definition 2.10. The set of *welded knots* is defined to be the set of virtual knots modulo upper forbidden move (UF-move).

Therefore, *welded isotopies* consist of virtual isotopies plus upper forbidden moves.

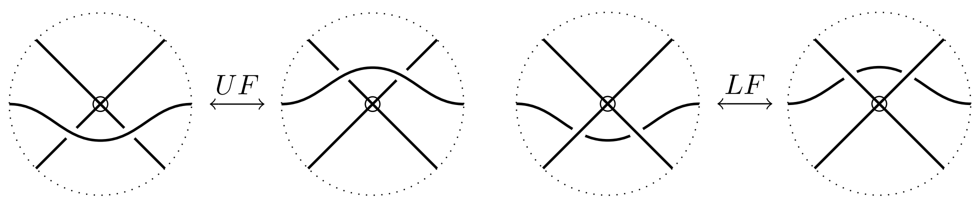


Figure 2.8: The upper and lower forbidden moves

2.2 Knot invariants

Knot equivalence is defined by isotopy. However, Reidemeister moves are not helpful in distinguishing inequivalent knots. Therefore, our strategy is to construct invariants that effectively tell us when two knots are inquivalent.

Definition 2.11. A mathematical object that does not change under classical (resp. virtual, welded) isotopy is called a classical (resp. virtual, welded) *knot invariant*.

Knot invariants take many different forms, such as numbers, polynomials, matrices, groups, etc. Note that a knot invariant ρ can only distinguish two knots K_1, K_2 if $\rho(K_1) \neq \rho(K_2)$; it can only tell us they are the same if ρ is a *complete* invariant. A knot invariant ρ is called complete if $K_1 = K_2 \Leftrightarrow \rho(K_1) = \rho(K_2)$.

Definition 2.12. The *knot group* of a classical knot K is the fundamental group of $S^3 \backslash K$.

The knot group is almost a complete knot invariant with respect to isomorphism. The knot group plus peripheral structure is a complete invariant; see [Wal68]. Knot groups are easy to write out using the Wirtinger presentation. For virtual knots, there is a definition of the virtual knot group; see [BDG+15] for a detailed explanation. However, the virtual knot group, even with peripheral structure, is not a complete invariant for virtual knots. Given two knot groups, it is still hard to check if they are isomorphic.

Definition 2.13. The *crossing number* $c(K)$ (resp. $c_v(K)$, $c_w(K)$) of a classical (resp. virtual, welded) knot K is defined by

$$\begin{aligned} c(K) = & \quad \min \{\# \text{ of crossings of } D \mid D \text{ is a classical diagram of } K\}, \\ c_v(K) = & \quad \min \{\# \text{ of crossings of } D \mid D \text{ is a virtual diagram of } K\}, \\ c_w(K) = & \quad \min \{\# \text{ of crossings of } D \mid D \text{ is a welded diagram of } K\}. \end{aligned}$$

Every classical knot bounds an oriented surface. Note that the surface is not unique but any two surfaces of a given knot will be surgery equivalent. A *surgery* cuts a surface F along a loop ∂D where $D \cap F = \partial D$, and then attaches two 2-disks to each boundary components after cutting. We will construct more algebraic invariants based on such surfaces of knots.

Definition 2.14. For a classical knot K in S^3, a surface F is called a *Seifert surface* of K if F is a connected oriented surface bounded by K.

Note that the result of a surgery on a Seifert surface of a knot is still a Seifert surface of the same knot.

One problem is whether or how to extend the definition of Seifert surface to virtual knots. As we mentioned above, a virtual knot, as a generalization of classical knots, can be viewed as a knot embedded in a thickened surface. A virtual knot is a classical knot if it can be realized as a knot in $S^2 \times I$. Indeed, given a classical knot $K \subset S^3$, we can carefully pick two 3-balls B_i such that $B_i \cap K = \emptyset$ for $i = 1, 2$. Delete B_1, B_2 from S^3. We obtain $S^2 \times I$ which contains K.

A classical knot always bounds an orientable surface. However, this is not always true for virtual knots. A virtual knot that bounds an oriented surface is called an *almost classical knot* or *AC knot*; see [BGH+17]. AC knots form a proper subset of the collection of virtual knots.

Definition 2.15. The 3-*ball genus* (or the Seifert genus) $g_3(K)$ of a knot K is the minimum genus over all Seifert surfaces. The 4-*ball genus* (or the slice genus) $g_4(K)$ of K is the minimal genus of a surface properly embedded in the 4-ball such that the surface is bounded by K. A knot K is said to be *slice* if $g_4(K) = 0$.

A number of algebraic invariants can be constructed from the homology of Seifert surfaces.

For a pair of disjoint simple closed curves x, y in S^3, the *linking number* of x and y is defined as follows. Consider a regular projection of $x \cup y$. At each point where x crosses over y, count as in Figure 1.1. The linking number $\mathrm{lk}(x, y)$ is defined by

$$\mathrm{lk}(x, y) = \sum \{ \text{ sign of crossing} \mid x \text{ is over } y \text{ on this crossing} \} . \tag{2.1}$$

Note that $\mathrm{lk}(x, y) = \mathrm{lk}(y, x)$ for classical knots.

Definition 2.16. For classical knots, given a Seifert surface F of genus g, Let $\{x_1, x_2, \ldots, x_{2g}\}$ be a basis for $H_1(F; \mathbb{Z})$. A *Seifert matrix* $V(F)$ for F is defined to be the matrix (v_{ij}) with $v_{ij} = \mathrm{lk}(x_i, x_j^+)$ for $i, j = 1, 2, \ldots, n$, where x_j^+ is the push-off of x_j in the positive direction of F.

If the generators are pushed in the other direction, the Seifert matrix changes to its transpose. However, for AC knots, it is not necessarily true. Hence there are two Seifert matrices defined on a Seifert surface F of an AC knot K. Denote them by V_F^+ and V_F^-; see [BGH+17]. Note that if K is a classical knot, it holds that $V_F^+ = (V_F^-)^T$.

Definition 2.17. Two matrices V, W are *congruent* if $W = PVP^T$ for a unimodular matrix P. A Seifert matrix W is called an *enlargement* of V if

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & M \\ 0 & N^T & V \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & x & M \\ 0 & N^T & V \end{pmatrix},$$

where M and N are row vectors. Meanwhile, V is a *reduction* of W. The S-*equivalence* class is generated by congruence, enlargements and reductions. The set of S-*equivalence classes* consists of all Seifert matrices modulo S-equivalence. Two knots are said to be S-*equivalent* if their Seifert matrices are S-equivalent.

Let $[V]$ denote the S-equivalence class of a Seifert matrix V. Note that all Seifert matrices of a given classical knot are S-equivalent. However, the converse is not true. For example, Seifert matrices of knots 8_{10} and 10_{143} (refer to [CL]) are S-equivalent.

With the Seifert matrix, one can calculate several knot invariants. The signature of classical knot is among those invariants.

Definition 2.18. Let F be a Seifert surface of some classical or AC knot K. The *signature* $\sigma(F,K)$ of F is the signature of the matrix $V^+ + (V^+)^T$, where V is the Seifert matrix of F with a basis.

It is easy to check that the signature of a classical knot is invariant under S-equivalence, hence the signature $\sigma(K) = \sigma(F,K)$ is a well-defined knot invariant. For an AC knot K, a pair of Seifert matrices V^+ and V^- of K has $\mathrm{sign}(V^+ + (V^+)^T) = \mathrm{sign}(V^{-1} + (V^{-1})^T)$; see [BCG19, Lemma 2.1]. Note that for AC knots, the signature $\sigma(F,K)$ depends on the choice of surface F, and also depends only on the S-equivalence class of the pair (V^+, V^-); see [BCG19] for a detailed explanation.

For the Arf invariant of knots, we refer to the discuss in [Sav12, Chapter 9]. In algebra, the Arf invariant classifies all non-degenerate quadratic forms over any finite dimension vector space V over $\mathbb{Z}/2$ into two types. A function $q : V \to \mathbb{Z}/2$ is called a quadratic form if $I(x,y) = q(x+y) - q(x) - q(y)$ is a bilinear form over $\mathbb{Z}/2$. It is called non-degenerate if the associated bilinear form I is non-singular.

Definition 2.19. The *Arf invariant of a quadratic form* $q : V \to \mathbb{Z}/2$ is defined by

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i),$$

where a_i, b_i $(i = 1, \ldots, n)$ is a symplectic basis in V, i.e. a_i, b_i satisfy $I(a_i, a_j) = I(b_i, b_j) = 0$ and $I(a_i, b_j) = \delta_{ij}$.

Definition 2.20. Let F be a Seifert surface of a classical knot K. The homology group $H_1(F, \mathbb{Z}/2)$ has a quadratic form q represented by $\frac{1}{2}(V + V^T) \pmod 2$. The *Arf invariant* of K is the Arf invariant of q.

The geometric interpretation of $q : H_1(F, \mathbb{Z}/2) \to \mathbb{Z}/2$ is that $q(a)$ measures the number of full twists modulo 2 of the band in a neighborhood of a. For example, the knot in Figure 2.9 has Arf invariant equal to 1. Note that the Arf invariant does not change if we choose a different basis. The Arf invariant has not been extended to virtual knots; see [Chr17] and [FIKM14, Sec.8.2.3]. But it is known not to extend to welded knots, and in Section 4.2 we will see that two knots are p-move equivalent if and only if they have the same Arf invariant.

Definition 2.21. For classical knots K, the *Alexander polynomial* $\Delta_K(t)$ is defined by $\Delta_K(t) \doteq \det(tV - V^T)$, where V is a Seifert matrix of K and $\doteq$ means equal up to multiplication by unit.

Extend this definition to AC knot Seifert matrices; see [BGH$^+$17]. The Alexander polynomial for an AC knot K is $\Delta_K(t) = \det(tV_K^- - (V_K^+)^T)$.

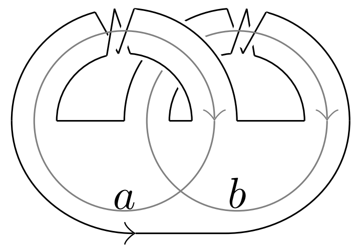


Figure 2.9: A Seifert surface bounded by a knot K, where $\mathrm{Arf}(K) = 1$. Denote by a, b the generators of $H_1(F;\mathbb{Z})$.

Remark 2.22. *A Laurent polynomial $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$ is an Alexander polynomial for some classical knot if and only if $\Delta(t) \doteq \Delta(t^{-1})$ and $\Delta(1) = 1$; see* [Sei35]. *The analogous result for AC knots and it states that a Laurent polynomial $\Delta(t) \in \mathbb{Z}[t, t^{-1}]$ occurs as the Alexander polynomial of an AC knot if and only if $\Delta(1) = 1$; see* [BCG19].

Note that Alexander polynomial can also be defined from other routes. For instance, it can be defined in terms of the knot group.

Definition 2.23. The knot *determinant* $\det(K)$ of a (classical or AC) knot K, is defined by $\det(K) = |\Delta_K(-1)|$, where $\Delta_K(t)$ is the Alexander polynomial of K.

Note that the determinant of a classical knot K is equal to the order of the first homology group for the double cover of S^3 branched along K.

Definition 2.24. Let K be a classical knot, M_K denote the double cover of S^3 branched along K. Its *linking form* $\lambda : H_1(M_K;\mathbb{Z}) \times H_1(M_K;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ is defined by $\lambda([x], [y]) \equiv \mathrm{int}(X, y)/n \pmod 1$, where $\partial X = nx$ for some 2-chain X with $n \in \mathbb{Z}$, and int denotes the intersection number.

Definition 2.25. Let $X(K)$ be $S^3\backslash K$ and $\widetilde{X}(K)$ be its infinite cyclic cover. Let Λ be $\mathbb{Z}[t, t^{-1}]$. The *Alexander module* A_K of a classical knot K is defined to be $H_1(\widetilde{X}(K);\mathbb{Z})$, considered as a Λ-module, where t acts on $\widetilde{X}(K)$ as the deck transformation. Its *rank* $r(A_K)$ (also called the *Nakanishi index*) is the minimum number of generators of A_K.

The Alexander module can also be given by $A_V = \Lambda^{2n}/(tV - V^T)\Lambda^{2n}$, where V is a $2n \times 2n$ Seifert matrix of a classical knot K. Note that $H_1(\widetilde{X}(K);\mathbb{Z})$ is a Λ-torsion module. For any $x \in H_1(\widetilde{X}(K);\mathbb{Z})$, there exists $p \in \Lambda$ such that $px = 0$. Hence it is possible to define the pairing form on $H_1(\widetilde{X}(K);\mathbb{Z})$. In the following, let $Q(\Lambda)$ be the quotient field of λ.

Definition 2.26. The *Blanchfield pairing* of a classical knot K is a map $\beta_K : H_1(\widetilde{X}(K);\mathbb{Z}) \times H_1(\widetilde{X}(K);\mathbb{Z}) \to Q(\Lambda)/\Lambda$, where $\beta_K([x],[y]) = \sum_{i\in\mathbb{Z}} \mathrm{int}(X, t^i y)t^i/p \in Q(\Lambda)/\Lambda$, where $\partial X = px$ for a 2-chain X, and $p \in \Lambda$.

Like the linking form of the double branched cover, the Blanchfield pairing can also be presented by a matrix related to a Seifert matrix of the knot. Further, the Blanchfield pairing is a complete invariant for S-equivalence classes; see [Tro73, p.179].

Note that the Alexander module can also be constructed from knot group; see [BDG$^+$15]. For virtual and welded knots, the Alexander module is a well-defined knot invariant. The rank of the Alexander module is an invariant for virtual and welded knots, too. Moreover, these invariants can be constructed in terms of Seifert pair V^+, V^- for AC knots.

Definition 2.27. Let K be an AC knot with Seifert surface F and Seifert pair V^+, V^- . The Alexander module A_K of K is defined to be $\Lambda^{2n}/(tV^- - (V^+)^T)\Lambda^{2n}$.

Chapter 3

Crossing change as an unknotting operation

In this chapter, we investigate the crossing change as an unknotting operation. For classical and welded knots, the unknotting number is bounded above by half of the crossing number. We use this to determine unknotting numbers of welded knots with $3, 4$ crossings. We discuss the difference between the sequential and diagrammatic definitions of unknotting numbers. At the end of this chapter, we define algebraic unknotting numbers analogously.

3.1 Unknotting classical, welded and virtual knots

We begin by showing that the crossing change is an unknotting operation for classical knots and welded knots.

Definition 3.1. The *(classical) unknotting number* $u(D)$ of a classical diagram D is the minimum number of crossing changes needed to turn D into U, where U is a diagram of the unknot. The *unknotting number* $u(K)$ of a classical knot K is defined by setting $u(K) = \min \{u(D) \mid D \text{ is a diagram of } K\}$.

Since every knot is unknottable by crossing changes, we have $u(K) \in \mathbb{N}$. We extend this notation to virtual knots. However, notice that not all virtual knots can be unknotted by crossing changes. A (generic) *homotopy* between two virtual knots is a sequence of isotopies and crossing changes; see [BC08]. A virtual knot that can be unknotted by crossing changes is called *null-homotopic*. For example, the Kishino knot in Figure 3.1 is not null-homotopic.

Definition 3.2. The *virtual unknotting number* $vu(D)$ of a virtual diagram D is the minimum number of crossing changes needed to turn D into U, where U is a virtual diagram of the unknot. If D is not null-homotopic, define $vu(D) = \infty$. The *virtual unknotting number* $vu(K)$ of a virtual knot K is the minimum number of crossing changes needed to deform a virtual diagram of K to a diagram of the unknot while allowing virtual equivalence at every step.

It is clear that $vu(K) \in \mathbb{N} \cup \{\infty\}$. The Kishino knot in Figure 3.1 has $vu(K) = \infty$.

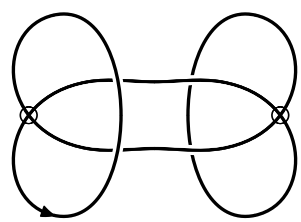


Figure 3.1: The Kishino knot

A welded knot, on the other hand, can always be unknotted by crossing changes. This is proved by showing that any *descending diagram* is welded equivalent to the unknot. Note that an oriented welded knot diagram D, is said to be *descending* if there is a base point $P \in D$ such that walking along D from P, for each crossing of the diagram, we always meet its over-crossing point firstly. The next two lemmas and one corollary are from [Sat18].

Lemma 3.3. *Using crossing changes, one can make any virtual knot diagram descending.*

Proof. For a Gauss diagram of a virtual knot, we can always change its crossings such that starting from the top along counterclockwise direction, every crossing is an over-crossing for the first time we meet it. □

Lemma 3.4. *Take a Gauss diagram of a welded knot K. If there is no under-crossing between the arrow head and arrow foot of one crossing, then this crossing can be removed.*

Proof. This is clear from Figure 3.2. □

Corollary 3.5. *A welded knot represented by a descending virtual diagram is trivial.*

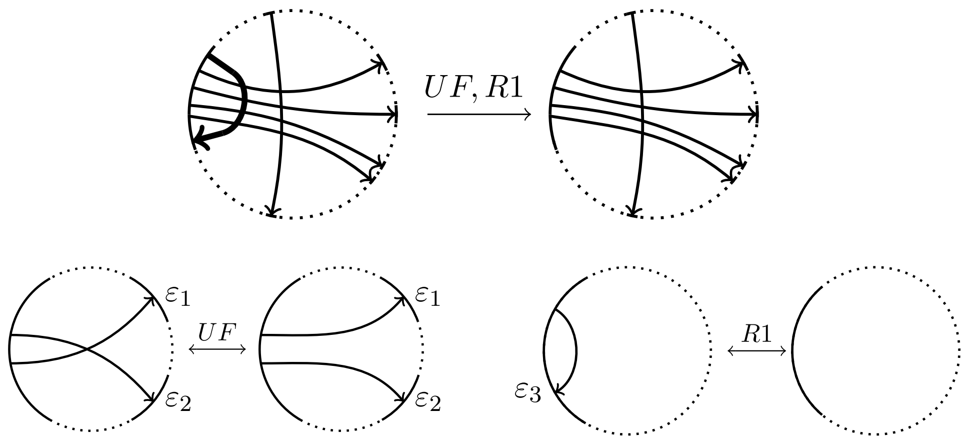


Figure 3.2: There are only over-crossings between the foot and head of the bold arrow. Hence we can remove it by UF-move and $R1$-move.

Proof. This follows by induction on the numbers of crossings using Lemma 3.4. □

Definition 3.6. The *welded unknotting number* $wu(D)$ of a welded diagram D is the minimum number of crossing changes needed to turn D into U, where U is a welded diagram of the unknot. The *welded unknotting number* $wu(K)$ of a welded knot K is the minimum number needed to turn a welded diagram of K into a diagram of the unknot while allowing welded equivalence at every step.

Clearly, both $wu(D)$ and $wu(K)$ are in $\mathbb{N}$.

As mentioned, a virtual knot is crossing-change-unknottable if and only if it is homotopic to the unknot. A *flat virtual knot* is an equivalence class of virtual knots modulo crossing changes. Alternatively, flat virtual knots can be defined by flat virtual diagrams modulo *flat Reidemeister moves*; see Figure 3.3. There are a number of invariants for flat knots, but none are known to be complete or even to detect the flat unknot. In theory, a strong invariant of flat knots would determine whether any given virtual knot is null-homotopic.

Recall the crossing number $c(K)$ (resp. $c_v(K)$, $c_w(K)$) of a classical (resp. virtual, welded) knot K in Definition 2.13. Clearly, we have that $c_w(K) \leq c_v(K) \leq c(K)$ for a classical knot K. Manturov proved that $c_v(K) = c(K)$ for classical knots in [Man13]. Further, we have $c_w(K) \leq c_v(K)$ for any virtual knot since applying an upper forbidden move may decrease the crossing number up to isotopy.

Question 3.7. *Is there classical knot for which $c_w(K) \neq c(K)$?*

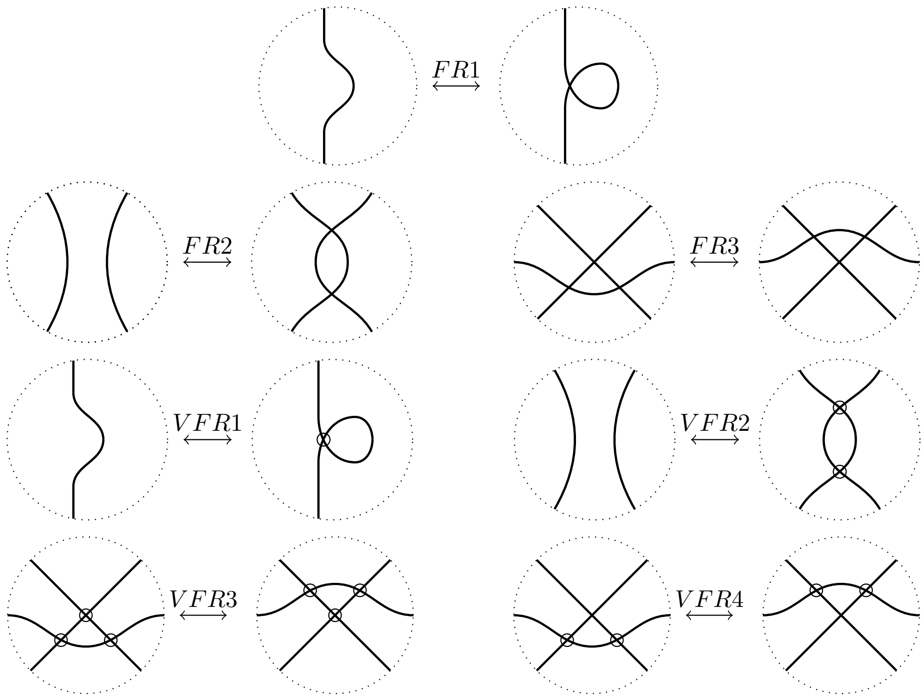


Figure 3.3: The flat Reidemeister moves

For virtual knots, on the other hand, we can easily find examples such that $c_w(K) < c_v(K)$. For example, the Kishino knot has $c_w(K) = 0$ and $c_v(K) = 4$.

Note that a minimal crossing diagram of a given knot may not realize its unknotting number. The first example was found independently by Nakanishi and Bleiler in [Ble84, Nak83]. Their example is a classical knot K with unknotting number $u(K) = 2$, but for which every minimal crossing diagram D has $u(D) \geq 3$, see Figure 3.4.

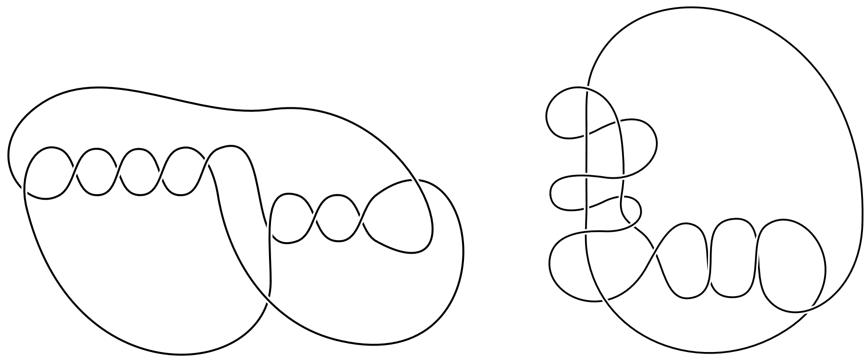
The lemma and theorem below are from [Sat18].

Lemma 3.8. *A welded knot K is non trivial if and only if $c_w(K) \geq 3$.*

Proof. If K has only one or two classical crossings, then its Gauss diagram only has at most two arrows. We can remove these arrows by an upper forbidden move (UF-move) and a virtual Reidemeister-I move (VR1-move). □

Theorem 3.9. *A non-trivial welded knot K has $wu(K) \leq (c_w(K) - 1)/2$. A non-trivial classical knot K has $u(K) \leq (c(K) - 1)/2$.*

Proof. Take a non-trivial welded knot K and we know it has $c_w(K) \geq 3$. The number of crossing changes needed to turn K descending or ascending is at most



Left:$c(D) = c(K) = 10, u(D) = 3$; Right:$c(D) = 14, u(D) = u(K) = 2$.

Figure 3.4: The Nakanishi-Bleiler example

$(c_w(K) - 1)/2$, since there is always one crossing that does not need to be changed. Pick a diagram D such that $c_w(D) = c_w(K)$. Then we have $wu(K) \leq (c_w(K)-1)/2$. The proof for classical knots follows in the same manner.

□

The next corollary follows immediately.

Corollary 3.10. *Every non-trivial welded knot K with $c_w(K) \leq 4$ has $wu(K) = 1$. Every non-trivial welded knot K with $c_w(K) \leq 6$ has $wu(K) \leq 2$.*

Remark 3.11. *There is a welded knot K with $c_w(K) = 5$ or 6 and $wu(K) = 2$. We will give an example and calculation in Section 5.1.*

By definition, we have the inequalities $wu(K) \leq vu(K) \leq u(K)$ for any classical knot K. See [Dye16, Chapter 4.4, Problem 2], an open question asks if the equality can be reached.

Question 3.12. *Is there an example for some classical knot K such that $wu(K) \neq u(K)$?*

3.2 Sequential unknotting numbers

Notice that the classical unknotting number is defined diagrammatically but the virtual and welded unknotting numbers are defined sequentially. The diagrammatic definition does not allow isotopies (i.e. Reidemeister moves and planar isotopy), though both definitions range over all diagrams.

For classical knots, these two definitions coincide; see [Ada94, p.58]. Indeed, we can keep track of where the crossing changes occur in the latter definition so that we get a set of arc pairs. Since we can stretch the arcs and put each over-crossing vertical to its under-crossing on the projection, we get a diagram that admits the minimal number of crossing changes in the first definition.

In contrast, the virtual unknotting number allows one to apply virtual equivalence at every step. Given a null-homotopic virtual knot, it is natural to ask if every diagram D of K is unknottable by crossing changes. Moreover, if we put $\overline{vu} = \min\{vu(D) \mid D \text{ is a virtual diagram of } K\}$, we can see that for any virtual knot K, $vu(K) \leq \overline{vu}(K)$; see [Dye16, Theorem 4.17]. There is an open question as follows; see [Dye16, Chapter 4.4, Problem 4].

Question 3.13. *For any virtual knot K, is $vu(K) = \overline{vu}(K)$?*

Similarly, for welded knots, we do not yet know if these two definitions are equivalent. Putting $\overline{wu}(K) = \min\{wu(D) \mid D \text{ is a welded diagram of } K\}$, we shall ask if $wu(K) = \overline{wu}(K)$ for every welded knot K.

3.3 Algebraic unknotting numbers

In this chapter, we will explore algebraic unknotting for classical and welded knots. Although virtual knots are not always unknottable, we will see they are all algebraic unknottable. These algebraic unknotting numbers bound the unknotting number from below.

For classical knots, there are several ways to define the algebraic unknotting number. Inspired by how crossing changes act on the Seifert matrix, *the algebraic unknotting operation* transforms a Seifert matrix V of a classical knot to

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 1 & x & M \\ 0 & N & V \end{pmatrix},$$

where $\varepsilon = \pm 1$ and x is some integer. Note that M, N are row and column vectors, respectively.

Definition 3.14. For a Seifert matrix V of a classical knot, there exists a sequence of algebraic unknotting operations and S-equivalences transforming V to a 0×0-matrix. The *classical algebraic unknotting number* $u_{alg}(V)$ of V (or $u_{alg}([V])$ for its S-equivalence class) is the minimum number of algebraic unknotting operations in such a sequence.

The original definition of the algebraic unknotting number is given on S-equivalence classes. Though a knot has infinitely many Seifert matrices, they are all S-equivalent to each other. Hence the algebraic unknotting number for a knot is also well-defined. There is another definition which concerns crossing changes rather than matrix operations. The algebraic unknotting number $u_{alg}(K)$ is defined to be the minimum number of crossing changes needed to turn K into a knot with trivial Alexander polynomial. Fogel [Fog93] and Saeki [Sae99] proved this definition coincides with the former one, i.e. $u_{alg}(K) = u_{alg}(V)$ if V is a Seifert matrix of K.

If a virtual knot K has knot group G_K isomorphic to $\mathbb{Z}$, then we say that K has a trivial knot group. For classical knots, the only knot with trivial knot group is the unknot. However, this is not true for virtual knots. For example, the virtual knot shown in Figure 2.5 is not trivial, but has trivial knot group. The next definition and two theorems are new observations.

Definition 3.15. The *algebraic unknotting number* $vu_{alg}(K)$ (resp. $wu_{alg}(K)$) for a virtual (resp. welded) knot K is the least number of crossing changes needed to turn K into a virtual (resp. welded) knot with trivial knot group.

By Lemma 3.3, together with the observation that any descending virtual knot diagram has trivial knot group, it follows that $vu_{alg}(K) \in \mathbb{N}$ for any virtual knot K.

Theorem 3.16. *If K is a virtual knot , then $vu_{alg}(K) \leq vu(K)$. If K is a welded knot, then $wu_{alg}(K) \leq wu(K)$.*

We will use this to get a lower bound for the virtual and welded unknotting number and also to get an upper bound from an unknotting sequence and then finally obtain the precise number.

Theorem 3.17. *If K is a virtual knot, then $vu_{alg}(K) \leq (c(K) - 1)/2$.*

Proof. An ascending or descending virtual diagram has trivial knot group. Then we can algebraically unknot a virtual knot by changing less than half of its crossings. The rest follows similarly as in Theorem 3.9. □

Chapter 4

Local transformations as unknotting operations

We have seen that crossing change is an unknotting operation for classical and welded knots. In Chapter 1, we defined several other diagrammatic moves. In this chapter, we check their capabilities in unknotting a given knot. We explain that the Δ-move and $\sharp$-move are unknotting operations for classical knots while the p-move, Γ-move, Δ^2-move are not. For welded knots, we show that the $\times$-move, Δ-move, $\sharp$-move and p-move are unknotting operations. We also explain why virtual knots are generally not unknottable. In addition, we compare the corresponding unknotting numbers of these moves and relate several knot invariants to them.

4.1 The Δ-move

For the sake of convention, take a diagrammatic move, say $\bullet$-move. A $\bullet$-move applied on classical diagrams is called a *classical $\bullet$-move*; Then following the same manner as we defined the unknotting operations regarding $\times$-moves, the *classical $\bullet$-unknotting number $u^{\bullet}(K)$* is the least number of $\bullet$-moves needed to turn the knot K into the unknot while allowing classical Reidemeister moves. Note that the $\bullet$-unknotting number can be infinite if the knot is not unknottable by $\bullet$-moves. Following the same logic, *the virtual and welded $\bullet$-unknotting numbers $wu^{\bullet}(K)$* and *$vu^{\bullet}(K)$* are defined.

We first discuss the Δ-move.

Remark 4.1. *All orientations assigned to Figure 1.2 can be generated by a single Δ-move in Figure 4.1. The mirror image of Figure 1.2 can be generated by a single*

Δ-move in Figure 4.1. This fact can be verified by drawing all possible diagrams. For example, Δ_2 and Δ_3 are both generated by Reidemeister moves and a single Δ_1-move.

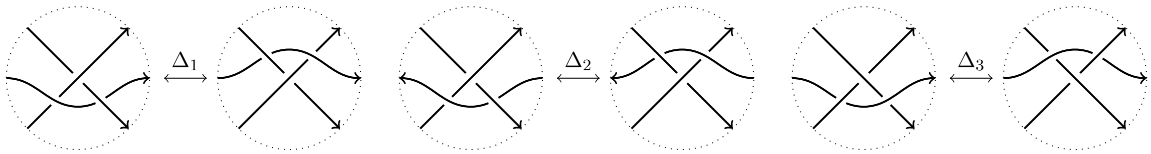


Figure 4.1:

For proving that the Δ-move unknots every welded knot, we need the following two lemmas from [NNSY18].

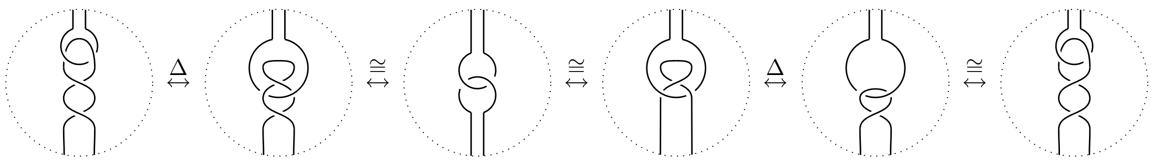


Figure 4.2: The banded Reidemeister-1 move

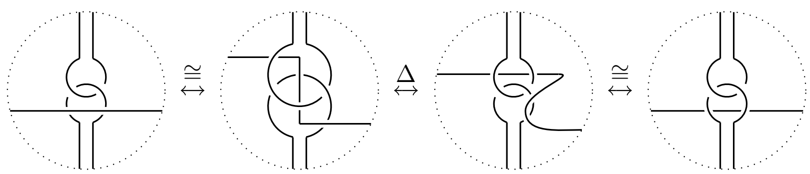


Figure 4.3: The banded crossing change

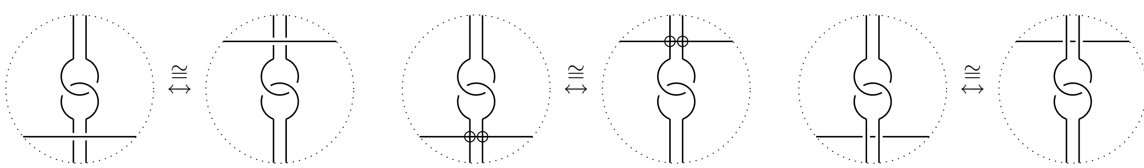


Figure 4.4: The Hopf link can go along the band

Lemma 4.2. *The banded Reidemeister-1 move (Figure 4.2) and the banded crossing change (Figure 4.3) are realized by a sequence of welded Reidemeister moves and a Δ-move, respectively.*

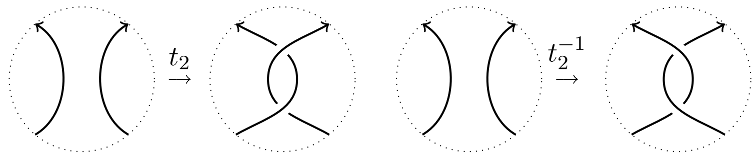


Figure 4.5: The t_2-move and t_2^{-1}-move

Note that a $t_2^{\pm 1}$-move in Figure 4.5 is equivalent to a X-move and a $R2$-move. We will use $t_2^{\pm 1}$-move in later proof. The next result is due to [MN89] for classical knots and [Sat18] for welded knots.

Theorem 4.3. *The Δ-move is an unknotting operation for classical and welded knots.*

Proof. For classical knots, a Δ-move can make a clasp leap over a hurdle as shown in Figure 4.6. Therefore, after a finite number of Δ-moves, the knot has only clasps with no hurdle. Then we can remove the clasp by isotopy. By the moves above, we complete a $t_2^{\pm 1}$-move. Since the $t_2^{\pm 1}$-move is an unknotting operation for classical knots, we conclude that the Δ-move is an unknotting operation for classical knots.

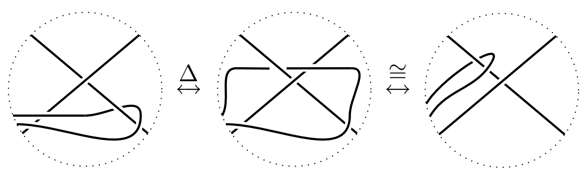


Figure 4.6: A clasp leaps over a crossing by a Δ-move

For a welded knot, similarly, we show that a $t_2^{\pm 1}$-move can be realized by Δ-moves and Reidemeister moves. The detailed proof can be found in [Sat18]. As shown in Figure 4.7, to realize a $t_2^{\pm 1}$-move, we consider a pair of linked bands on the diagram. The two bands are hooked with each other as a Hopf link. Notice that the Hopf link can go along the band without changing any other part of the diagram as shown in Figure 4.4. This is realized by welded Reidemeister moves.

Now perform banded crossing changes to make all arcs only cross below the bands. Further, by performing banded crossing changes, we make the two bands descending, i.e. from one side, the band always meets its over-crossings first as shown in Figure 4.7 (left).

Then similar to the proof of Lemma 3.4, by performing banded Reidemeister-1 moves and banded forbidden moves, we obtain a banded arc with no classical crossing.

Let x_1, x_2 denote the two crossings of the Hopf link as in Figure 4.7 (middle). Notice that the crossing x_1, satisfies that one of the arcs starting and ending at x_1 has no under-crossing except x_1. Then by Lemma 3.4, we can replace x_1 by a virtual crossing. This makes x_2 now satisfy the condition Lemma 3.4 requires, so we can apply another replacement. Finally, we can unlink the two bands and stretch it back to the horizontal arc. Thus we finish a $t_2^{\pm 1}$-move. Since any welded knot can be unknotted by a sequence of $t_2^{\pm 1}$-moves, we conclude that the Δ-move is an unknotting operation.

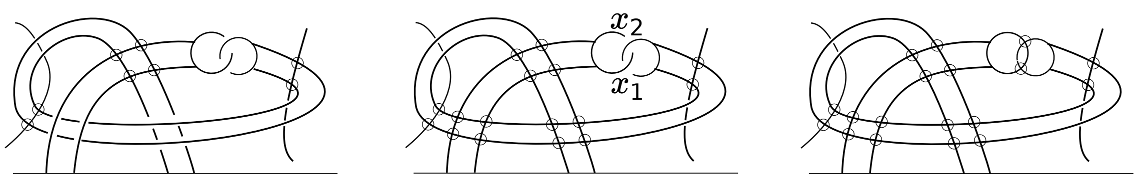


Figure 4.7: Unlink two bands on a welded diagram

□

Theorem 4.4. *For a classical knot K, it holds that $u^{\Delta}(K) \geq u(K)/2$.*

Proof. All possible oriented Δ-moves can be realized by only one Δ_1-move in Figure 4.1. Further, a Δ_1-move can be realized by two crossing changes as shown in Figure 4.8. □

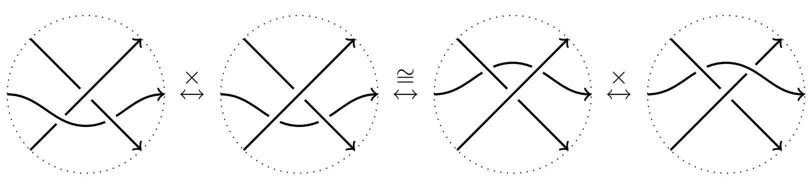


Figure 4.8: The Δ-move can be accomplished by two $\times$-moves

For example, the connected sum $3_1 \# 3_1$ has $u^{\Delta}(K) = 2 \geq u(K)/2 = 1$.

Remark 4.5. *A single Δ-move necessarily changes the classical knot type. However, this is not true for welded knots. As we showed in Figure 4.9, a Δ-move results in a local change to the diagram that can be realized by generalized Reidemeister moves instead.*

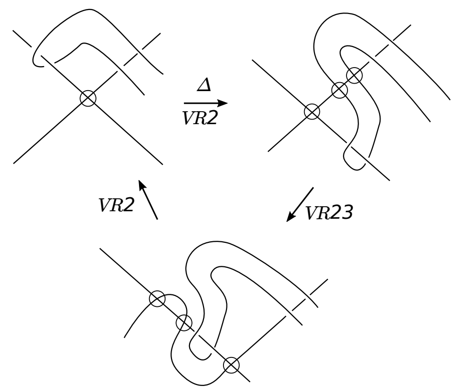


Figure 4.9: Δ-move that does not necessarily change the knot type

4.2 The $\sharp$-move and p-move

We show that the $\sharp$-move is an unknotting operation for classical knots following the proof in [Mur85]. For welded knots, Satoh showed the $\sharp$-move is an unknotting operation in [Sat18]. The next theorem is due to them.

Theorem 4.6. *The classical $\sharp$-move is an unknotting operation for classical knots. The welded $\sharp$-move is an unknotting operation for welded knots.*

Proof. For classical knots, consider a non-orientable surface F bounded by a knot K. The surface can be represented as a disk with a finite number of bands attached. By $\sharp$-moves, two linked bands can be unlinked and every band can be untwisted to only one half twist as shown in Figure 4.10. By the above argument, we deform F into a surface F' made by a disk with only half-twisted bands. Since F' is bounded by the unknot, we conclude that $\sharp$-moves unknot the given knot K.

For welded knots, we need Corollary 4.15, that the p-move is an unknotting operation for welded knots. Observe that a p-move is realized by classical Reidemeister moves and $\sharp$-moves; see [Mur85, Theorem A.2]. Hence $\sharp$-moves can unknot all welded knots. □

The next theorem is due to [Sat18].

Theorem 4.7. *For a welded knot K, it holds that $wu^p(K) \geq wu^\sharp(K)/4$.*

Proof. A $\sharp$-move can be realized by 4 p-moves as shown in Figure 4.11. □

One may wonder if two given knots are related by a single $\bullet$-move. Some algebraic invariants can be used as obstructions for knots to be one $\bullet$-move related. For

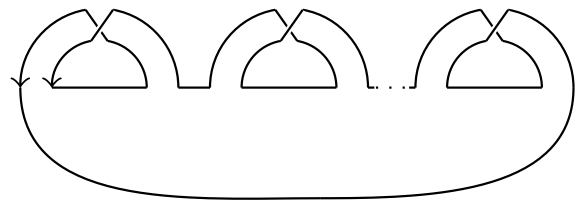


Figure 4.10: Non-orientable surface bounded by a knot

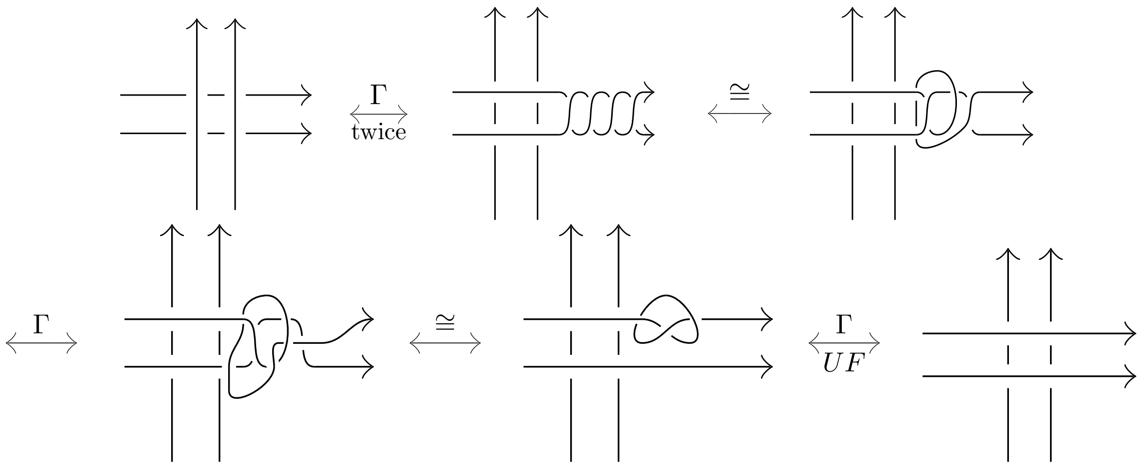


Figure 4.11: The $\sharp$-move can be accomplished by 4 p-moves

example, the Arf invariant has the following property. The following theorem is from [Kau83, p.149] and [Mur85].

Theorem 4.8. *For a classical knot K,*
(a) if K' is obtained from K by a Δ-move, then $\mathrm{Arf}(K) \neq \mathrm{Arf}\,(K')$;
(b) if K' is obtained from K by a p-move, then $\mathrm{Arf}(K) = \mathrm{Arf}\,(K')$;
(c) if K' is obtained from K by a $\sharp$-move, then $\mathrm{Arf}(K) \neq \mathrm{Arf}\,(K')$.

Proof. A band move between two different components is called a *fusion.* A Δ-move can be realized by the result of fusion with Borromean rings as in Figure 4.12.

We call an oriented link *proper* if the sum of the linking numbers of any component of L with all the other components is even. For instance, the Borromean rings is a proper link. Hoste proved that if K is obtained from K' by fusion with a proper link L, then $\mathrm{Arf}(K) \equiv \mathrm{Arf}(K') + \mathrm{Arf}(L) \pmod 2$; see [Hos84]. Note that the Borromean ring has Arf invariant equal to one, hence we conclude that the Δ-move changes the Arf invariant by one.

A p-move on a knot can be seen as a fusion of the link in Figure 4.13 (left). Hence the p-move does not change the Arf invariant.

Similarly, a $\sharp$-move on a knot can be seen as a fusion of the link in Figure 4.13 (right). Hence the $\sharp$-move always changes the Arf invariant by one. □

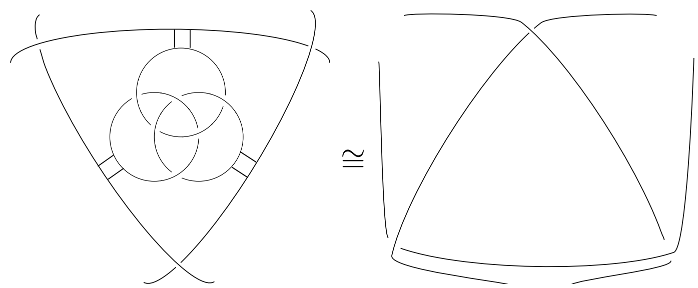


Figure 4.12: Fusion with a Borromean rings

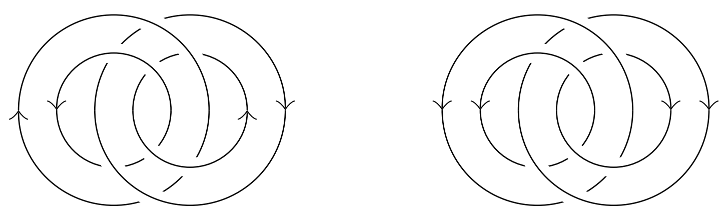


Figure 4.13: A link with $\mathrm{Arf}(L) = 0$ (left) and a link with $\mathrm{Arf}(L) = 1$ (right)

Note that though the diagrams for the $\sharp$-move and p-move are similar, they can not generate each other. Moreover, we show that they are different in the capabilities of unknotting classical and welded knots. Firstly, in contrast to the $\sharp$-move, the p-move requires opposite orientations on every band. Hence it is easy to see how the p-move works on ribbon knots. A *ribbon knot* is a knot that bounds an immersed disk $f : D^2 \to S^3$, where each component of self-intersection is an arc $A \subset f(D^2)$ for which $f^{-1}(A)$ is two arcs in D^2, one of which is interior.

The next lemma and theorem are from [Kau83, Theorem 10.6].

Lemma 4.9. *The p-move is an unknotting operation for ribbon knots.*

Note that the connected sum of a knot and its mirror image is a ribbon knot. In particular, the right hand trefoil is p-move equivalent to the left hand trefoil.

Theorem 4.10. *Every classical knot is p-equivalent to either the trefoil or the unknot.*

Proof. A knot can be deformed by p-moves such that the result admits a Seifert surface as in Figure 4.14, whose boundary is a connected sum of finite trefoils. The connected sum of two adjacent trefoils is p-move equivalent to a connected sum of a trefoil and its mirror image. Then it can be p-move unknotted by Lemma 4.9. Hence we conclude that a classical knot is p-move equivalent to either the trefoil or the unknot. □

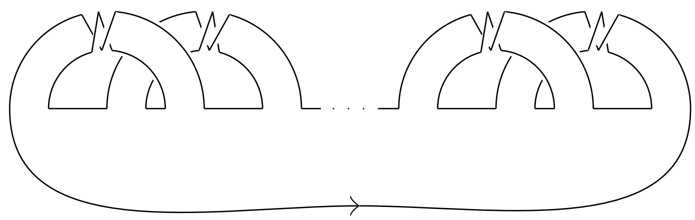


Figure 4.14: Seifert surface after p-moves

With Theorems 4.8, 4.10, the next two corollaries follow immediately.

Corollary 4.11. *Two classical knots are p-move related if and only if they have the same Arf invariant.*

Since the Arf invariant takes value in $\mathbb{Z}/2$, the p-move classifies the classical knots into two sets. One set consists of all knots p-equivalent to trefoil. The other one set consists of knots which are unknottable by p-moves.

Corollary 4.12. *The p-move is an unknotting operation for classical knots with Arf invariant equal to zero.*

Kauffman defined the Γ move, and he showed it is generated by a single p-move. Hence it is not an unknotting operation for classical knots. This move, will be used to show that the p-move is an unknotting operation for welded knots.

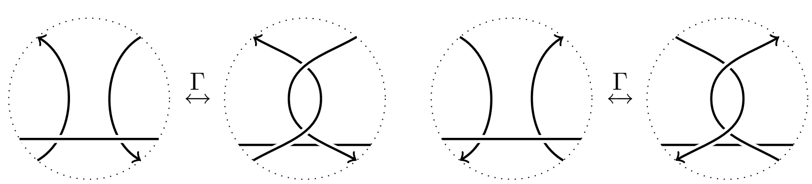


Figure 4.15: The Γ-moves

Lemma 4.13. *A Γ move can be generated by a single p-move.*

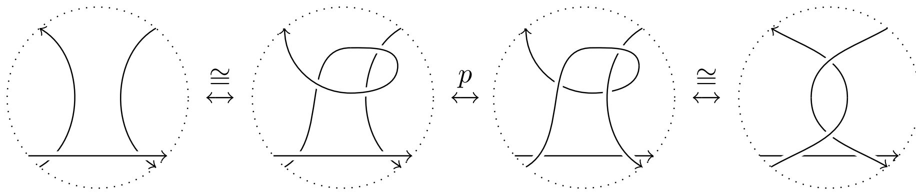


Figure 4.16: A Γ move is generated by a single p-move.

Proof. Take one case for example, the Γ move is completed in Figure 4.16. The other cases follow similarly. ☐

The next theorem is from [NNSY18].

Theorem 4.14. *For welded knots, the lower forbidden move can be generated by p-moves up to welded isotopy.*

Proof. As shown in Figure 4.17, a lower forbidden move is generated by a sequence of Γ-moves and p-moves. By Lemma 4.13, a lower forbidden move can be accomplished by a sequence of p-moves and welded isotopies. ☐

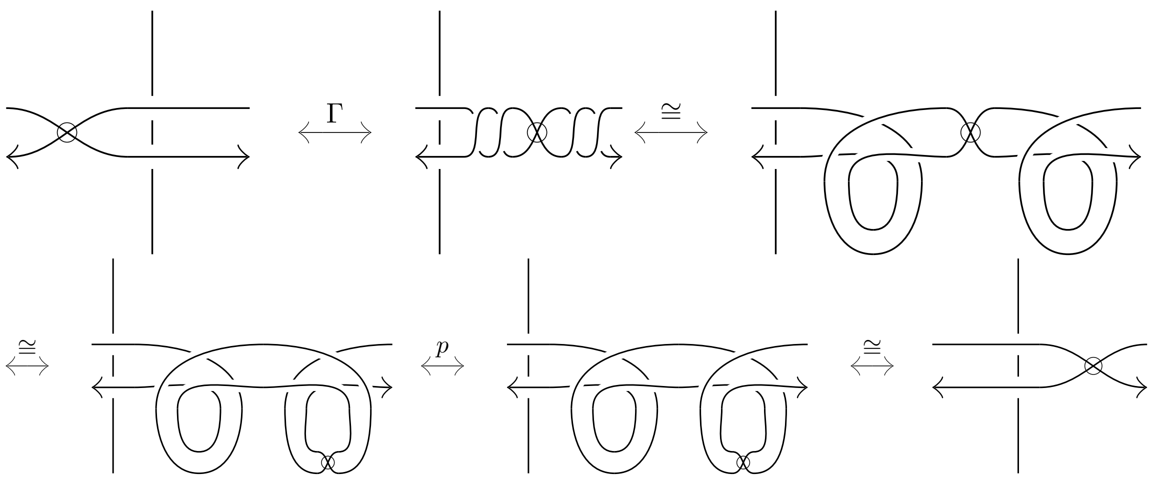


Figure 4.17: A lower forbidden move is accomplished by welded p-moves

Since the two forbidden moves together unknot all virtual knots, we immediately obtain the following corollary.

Corollary 4.15. *The p-move is an unknotting operation for welded knots.*

Remark 4.16. *Virtual knots are in general unknottable (except for the move in Figure 4.18). Indeed, none of the $\times, \Delta, \sharp, p$-moves is an unknotting operation for virtual knots. So instead we consider algebraic unknotting operations for virtual knots.*

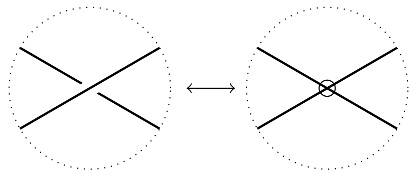


Figure 4.18: The virtualization moves

Question 4.17. *Are $\Delta, \sharp, p$-moves algebraic unknotting operations for virtual knots?*

4.3 The Δ^2-move

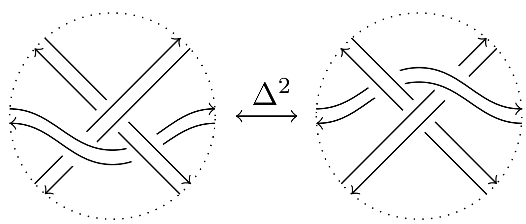


Figure 4.19: The Δ^2-move

The Δ^2-move (or double-delta move) is defined in Figure 4.19. For discussing properties about Δ^2-move, we need define a new topological object called string link.

Let $I = [0, 1]$ be the unit interval and $D \subset \mathbb{R}^2$ the standard 2-disk. Choose distinct points $p_1, \ldots, p_k \in D$. A *(k-component) string link* is a smooth proper embedding $\sigma : \bigsqcup_{j=1}^{k} I_j \to D \times I$, such that $\sigma \mid_{I_j}$ is a path in $D \times I$ from $\{p_j, 0\}$ to $\{p_j, 1\}$ for $1 \leq j \leq k$, where I_j is a copy of I. Two string links σ_1, σ_2 are said to be *equivalent* if one can deform into the other via a sequence of Reidemeister moves preserving the end-point $\{p_i\} \times \{0, 1\}$.

Lemma 4.18. *Two string links are equivalent by a sequence of Δ^2-moves, if and only if they have the same pairwise linking numbers.*

The "if" part is clear since Δ^2 moves does not change pairwise linking number. We need this lemma to prove the next theorem. The proof for the "only if" part can be found in [NS03]. The next theorem is from the same paper.

Theorem 4.19. *Two classical knots are S-equivalent if and only if they can be related by a sequence of Δ^2-moves.*

Proof. Assume two classical knots K_1 and K_2 are related by a single Δ^2-move. We construct a Seifert surface as follows. First cut the knot at the pass crossings and glue each pair up as in Figure 4.20, so that we obtain some oriented surface(s). Then

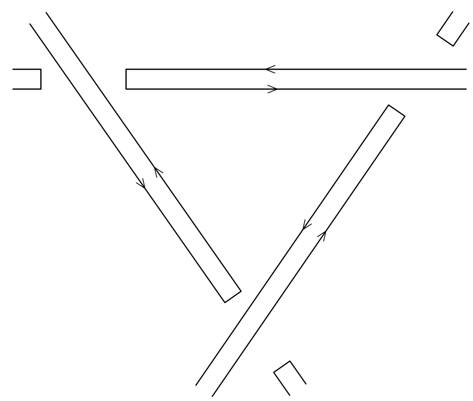


Figure 4.20: An oriented surface

add back the bands at where we cut the knot. We obtain a Seifert surface and we can choose the generators of $H_1(F;\mathbb{Z})$ such that the Seifert matrices before and after the Δ^2-move are the same.

Conversely, assume two classical knots K_1 and K_2 are S-equivalent. Then by enlargement, reduction and congruent we can obtain a matrix M such that M is a Seifert matrix for both K_1 and K_2. Assume M is an $2n \times 2n$ matrix, so K_1 and K_2 both have Seifert surfaces S_1 and S_2 of genus n .

Take a standard Seifert surface F_n as shown in Figure 4.21. Assume $H_1(F_n;\mathbb{Z})$ has basis $\{a_1,\dots,a_{2n}\}$. For $i=1,2$, there exist map $\phi_i : F_n \to S_i$ such that $\{\phi_i(a_1),\dots,\phi_i(a_{2n})\}$ is a basis for S_i. Assume with these bases, the linking form of $H_1(S_i;\mathbb{Z})$ is represented by N_i for $i=1,2$, so $N_i = P_i M P_i^T$ for some unimodular matrix P_i. Then $N_1 = P_1P_2^{-1}N_2(P_1P_2^{-1})^T$. By $N_1 - N_1^T$ is a matrix with blocks of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the diagonal and zeros elsewhere. Since $N_1 - N_1^T = P_1P_2^{-1}(N_1 - N_1^T)P_2^{-T}P_1^{-1}$ we get $P_1P_2^{-1}$ is symplectic. Then there exists a homeomorphism g :

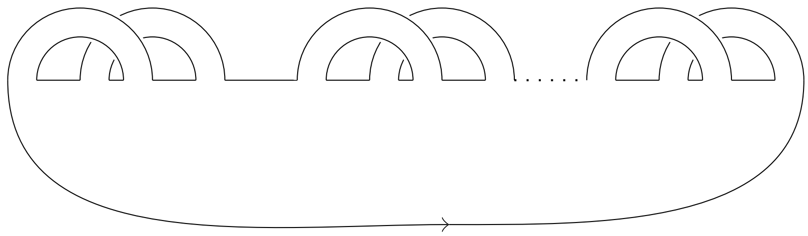


Figure 4.21: A standard Seifert surface

$S_2 \to S_2$ such that $g(\phi_2(a_i)) = \sum_{j=1}^{2n} [P_1 P_2^{-1}]_{i,j} \phi_2(a_j)$. For all $1 \leq i \leq 2n$, and N_1 represents the linking form of S_2 with respect to the basis $(g(\phi_2(a_1)), \ldots, g(\phi_2(a_{2n})))$. We use ϕ_1 and $g\phi_2$ to put S_1, S_2 into a handle body as in Figure 4.22. The knots

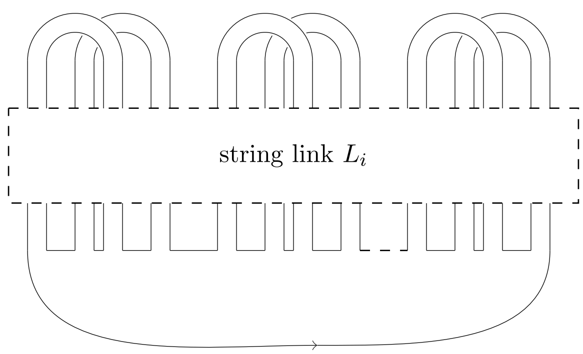


Figure 4.22: A standard Seifert surface

K_1, K_2 are as shown in Figure 4.22 with $2n$-string links L_1, L_2 respectively. By construction, the pairwise linking numbers of the two string links L_1 and L_2 are the same. According to Lemma 4.18, L_1, L_2 are Δ^2-related. This completes the proof. □

Immediately, we have that if a classical knot is of the same S-equivalence class of the unknot, then it is unknottable by the Δ^2-move.

The next lemma was proved by Trotter using linear algebra; see [Tro62] for a detailed proof.

Lemma 4.20. *Suppose V is an $n \times n$ matrix for $n > 1$ and is singular, then it is*

S-equivalent to

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & * & W \end{pmatrix}$$

where W is an $(n-2) \times (n-2)$ matrix.

The next lemma is also due to Trotter; see [Tro73].

Lemma 4.21. *A knot K is S-equivalent to the unknot if and only if it has Alexander polynomial $\Delta_K(t) = 1$.*

Proof. Take a knot K with Alexander polynomial 1. Any Seifert matrix V of K satisfies $\det(tV - V^T) \doteq 1$ for any t so V has to be singular and hence can be reduced. Therefore, there is only one S-equivalence class of classical knots with Alexander polynomial 1. □

Then the next corollary follows immediately.

Corollary 4.22. *The Δ^2-move is an unknotting operation for classical knots with Alexander polynomial* 1*.*

Proof. Any knot with trivial Alexander polynomial, is S-equivalent to the unknot. By Theorem 4.19, it is unknottable. □

Note that any classical knot K with $\Delta_K(t) = 1$ is topological slice; see [FQ90].

We now discuss the existence of an unknotting number one knot. For the $\times$-move, it is true that we can construct the classical unknotting number one knot with any Alexander polynomial; see [Kon79, Theorem 3, p.558]. Moreover, we know that for certain Alexander polynomials, there exists only one S-equivalence class. This can be done by checking the class numbers of the corresponding binary quadratic forms, see [Che19].

Lemma 4.23. *For a Seifert matrix V, if $\Delta_V(t) = ht + ht^{-1} + 1 - 2h$ with $h \in \{1, 2, 3, 5\}$, then $u_{alg}(V) = 1$.*

The next corollary is an original contribution.

Corollary 4.24. *Any classical knot K with Alexander polynomial $\Delta_K(t) = ht + ht^{-1} + 1 - 2h$ and $h \in \{1, 2, 3, 5\}$, can be turned into the unknot by a sequence of Δ^2-move and one single $\times$-move.*

Moreover, Theorem 4.19 gives a diagrammatic definition of algebraic unknotting number of classical knots. The next theorem is a new observation.

Theorem 4.25. *Every knot can be unknotted by a sequence of Δ^2-moves and crossing changes. Let $\tilde{u}(K)$ be the least number of crossing changes in this sequence. Then we have $\tilde{u}(K) = u_{alg}(K)$.*

Proof. Assume $u_{alg}(K) = n$. By definition, it is equivalent to say that we need at least n algebraic unknotting operations on a matrix in the S-equivalence class of K to deform it to a Seifert matrix of the unknot. The S-equivalences in this sequence can be realized by Δ^2-moves. The algebraic unknotting operation relates two Seifert matrices, based on which we can construct two knots differ by one crossing change. Hence we have $\tilde{u}(K) \leq u_{alg}(K)$.

Conversely, every crossing change has a matrix interpretation, so we have $u_{alg}(K) \leq \tilde{u}(K)$. □

Note that in [BF14, BF15], Borodzik and Friedl showed that the numbers below are equal to $u_{alg}(K)$ as well:

(a) The minimal size of a square hermitian matrix A over $\mathbb{Z}[t^{\pm 1}]$ such that the sesquilinear form $l(A)$ is isometric to the Blanchfield pairing of K and such that $A(1)$ is diagonalizable over $\mathbb{Z}$.

(b) The minimal second Betti number of a topological 4-manifold that strictly cobounds $M(K)$, where $M(K)$ is the zero-framed surgery along K.

There is a result regarding the existence of p-unknotting number one welded knot. This result is similar to Kondo's for classical knots; see [Kon79, Theorem 3]. It is proved by Satoh in [Sat18] by a constructive method.

Theorem 4.26. *For any Alexander polynomial $\Delta(t)$ there exists a classical knot K with $wu^p(K) = 1$ and $\Delta_K(t) = \Delta(t)$.*

4.4 Twist moves

We discuss local transformations involving twists. We show how a t_2-move is equivalent to a crossing change. Referring to [Kir, Problem 1.59], there are a range of t_n-involved conjectures. For an integer m and an even integer n, we extend the t_m-move, $\bar{t}_n$-move to their algebraic versions.

The t_n-move is generated by Figure 4.23. It is accomplished by adding n positive half-twists to two arcs of the same orientation on the knot diagram; see [Prz88] for more details of the t_n-move. By this definition, a family of twisting moves are defined. The t_2^{-1}-move is the inverse of t_2. As we show in the next theorem, the $t_2^{\pm 1}$-move is equivalent to the $\times$-move, and hence it is an unknotting operation for classical and welded knots.

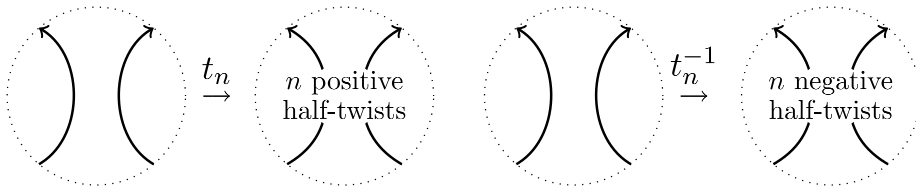


Figure 4.23: The t_n-move and t_n^{-1}-move

The next result is from [Prz88].

Theorem 4.27. *A $t_2^{\pm 1}$-move is equivalent to a $\times$-move.*

Proof. A $t_2^{\pm 1}$-move can be realized by applying one Reidemeister II move (R2-move) and one $\times$-move. Conversely, an $\times$-move can be realized by one $t_2^{\pm 1}$-move and one Reidemeister II move (R2-move). □

We wonder for $n \geq 2$, are there any other t_n-moves that unknot every knot. The next conjecture was first posed by Nakanishi and Kawauchi, according to [Prz88, Conjecture 3.6].

Conjecture 4.28. *Denote the move in Figure 4.24 by $\bar{t}_n$. Then any classical knot can be unknotted by a sequence of t_4 and $\bar{t}_4$-moves.*

Conjecture 4.28 has been verified for several types of classical knots; see [Kir, p.39, Remarks].

Note that $\bar{t}_4$ is not equivalent to t_4^{-1}.

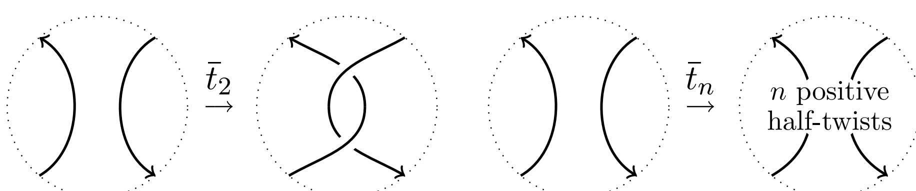


Figure 4.24: The $\bar{t}_n$-move

Question 4.29. *For $k > 4$, is there an unknotting operation as a combination of some t_k's for classical (resp. virtual, welded) knots?*

Observe that a t_2-move is equivalent to a $\bar{t}_2$-move. Though this is not true for $n \geq 2$, their effect on the Seifert matrix are similar for even n. Indeed, the t_n, $\bar{t}_n$-move adds n half-twists to a classical knot diagram and the other part of the diagram stays the same. They do not change generators of the Seifert surface except adding new ones as shown in Figure 4.25. Therefore, we can study their Seifert matrices instead and define two algebraic moves. For n even, it is a generalization of the algebraic unknotting operation in Section 3.3.

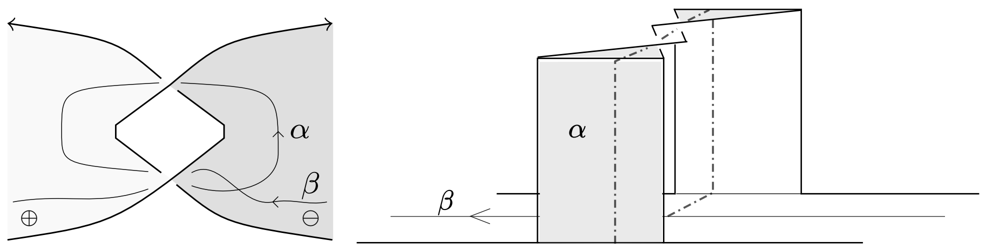


Figure 4.25: Seifert surface after a t_2-move (left), or a $\bar{t}_2$-move (right)

The following definition and theorem are new observations.

Definition 4.30. Given a positive integer m and $\varepsilon = \pm 1$, the *algebraic t_m^ε-move* associates to V the matrix

$$\begin{pmatrix} \varepsilon & 0 & \cdots & 0 & 0 & 0 \\ 1 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ddots & \varepsilon & 0 & 0 & 0 \\ \vdots & \ddots & 1 & \varepsilon & 0 & 0 \\ 0 & \cdots & 0 & 1 & x & M \\ 0 & \cdots & 0 & 0 & N & V \end{pmatrix},$$

where x is some integer, M and N are row and column vectors. Note that the size of the Seifert matrix is increased by m.

Similarly, take a positive even integer n. Given a Seifert matrix V of a classical knot, the *algebraic $\bar{t}_n^\varepsilon$-move* associates to it the matrix

$$\begin{pmatrix} n\varepsilon/2 & 0 & 0 \\ 1 & x & M \\ 0 & N & V \end{pmatrix},$$

where x is some integer, $\varepsilon = \pm 1$, M and N are row and column vectors.

Definition 4.31. Let $m, n \in \mathbb{Z}_{>0}$, m be even. Denote by $u^{t_m,\bar{t}_n}(K)$ the minimal operation numbers in a sequence consisting of $t_m^{\pm 1}$, $\bar{t}_n^{\pm 1}$-moves to turn K into the unknot. Note that $u^{t_m,\bar{t}_n}(K) = \infty$ when K is not unknottable by $t_m^{\pm 1}$, $\bar{t}_n^{\pm 1}$-moves.

Definition 4.32. Let $m, n \in \mathbb{Z}_{>0}$, m be even. For a Seifert matrix V, denote by $u_{alg}^{t_m,\bar{t}_n}(V)$ the minimal operation numbers in a sequence consisting of algebraic $t_m^{\pm 1}$, $\bar{t}_n^{\pm 1}$-moves to turn V into the 0×0-matrix. Note that $u_{alg}^{t_m,\bar{t}_n}(V)$ can be equal to ∞.

Theorem 4.33. *Let $m, n \in \mathbb{Z}_{>0}$, m be even. For a classical knot K and a Seifert matrix V of K, it holds that $u_{alg}^{t_m,\bar{t}_n}(V) \leq u^{t_m,\bar{t}_n}(K)$.*

We obtain a weak version of [Prz88, Conjecture 3.6].

Conjecture 4.34. *For classical knots, any Seifert matrix can be turned into a trivial matrix by a sequence of algebraic $t_4^{\pm 1}$, $\bar{t}_4^{\pm 1}$-moves and S-equivalences.*

Equivalently, this conjecture asserts that $u_{alg}^{t_4,\bar{t}_4}(V) < \infty$ for any Seifert matrix V.

Let $\bar{u}_{alg}^{t_4,\bar{t}_4}(V)$ be the least number of algebraic $t_4^{\pm 1}, \bar{t}_4^{\pm 1}$-moves needed to turn V into a Seifert matrix with trivial Alexander polynomial. Clearly, it holds $u_{alg}^{t_4,\bar{t}_4}(V) \leq \bar{u}_{alg}^{t_4,\bar{t}_4}(V)$.

Question 4.35. *Is $u_{alg}^{t_4,\bar{t}_4}(V) = \bar{u}_{alg}^{t_4,\bar{t}_4}(V)$?*

In [Prz88] and [Kir, Problem 1.59], it was asked (a) whether $t_3^{\pm 1}, \bar{t}_4^{\pm 1}$ can be used to unknot classical knots, and (b) whether $t_3^{\pm 1}, \bar{t}_6^{\pm 1}$ can be used to unknot classical knots.

In [Tuc19], Tucker gives a negative answer to (a). The following conjecture gives a weak version of (b).

Conjecture 4.36. *For classical knots, any Seifert matrix can be turned into a trivial matrix by a sequence of algebraic $t_3^{\pm 1}$, $\bar{t}_6^{\pm 1}$-moves and S-equivalences.*

Based on the algebraic version of these conjectures, we can work on matrices and search for a counterexample. Moreover, if the t_n-move fails to be an unknotting operation, we still can possibly find a criterion to tell which knot can be unknotted by t_n-moves using its algebraic move and Seifert matrices.

Chapter 5

Invariants as obstructions to unknotting

After defining these diagrammatic moves, we hope to calculate their unknotting numbers. To that end, we are interested in using algebraic invariants to obtain bounds on the unknotting numbers. We introduce the Gordian distance, which gives a well-defined distance function on knots. We also give some obstructions on Gordian distance using various knot invariants.

5.1 Bounds on unknotting numbers

As mentioned in the introduction, the rank of the Alexander module gives a lower bound on the unknotting number of classical knots. More precisely, it gives a lower bound on the algebraic unknotting number; see [Nak81].

Theorem 5.1. *If V is a Seifert matrix of some classical knot, then $u_{alg}(V) \geq r(A_V)$, where $r(A_V)$ is the rank of the Alexander module A_V.*

Proof. The rank of the Alexander module can be equivalently defined as the minimal dimension of the matrix presenting the Alexander module. For any Seifert matrix V with algebraic unknotting number n, we need to show that there exists such a matrix of dimension less than or equal to n. We prove this by induction.

For algebraic unknotting number zero Seifert matrix, the Alexander module is presented by a 0×0-matrix.

Assume that every Seifert matrix V with algebraic unknotting number $u_{alg}(V) < n$, the rank of the Alexander module $r(A_V) \leq u_{alg}(V)$. By the definition of algebraic unknotting operation, any Seifert matrix V' with $u_{alg}(V') = n$ is S-equivalent to

$$\begin{pmatrix} \varepsilon & 0 & \mathbf{0} \\ 1 & x & * \\ \mathbf{0} & * & V \end{pmatrix},$$

where $\varepsilon = \pm 1$ and x is some integer. The block V is a Seifert matrix with $u_{alg}(V) = n - 1$. The Alexander module is presented by

$$A_{V'} = \begin{pmatrix} \varepsilon(t-1) & t & \mathbf{0} \\ -1 & x(t-1) & * \\ \mathbf{0} & * & A_V \end{pmatrix},$$

where A_V is a presentation matrix for Alexander module of V. We can transform $A_{V'}$ to $A'_{V'}$ by an invertible matrix over $\mathbb{Z}[t, t^{-1}]$, such that

$$A'_{V'} = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & y & * \\ \mathbf{0} & * & A_V \end{pmatrix}.$$

Since $r(A_V) \leq n - 1$, A_V is $\mathbb{Z}[t, t^{-1}]$-congruent to an $(n-1) \times (n-1)$-matrix. Therefore $A'_{V'}$ is $\mathbb{Z}[t, t^{-1}]$-congruent to an $n \times n$ matrix. Hence $r(A'_V) \leq u_{alg}(V')$. □

Corollary 5.2. *For any classical knot K, we have $u_{alg}(K) \geq r(A_K)$.*

This bound is often used in calculating the unknotting numbers and algebraic unknotting numbers of knots; see [CL, BF] for more examples.

For virtual knot, the Alexander module can be calculated from the knot group. The knot group is invariant under upper forbidden moves and hence is also an invariant for welded knots. Therefore, Alexander module is well-defined for welded knots. In [KKKS17, Lemma 5.2], it is proved that a crossing change can only change $r(A_K)$ by one. Immediately, we have the next corollary.

Corollary 5.3. *For any virtual or welded knot K, we have $r(A_K) \leq wu(K)$.*

The Alexander module is a $\mathbb{Z}[t, t^{-1}]$ module. For a classical, virtual or welded knot K and a prime number p, the order of $\mathrm{Hom}(A_K|_{t=-1}, \mathbb{Z}/p)$ is also an invariant. It tells us how many ways we can p-color the knot (including the trivial coloring). We know that $|\mathrm{Hom}(A_K|_{t=-1}, \mathbb{Z}/p)| = p^{n+1}$, where n is the mod p rank of $A_K|_{t=-1}$. The mod p rank is bounded above by the rank of the Alexander module. (See [Liv93, Chapter 4] and [EN15, chapter 3] for the mod p rank of knots.)

Corollary 5.4. *For any virtual (or welded) knot K, we have $vu(K)$ ($wu(K)$) is bounded below by the mod p rank of $A_K|_{t=-1}$.*

We now use the mod 3 rank to give an example to Remark 3.11. Let K be the AC knot 6.87262 in [BCG19, Table 2]. Note K is a virtual knot with $c_v(K) = 6$. We now show it has $5 \leq c_w(K) \leq 6$ and $wu(K) = 2$.

By Definition 2.27, equivalently we need to calculate the mod 3 nullity of the presentation matrix $V^- + (V^+)^T$. A pair of Seifert matrices of K are

$$V^+ = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V^- = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

due to [BCG19, Table 3]. We get the mod 3 nullity of the matrix is 2. The unknotting number is at least 2. By Theorem 3.9, we conclude $wu(K) = 2$. Also by Theorem 3.9, we have $c_w(K) > 4$.

By the same method, one can show that the AC knot 6.87269 also has $5 \leq c_w(K) \leq 6$ and $wu(K) = 2$.

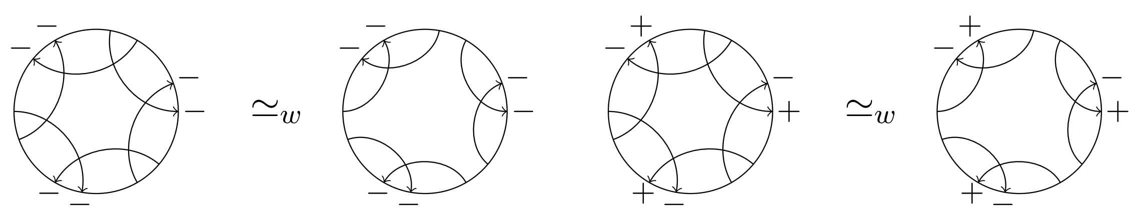


Figure 5.1: Gauss diagrams of 6.87262 (left) and 6.87269 (right) as welded knots

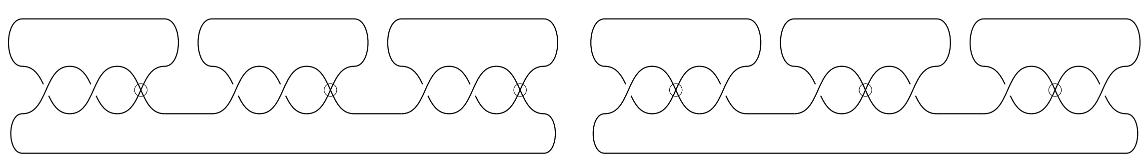


Figure 5.2: Welded diagrams of 6.87262 (left) and 6.87269 (right)

In [Kir, Problem 1.69], it is conjectured that the unknotting number is additive under connected sum for classical knots. We notice that 6.87262 is a connected

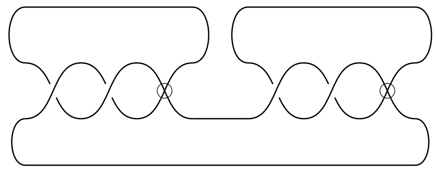


Figure 5.3: $wu(K) = 1$

sum of three trivial welded knots, and that the knot K in Figure 5.3 is a connected sum of two trivial welded knots. We also notice that $wu(K\#K) = 3$ even though $wu(K) = 1$. This example tells us that this conjecture does not hold for welded knots.

Theorem 5.5. *The unknotting number is not additive under connected sum for welded knots.*

As a weaker result related to the conjectured additivity of unknotting numbers for classical knots, the following theorem is given in [Sch85].

Theorem 5.6. *If a classical knot is composite, then it has unknotting number larger than or equal to* 2.

Note that for other local transformations, this is not necessarily true. For example, there are infinitely many composite knots with $\sharp$-unknotting number one; see [MS93].

Let p, q be relatively prime integers. In [Mil68], Milnor conjectured that the 4-ball genus of the p, q torus knot $T(p, q)$ is $(p-1)(q-1)/2$. Since the unknotting number of a knot is bounded below by its 4-ball genus, and since $T(p, q)$ can be unknotted with $(p-1)(q-1)/2$ crossing changes, the Milnor conjecture implies that $u(T(p, q)) = (p-1)(q-1)/2$. It was proved by Kronheimer and Mrowka in [KM93, KM95] using gauge theory. Since the unknotting number of a torus knot is known, it is natural to consider turning a knot into a torus knot to get a bound on its unknotting number. Some results on this can be found in [SM15] for classical knots and [IY17] for virtual knots.

More studies have been done on unknotting numbers, using various knot invariants to give obstructions. Miyazawa found an obstruction in terms of the Jones polynomial for a classical knot to be unknotted with one crossing change; see [Miy98]. Ozsváth, Szabó and Owens used Heegaard Floer homology in [OS05, Owe08] and obtained new bounds which help in calculating unknotting numbers for several knots of crossing number 9 and 10.

5.2 Gordian distances of knots

A natural question is to determine how many transformations are needed to relate two given knots. The Gordian distance is the least number of crossing changes needed to turn one knot into another. This defines a distance function on the set of knots. We call it the *Gordian distance* of knots.

Definition 5.7. By convention, take a reversible $\bullet$-move. For two knots K_1 and K_2, if there exist a sequence of $\bullet$-moves and classical (resp. welded, virtual) isotopies that converts K_1 into K_2, then the *classical (resp. welded, virtual) $\bullet$-Gordian distance* $d^\bullet(K_1, K_2)$ (resp. $wd^\bullet(K_1, K_2)$, $vd^\bullet(K_1, K_2)$) of K_1 and K_2 is the least number of the $\bullet$-moves in this sequence. If the sequence does not exist, we set $d^\bullet(K_1, K_2) = \infty$.

We check that this gives a well-defined metric on knots. Take the classical Gordian distance $d^\times : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{N}$ for example, where $\mathcal{K}$ is the set of all classical knots. It is clear that $d^\times(K_1, K_2) = d^\times(K_2, K_1)$, as we can always reverse the $\times$-move sequence. It satisfies the triangle inequality $d^\times(K_1, K_2) + d^\times(K_2, K_3) \geq d^\times(K_1, K_3)$, and also that $d^\times(K_1, K_2) = 0$ if and only if $K_1 = K_2$. Therefore, the Gordian distance defines a metric on classical knots. Notice that $d^\times(K, \text{unknot}) = u(K)$. Using a similar argument, we see that d^Δ and $d^\sharp$ determine metrics on the space $\mathcal{K}$ of all classical knots; and that $wd^\times$, wd^Δ, $wd^\sharp$ and wd^p determine metrics on the space $\mathcal{WK}$ of all welded knots.

We now introduce an algebraic analogue of the Gordian distance.

Definition 5.8. Suppose K_1 and K_2 are two knots with Seifert matrices V_1 and V_2. The *algebraic $\times$-Gordian distance* $d^\times_{alg}(K_1, K_2)$ is defined to be the minimum of algebraic $\times$-operations (see section 3.3) required to transform $[V_1]$ to $[V_2]$, where $[V_1]$ and $[V_2]$ denote the S-equivalence classes of V_1 and V_2, respectively.

Note that $d^\times_{alg}$ determines a metric on the space consisting of S-equivalence classes of classical knots. It is clear that $d^\times_{alg}(K_1, K_2) = d^\times_{alg}(K_2, K_1)$ and $d^\times_{alg}(K, U) = u_{alg}(K)$ where U is the unknot.

Using the Seifert pairs associated to AC knots, one can ask the following question.

Question 5.9. *Can one define an algebraic Gordian distance for AC knots? Virtual knots have algebraic unknotting numbers as in Definition 3.15. Is there a relation between algebraic Gordian distances and algebraic unknotting numbers of virtual knots?*

It can be difficult to calculate the Gordian distance between two given knots. For example, it is hard to show that $d^\times(3_1, 4_1) > 1$ using diagrammatic methods due to the fact that there exist infinitely many diagrams for a given knot. Hence, we need obstructions provided by invariants.

The next theorem is called the signature criterion; see [Mur85, Mur70]. This criterion is obtained by constructing the Seifert matrices before and after a $t_2^{\pm 1}$-move. Then the signature of the Seifert matrix only changes by at most 2.

Theorem 5.10. *Given two classical knots K, K', we have $|\sigma(K) - \sigma(K')|/2 \leq d^{\times}(K, K')$, where σ is the knot signature.*

Remark 5.11. *AC knots are preserved under crossing changes. For a Seifert surface F bounded by K, we have $|\sigma(F, K)| \leq |\sigma(F', K')| + 2$ if K' is obtained from K by one crossing change. Therefore, $|\sigma(F, K)|/2 \leq vu(K)$ for every F bounded by K.*

Every obstruction on Gordian distances can be used to calculate unknotting numbers. For example, the torus knot $T(p, 2)$ has signature $\sigma(T(p, 2)) = p - 1$. We also know $T(p, 2)$ can be unknotted by $(p-1)/2$ crossing changes. Hence we conclude that $u(T(p, 2)) = (p-1)/2$. Since the same obstruction works for AC knots, we have $vu(T(p, 2)) = (p-1)/2$. In general, one expects that $vu(K) < u(K)$ or $wu(K) < u(K)$ for some classical knot K, but no such example is known. Specially, torus knots have $wu(K) = u(K)$.

It is known that any pair of knots with Gordian distance one have knot diagrams which only differ at one crossing. Thus, the Seifert matrices are all the same except for one entry. Murakami [Mur85] used this method and obtained a result on the double branched cover of a knot as in the theorem below.

Theorem 5.12. *Let K and K' be two classical knots. Let M_K and $M_{K'}$ denote the double covers of S^3 branched along K and K', respectively. Denote by λ the linking form on the double branched cover. If $d^{\times}(K, K') = 1$, then there exist elements $a \in H_1(M_K; \mathbb{Z})$ and $a' \in H_1(M_{K'}; \mathbb{Z})$ such that*

$$\lambda(a, a) \equiv \pm \frac{\det(K) - \det(K')}{2 \det(K)} \pmod 1,$$
$$\lambda(a', a') \equiv \pm \frac{\det(K) - \det(K')}{2 \det(K')} \pmod 1.$$

It is well-known that the double cover of S^3 branched along a 2-bridge knot $B(p, q)$ is the lens space $L_{p,q}$. Hence we obtain $H_1(M_{B(p,q)}; \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$; see [Sch56].

Corollary 5.13. *If $d^{\times}(K, B(p, q)) = 1$ for a knot K, then there exists some integer s such that $\pm q s^2 \equiv \frac{1}{2} \mid p - \det(K) \mid \pmod p$.*

The criterion above gave the first proof that $d^{\times}(3_1, 4_1) = 2$ in [Mur85]. It is often used in calculating the lower bound for Gordian distances and unknotting numbers; see [Moo10, CL] for more examples. We use this as an example to show how the Gordian distance can be calculated. We know that 4_1 is the 2-bridge knot $K_{5,2}$ and that the determinant of 3_1 is 3. Since there does not exist an integer s such that $\pm 2s^2 \equiv \frac{1}{2}|5-3| \pmod 5$, we have $d^{\times}(3_1, 4_1) > 1$. Since the unknotting sequence gives an upper bound, we conclude that $d^{\times}(3_1, 4_1) = 2$.

A generalization of this is given in [Che19] using Blanchfield pairings of classical knots. Recall that β_K denotes the Blanchfield pairing of classical knot K as in Definition 2.26.

Theorem 5.14. *Let K and K' be two classical knots. If the algebraic Gordian distance $d^{\times}_{alg}(K, K')$=1, then there exist elements $a \in A_K$ and $a' \in A_{K'}$ such that $\beta_K(a, a) \equiv \pm \dfrac{\Delta_{K'}(t)}{\Delta_K(t)} \pmod \Lambda$ and $\beta_{K'}(a', a') \equiv \pm \dfrac{\Delta_K(t)}{\Delta_{K'}(t)} \pmod \Lambda$.*

This result can help in calculating Gordian distances. The next theorem is original.

Theorem 5.15. *If $d^{\times}_{alg}(K, T(p, q)) = 1$ and if g is the generator of Alexander module $A_{T(p,q)}$, then $\pm \dfrac{\Delta_K(t)}{\Delta_{T(p,q)(t)}} \equiv c\bar{c}\beta_{T(p,q)}(g, g) \pmod \Lambda$ for some Laurent polynomial c.*

Proof. The Alexander modules are cyclic for torus knots. If $d^{\times}_{alg}(K, T(p, q)) = 1$ and g is the generator of $A_{T(p,q)}$, then by Theorem 5.14, $\pm \dfrac{\Delta_K(t)}{\Delta_{T(p,q)}(t)} \equiv \beta_{T(p,q)}(cg, cg) \equiv c\bar{c}\beta_{T(p,q)}(g, g) \pmod \Lambda$ for some Laurent polynomial c. □

For instance, if $d^{\times}_{alg}(K, T(2, p)) = 1$, then $\pm\Delta_K(t) \equiv c\bar{c}\, (t^{\frac{p-1}{2}} + t^{\frac{1-p}{2}}) \pmod{\Delta_{T(2,p)}(t)}$ for some Laurent polynomial c.

There are some well-known results regarding other local transformations. The theorem below follows in the same manner as in Theorem 4.4.

Theorem 5.16. *For two classical knots K, K', we have $d^{\times}(K, K')/2 \leq d^{\Delta}(K, K')$.*

It is hard to relate the $\times$-distance with the Arf invariant. But for Δ and $\sharp$-moves, there is an immediate result of Theorem 4.8.

Theorem 5.17. *For two classical knots K, K', we have*

$$d^{\Delta}(K, K') \equiv \mathrm{Arf}(K) - \mathrm{Arf}(K') \pmod 2,$$

$$d^{\sharp}(K, K') \equiv \mathrm{Arf}(K) - \mathrm{Arf}(K') \pmod 2.$$

Gordian distances have not been extensively studied for virtual or welded knots. The next theorem follows from Theorem 4.7.

Theorem 5.18. *For two welded knots K, K', we have $wd^{\sharp}(K, K')/4 \leq wd^{p}(K, K')$.*

Problem 5.19. *Find algebraic invariants that give bounds on the Δ-Gordian distance and the $\sharp$-Gordian distance for welded knots.*

Chapter 6

Conclusion

Local transformations reveal abundant knot properties. If a local transformation is an unknotting operation, it turns the set of knots into a metric space. And the whole set of knots is a complex, while different transformations give different edges. If the local transformation is not an unknotting operation, it gives us a classification of knots. For example, two classical knots are p-move equivalent if and only if they have the same Arf invariant. Moreover, crossing change is not an unknotting operation for virtual knots, so the set of virtual knots are divided into connected components, where each component is a homotopy class. The unknot belongs to the component consisting of all null-homotopic knots. There is a bijection between flat knots and stable-homotopy classes of virtual knots.

The algebraic unknotting of virtual knots does not preserve the underlying flat knot type. Consequently, every virtual knots is algebraically unknottable. The algebraic unknotting of classical knots treats the unknotting problem in terms of Seifert matrices. In the same manner, we consider to define an algebraic version for other local transformations.

Question 6.1. *Is it possible to define an algebraic Δ-unknotting operation according to how the Seifert matrix changes?*

The unknotting numbers are related to a range of knot invariants in many ways. In exploring these, we developed more understanding of their topological and algebraic behavior. In return, new results offer more obstructions that could be used to tabulate knots and their unknotting numbers.

Corollary 4.24 gives examples of many knots which can be unknotted by a sequence of Δ^2-moves and one single $\times$-move. This method is based on the classification of the binary quadratic forms. One can obtain more examples of Alexander polynomials using the same method. It would be interesting to determine how many Alexander polynomials occur for knots of a fixed algebraic unknotting number.

Question 6.2. *Are there infinitely many primes p such that any Seifert matrix with Alexander polynomial $\Delta_V(t) = -pt^2 + (2p+1)t - p$ has $u_{alg}(V) = 1$?*

Note that Question 6.2 is related to the Gauss class number problem. Any answer to this question may also offer a new angle to treat the open problem.

For any virtual knot, making its diagram descending can unknot it weldedly and algebraically. However, it only admits an upper bound on $wu(K)$ and $vu_{alg}(K)$, so we do not know if they are equal.

Question 6.3. *Is $vu_{alg}(K) = wu(K)$ for a virtual knot K? More basically, if a welded knot K' has trivial knot group, is K' necessarily trivial?*

To derive more results from Theorem 5.14, we hope to learn more about the Blanchfield pairing. Note that the Alexander module can also be defined using Fox differentiation, and the Alexander module has been extended to AC knots in [BDG$^+$15]. However, the Blanchfield pairing has not been extended to the virtual setting and we hope to define and study it for AC knots.

Question 6.4. *Can a Blanchfield pairing be defined for AC knots?*

AC knots resemble classical knots in many ways. Aside from Blanchfield pairing, there are other classical invariants that are expected to extend to the set of AC knots.

Question 6.5. *For an AC knot K, can we define an algebraic unknotting number in terms of its Seifert matrices? What relationship does this number have with $vu_{alg}(K)$ given in Definition 3.15?*

In Corollary 3.10, we saw that every non-trivial welded knot K with $c_w(K) \leq 6$ has $wu(K) \leq 2$. The calculation of $wu(K)$ for 5 and 6 crossing knots remains to be done. These calculations may help to find answers to a range of interesting questions.

For other local transformations, we have not yet obtained many useful obstructions in regard to welded knots. We will work on answering the following question.

Question 6.6. *Is there an algebraic invariant to tell if two welded knots (or virtual knots) are related by one Δ-move or $\sharp$-move?*

Recall that [Kir, Problem 1.69] conjectured that the unknotting number is additive under connected sum for classical knots. Related to this conjecture, we ask the question below.

Question 6.7. *Consider the direct sum $\oplus$ of Seifert matrices. Is $u_{alg}(V \oplus V') \geq 2$ for any two non-trivial Seifert matrices V, V' of classical knots?*

In future work, we hope to tabulate the unknotting numbers for AC knots. Notice that any AC knots represented in the thickened torus is automatically null-homotopic. It would be extremely helpful to have invariants to determine when a given virtual knot is null-homotopic. At its heart, this is a problem about invariants of flat knots, and in future work we hope to make progress on tabulating and classifying flat knots.

Problem 6.8. *Find an invariant that is easy to calculate and can detect the trivial flat knot.*

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