PARITIES FOR VIRTUAL BRAIDS AND STRING LINKS

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A thesis submitted to the faculty of Mathematics
in partial fulfillment of the requirement for the degree
Master of Science.

McMaster University, Hamilton, Ontario

Master of Science (2016) (Mathematics)

Title: Parities for virtual braids and string links

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Number of pages: v, 42

Abstract

Virtual knot theory is an extension of classical knot theory based on a combinatorial presentation of crossing information. The appropriate extensions of braid groups and string link monoids have also been studied. While some previously known knot invariants can be evaluated for virtual objects, entirely new techniques can also be used, for example, the concept of index of a crossing, and its resulting (Gaussian) parity theory. In general, a parity is a rule which assigns 0 or 1 to each crossing in a knot or link diagram. Recently, they have also been defined for virtual braids. Here, novel parities for knots, braids, and string links are defined, some of their applications are explored, most notably, defining a new subgroup of the virtual braid groups.

Contents

List of Figures

List of Tables

Introduction

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The theory of virtual knots is constructed on this combinatorial
            basis in terms of the generalized Reidemeister moves.
                                          - L. H. Kauffman, [29].
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The classical definition of a knot is that of a smooth embedding of the circle $\mathbb{S}^1$ in the three dimensional sphere $\mathbb{S}^3$, up to orientation-preserving ambient isotopies. Intuitively, this models a rubber band in space, which may have been created entangled. The convenient thing to do to represent such an equivalence class is to construct a decorated 4-valent graph, by projecting the knot orthogonally on a disk, such that the resulting immersion of $\mathbb{S}^1$ is also smooth, except for a finite number of transverse double points, acting as vertices, and decorate said double points with over and under crossing information. These graphs are called knot diagrams, and the cyclic ordering of edges around is fixed.

This kind of presentation is very natural and quite familiar as it is heavily featured in traditional art all over the world. The part of the knot going under a crossing is broken near the part going over it, and the tangents to the diagram at the undercrossing are colinear and perpendicular to the overcrossing. The ambient isotopies of the embedding of the circle in space translate to a variety of changes to the planar diagram. The allowable moves on planar diagrams were fully codified in the 1920s by multiple authors, most notably Reidemeister [41], hence those operations are called Reidemeister moves. They are illustrated by Figures 3, 4, and 5.

However, the first prominent mathematician to develop the combinatorial approach to knot theory was C. F. Gauss. In [20], he introduced codes, called *intersection sequences* which denote the way crossings of a knot are encountered from the point of view of the knot itself. For example, 123123 is the code associated to a trefoil, which is the knot denoted 3.6 in Green's table [22]. However, the virtual knot 3.7 has the same intersection sequence, as illustrated in Figure 1.

The crossings are numbered consecutively, and met cyclically as one travels once along the knot. In 3.7, there are also double points which are circled. Those are *virtual crossings*, and they do not appear in the intersection sequence. Crossings that do are called *classical*. Gauss was already aware that there were some limitations on which sequences could be produced as intersection sequences of knots. For example, it is essential

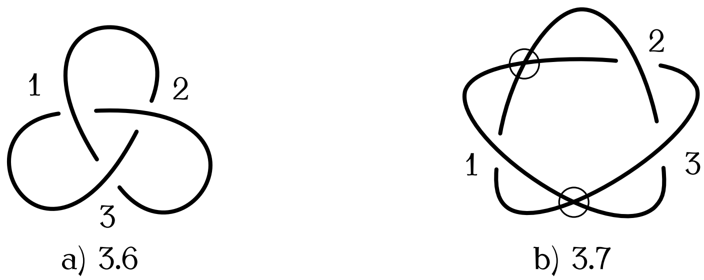

Figure 1: The inequivalent knots 3.6 and 3.7.

that the sequence be *evenly intersticed*, meaning that any two occurrences of the same crossing be separated by an even number of other crossings in the sequence. Sufficient criteria have since been found. A short history of the solutions is featured in [11].

Ignoring the requirement that the knot diagram be planar makes the intersection sequence too weak to uniquely define a given knot. Further decorations are added, yielding objects call *Gauss words*. Classical crossings have two different types, distinguished by the sign of the rotation which brings the overcrossing arc to the undercrossing one in a way that they both point the same way. These positive (+) and negative (−) crossings are shown in Figure 2. After numbering the crossings of a diagram, one determines their sign, and writes down the Gauss word as a sequence of triples consisting of a letter, a number and a sign. The letter `O` is used when going over a crossing, `U` when going under, and the sign is that of the crossing. The knots in Figure 1 then have distinct Gauss codes `O1+U2+O3+U1+O2+U3+` and `O1-U2+O3-U1-O2+U3-`.

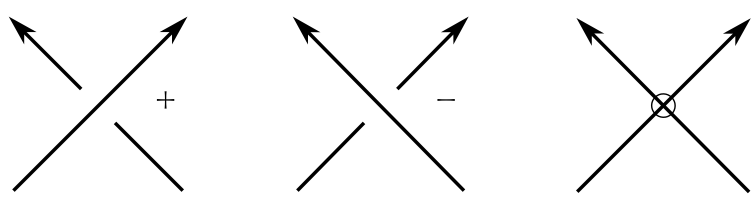

Figure 2: Positive, negative, and virtual crossings.

Virtual knot theory involves the study of all signed Gauss words, not just the ones corresponding to planar graphs, modulo the analogues of the three Reidemester moves. The present thesis uses the combinatorial ideas of *parity*, a theory spanned by Gauss' observation about intersection sequences, to investigate the different generalizations of virtual knot theory, and their differences with classical objects.

The first section contains contains an introduction to virtual knot theory, assuming some familiarity with graphs and groups. The second is concerned with the topological interpretation of virtual knots and links. Finally, the last two sections define parities for braids and string links, and explore their applications.

Acknowledgments. This thesis is the culmination of over three years of collaborating with and studying under the helpful guidance of my supervisor, Dr. H. Boden, and his colleague, Dr. A. Nicas. They nurtured my interest in virtual knot theory and mathematical research by patiently answering my repetitive questions, and encouraging me to participate in international conferences, schools, and meetings. The 2016 Winter Braids school and its organizers were important influence in shaping this written work. Moreover I am immensely grateful to my friends, in particular R. Nieuwenhuis, for believing in me even when I could not.

1 Background material

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Like the complex numbers arising from missing roots of
real polynomials, the new generalized knot types appear
as abstract solutions in knot equations that have
no solutions among the classical geometric knots.
- S. Nelson, [37].
```

1.1 Vocabulary

Knots are not the only objects that can be represented by 4-valent decorated graphs. Let the two dimensional ball be $I \times I$, and $T \subset I \times I$ be a 4-valent graph with classical crossing decorations at the vertices, and maybe end vertices, located on the boundary of the ball. Such a T is called a tangle diagram. If the edges of the graphs can be drawn without intersection, T is *classical*. Otherwise, it is virtual. In both cases, edges are assigned an orientation, where opposite edges at a vertex must have consistent orientations, that is, an edge oriented towards the vertex must be opposite to one oriented away from the vertex. Certain types of tangles have special names, which depend on the number C of closed paths, and the number E of paths with end vertices, and the way those endpoints are situated. The main vocabulary used here is summarized in Table 1.

E	C	Name
1	0	long knot
0	1	round knot
1	> 0	long link
0	> 1	round link
> 0	0	string link[1] or braid[2]
≥ 0	≥ 0	tangle

Table 1: Vocabulary

1.1.1 Gauss diagrams

A Gauss diagram, or GD is a visual presentation of the preimage of a knotted diagram. For knots, one can start with the Gauss word corresponding

[1] If the ith component connects $(1, i)$ to $(0, i)$.

[2] If the components are oriented monotonically down.

to the diagram, and write it around a circle, going counter-clockwise, to preserve the orientation. Occurrences of the same number are connected by arrows pointing from the over crossing to the under crossing. The sign of the crossing is written near the arrow, usually outside the circle at the head or foot. Throughout the text, this core circle is drawn with a thick line while the arrows are thinner. Representations of the first and second Reidemeister moves of GDs appear in Figures 3 and 4 respectively.

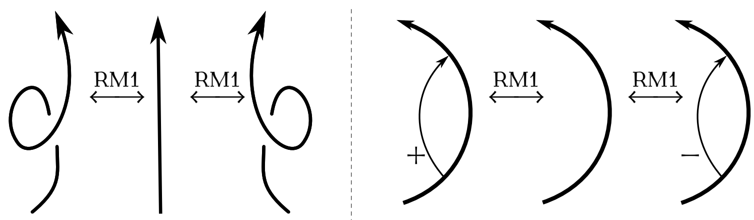

Figure 3: The first Reidemeister move (RM1).

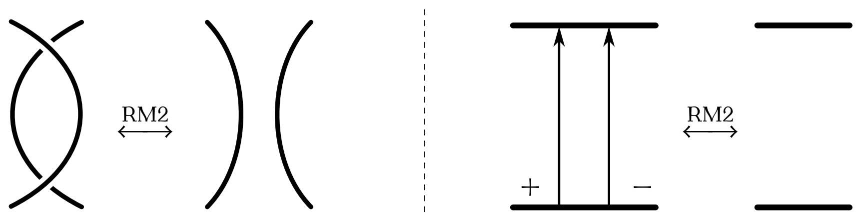

Figure 4: The second Reidemeister move (RM2).

In Figure 5, there are four examples (since ϵ takes value $-$ or $+$) of Reidemeister 3 move on the Gauss diagram, when the illustration for the move on a knot or link diagram allows for eight structurally different choices of orientations of the parts of the diagram involved. The work of Polyak in [40] shows that the four moves pictured here, along with Reidemeister moves of types 1 and 2 are sufficient to generate the other four relations.

For links, it is possible to write a *Gauss paragraph*, consisting or a collection of the Gauss words of each component separated by //. Again, from the paragraph one obtains a Gauss diagram by writing each word around a distinct core circle and connecting the two occurrences of a crossing with an arrow. Core circles need be disjoint but may be nested.

Definition 1. *Let D be a Gauss diagram. If D' can be obtained from D by erasing arrows, then D' is called a* subdiagram *of D, and it is denoted by $D' \subset D$.*

Subdiagrams are the building blocks of GPV finite-type invariants introduced in [21].

In Figure 6, T can be any tangle diagram.

1.2 Braid groups

Classically, braids are tangles consisting of n strands (open components), such that each strand is oriented monotonically from $(i/(n+1), 1)$ to $(j/(n+1), 0)$, for some $i, j \in \{1, 2, ..., n\}$, and such that for each i and each j, there is a unique strand starting or ending at that point, and the strands admit crossings consisting of transverse double points decorated with over and under crossing information. Braid are defined in terms of braid diagrams up Reidemeister moves 2 and 3, which preserve the monotone orientations, and they can be regarded as elements in the Artin braid group B_n. A standard reference about that theory is [1].

The braid group B_n is finitely generated by the elements $\{\sigma_i\}_{i=1}^{n-1}$, corresponding to the elementary braid diagrams shown in the first part of Figure 7. A word in a braid group is represented by stacking the corresponding pictures from top to bottom. The algebraic counterpart to the Reidemeister moves are the natural $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$, and Equation 1. Moreover, ambient isotopies allow Equation 4.

The inverse of a braid word is obtained by reading the word backwards and taking the negative exponent for each classical generator. Geometrically, this is a reflection with respect to a horizontal line.

The first published reference to virtual crossings is the 1997 paper [18] of Fenn, Rimyáni, and Rourke where they introduce the welded braid groups wB_n, which they proved to be isomorphic to a subgroup of $\mathrm{Aut}(\mathcal{F}_n)$ consisting of the automorphisms of permutation-conjugacy type. Later, it became more natural to see welded braids as a quotient of virtual braid groups, vB_n, however much of the original notation has been preserved. Classical generators are $\{\sigma_i\}_{i=1}^{n-1}$, and virtual ones are $\{\tau_i\}_{i=1}^{n-1}$, where $i = 1, ..., n-1$. In vB_n, the following relations hold:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \tag{1}$$

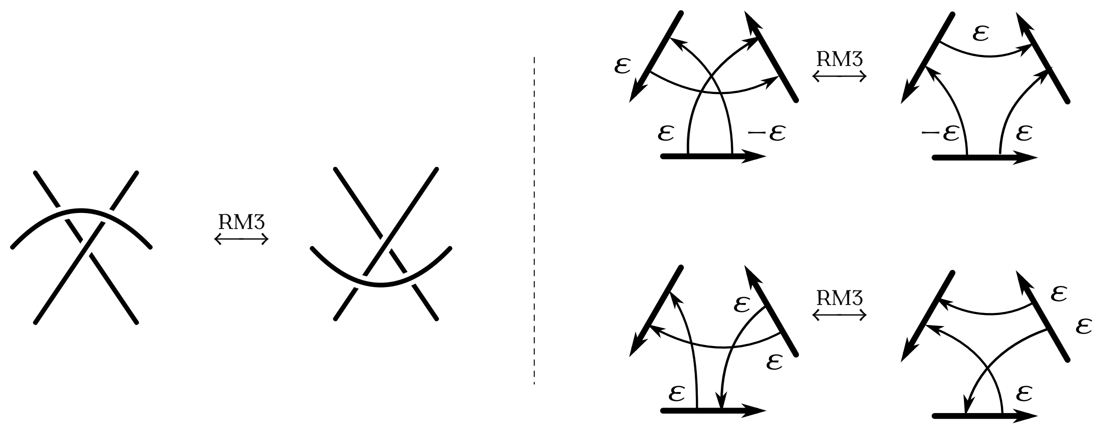

Figure 5: The third Reidemeister move (RM3).

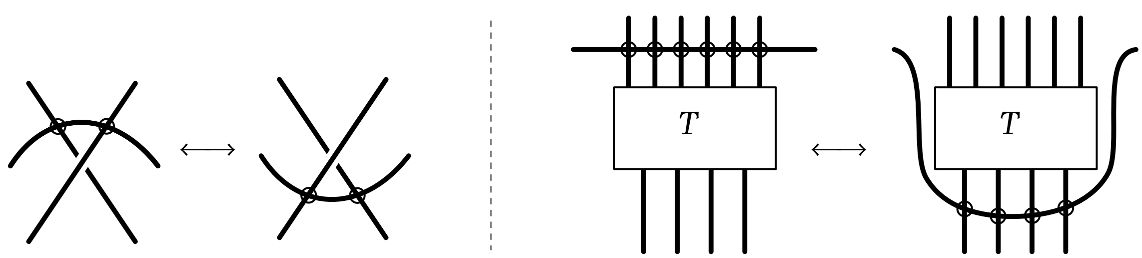

Figure 6: Virtual equivalent to RM3, and the general detour move.

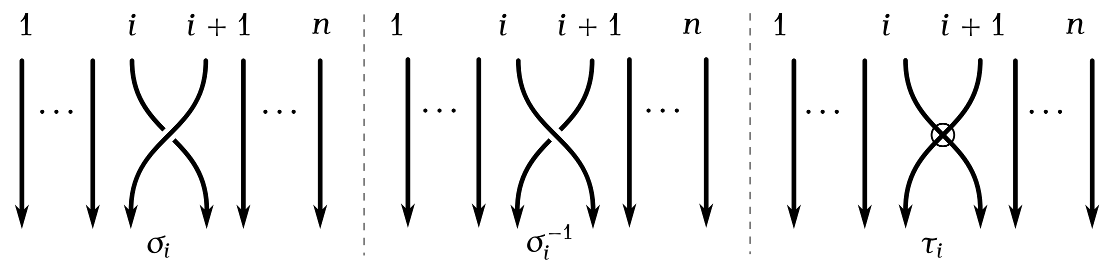

Figure 7: Classical and virtual braid group generators.

$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, \tag{2}$$

$$\tau_i\tau_{i+1}\sigma_i = \sigma_{i+1}\tau_i\tau_{i+1}, \tag{3}$$

for $i = 1, ..., n-2$, and

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \tag{4}$$

$$\tau_i\tau_j = \tau_j\tau_i, \ \sigma_i\tau_j = \tau_j\sigma_i, \tag{5}$$

for $|i-j| \geq 2$, and $\tau_i^2 = 1$ for all $0 < i < n$.

Another interesting subgroup of vB_n is the pure braid group vP_n. It is the kernel of $\Phi : vB_n \to S_n$, $\sigma_i \mapsto \tau_i$, $\tau_i \mapsto \tau_i$. As asked in [3], it remains an open problem to find a topological interpretation of the virtual braid groups.

Classical braids can be represented by Gauss diagrams with one core interval for each strand, and signed arrows for each crossing. For virtual braids, more information is required. Virtual crossings that are found at the end of the braid, are denoted by recording the final position of stands at the bottom of their representative in the Gauss diagram. Virtual crossings that are found before classical crossings influence the position of arrows in the Gauss diagram. With such a system, pure braids are the ones for which the resulting permutation is the identity.

The following sections define and discuss various braid-like groups.

1.2.1 Welded

The welded braid group wB_n is a quotient of vB_n by the relation:

$$\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}, \tag{6}$$

The relation is called the first forbidden move or the overcrossings-commute (OC) relation. It appears in Figure 8. Applying that relation to vP_n yields wP_n, the welded pure braid group.

Welded braid groups appear under many other guises, including braid-permutation groups and loop braid groups. The recent survey article [15] provides a unified view of the work that has been done on those groups, including some important topological perspective.

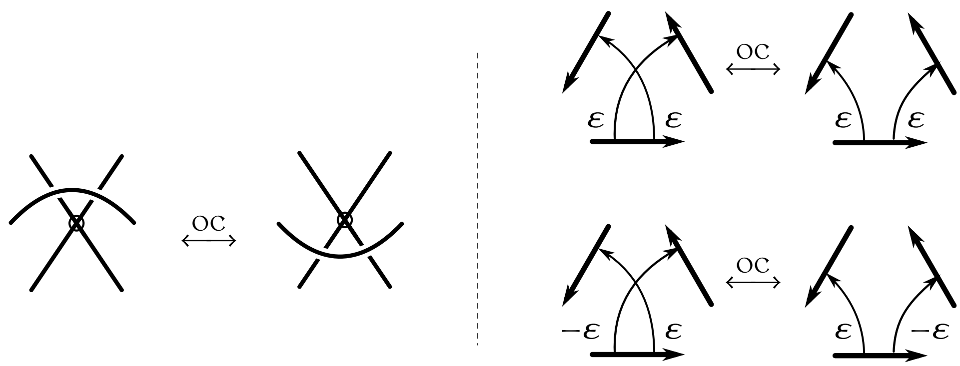

Figure 8: The first forbidden move (OC).

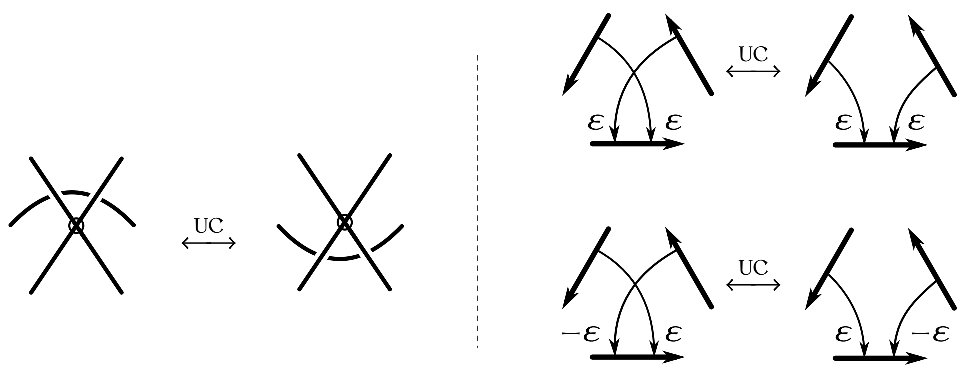

Figure 9: The second forbidden move (UC).

The classical braid group B_n is isomorphic to the subgroup of wB_n generated by the classical generators, as proved in [18]. Hence, the use of the same generator is acceptable. It then easily follows that the inclusion of classical braids in virtual braids at the word level is also an injection.

1.2.2 Unwelded

The second forbidden move, in Figure 9 is also called the undercrossings commute (UC) move. It generates the unwelded braid group uwB_n.

$$\sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1}, \tag{7}$$

It is shown in [5] that uwB_n is not isomorphic to the symmetric group on n elements, which, for a braid group, makes it trivial.

The quotient of wP_n by Relation 7 is the unwelded pure braid group denoted uwP_n.

1.2.3 Flat

Flat braid groups, xB_n, are defined as the quotient of vB_n with the additional relation

$$\sigma_i^2 = 1. \tag{8}$$

Geometrically, it corresponds to the move CC in Figure 10. When this move is allowed, flat crossings, as on the second line, may be used. This is an unknotting operation for classical links, and in particular, allowing this move on B_n, the classical braid group, maps it homomorphically onto S_n, the symmetric group on n elements. This map is equivalent to the one obtained by projecting every crossing in the braid to a virtual one.

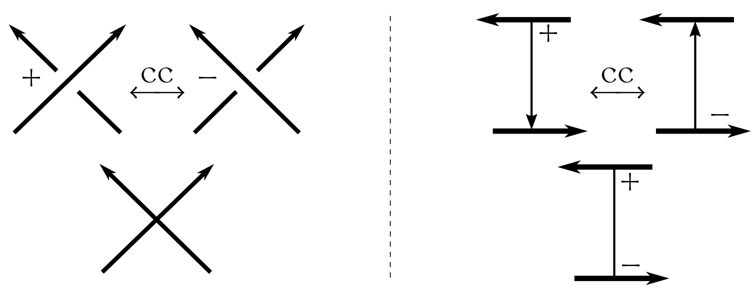

Figure 10: The crossing change (CC).

The name xB_n, and its pure analogue, xP_n are chosen since the letter x evokes the representation of flattened classical crossings as undecorated transverse double points in Figure 10. Their Gauss diagrammatic counterparts bear two signs, + for the strand crossing towards the right, and − on the other end.

1.2.4 Free

The names *oriented* and *unoriented* of the virtualisation moves, as in Figures 11 and 12 are purposely similar to the ones used in [17], but use a different logic. Here, oriented virtualisation means that the move preserves the orientation of the chord in the GD.

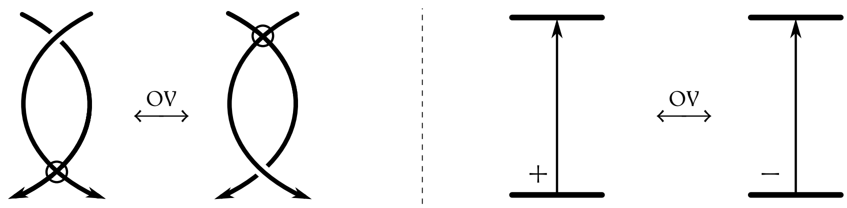

Figure 11: The oriented virtualization move.

They each generate relations on the virtual braid groups,

$$\sigma_i^{-1}\tau_i = \tau_i\sigma_i^{-1}, \text{ (UV), and} \tag{9}$$

$$\sigma_i^{-1}\tau_i = \tau_i\sigma_i, \text{ (OV)}. \tag{10}$$

The quotient of vB_n by the UV relation is called the free braid group and it is denoted fB_n. Again, the analogous pure braid group is fP_n.

The UV equivalence is also called Z-equivalence in [6], particularly when applying a similar move to knot diagrams. The term free braid may not be particularly standard, but the adjective *free* has been extensively used to talk about knots where this move is allowed, notably in [31].

Definition 2. *Free braids on n strands are equivalence classes of Gauss diagrams with n vertical core intervals where chords are horizontal, have neither orientation nor signs, up to the Reidemeister-like moves obtained by ignoring that information for the diagrams of the moves too.*

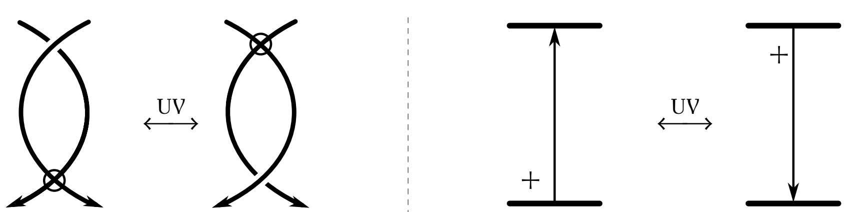

Figure 12: The unoriented virtualization move.

1.3 Virtual knots and quotients

The term virtual knot was introduced in [29]. There currently exists a unique survey of the work on the topic, [36].

As was done for the braid groups, once one defines virtual link diagrams, operations beyond the Reidemeister moves can be allowed. Sometimes this will lead to a trivial theory, meaning that one can use the new moves to unknot all possible diagrams. Otherwise, it may lead to a non-trivial theory. For instance, starting with a virtual link diagram or a Gauss diagram with only closed components, the equivalence class of diagrams up to Reidemeister moves and OC moves is called a welded link. Similarly, the equivalence class of such a diagram up to RMs and CC is a flat link.

Not every quotient theory is interesting. For example, any knot can be unknotted, that is, transformed to a knot with no crossings called the *unknot* by a sequence of Reidemeister moves and forbidden moves (OC and UC). This is proved in Section 4.3.

1.3.1 Alexander numbering

Let D be an oriented virtual knot diagram. *Arcs* of D are subsets of the diagram consisting of curves that go from one classical crossing to the next.

An *Alexander numbering* of D is a assignment of $\mathbb{Z}$-valued labels to the arcs of the diagram up to a global shift in those indices by any integer. The numbering is obtained by as illustrated by Figure 13.

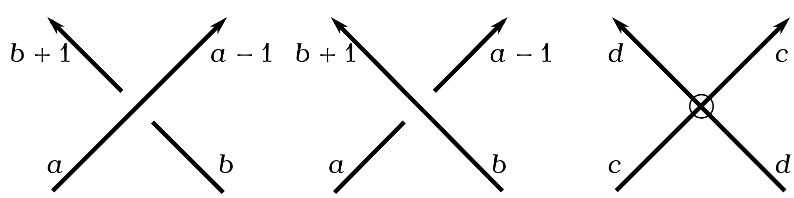

Figure 13: Local values of the Alexander numbering of a diagram.

It follows by the Jordan curve theorem that any knot diagram without virtual crossings admits an Alexander numbering with exactly two labels meeting at each vertex.

Definition 3. *A knot is called almost classical (AC) if and only if it admits a diagram for which the Alexander numbering satisfies the addition condition that for every classical crossing as in Figure 13,* $a = b + 1$*.*

The question of extending Alexander numberings and almost classicality to braids, links, and string links is one of the themes of the present work. The reason the present definition cannot directly be applied is that numberings of a diagram depends on the choice of a basepoint for each of its components.

2 Knots and links

```
[...]  for some purposes it is easier just to ignore the problem
                    of whether a Gauss diagram represents a knot,
                                   rather than trying to solve it.
                    - M. Goussarov, M. Polyak, and O. Viro [21].
```

Under the name *abstract links*, virtual knots and links were first studied in 1993 by N. Kamada, but that idea did not appear in print until the paper [28], which establishes a correspondence between abstract link diagrams and Kauffman's theory of virtual knots. An alternative proof is found in [32], and has the advantage of considering the genera of the surfaces involved, showing that there is a unique minimal genus abstract representative of a virtual knot.

However, the other virtually knotted objects discussed here have not been so extensively studied as topological objects. For example, in [14], braids are set to live on orientable surfaces with two boundary components. Setting a consistent interpretation of virtual braids, string links, and tangles as one-dimensional sub-manifolds of thickened surfaces allows one to construct more invariants and parities for them.

Analogously to the quotients of the virtual braid group defined in the introduction, one can construct coarser knot theories by allowing the additional unoriented moves on link diagrams. A knot theory [25] is a collection of all finite four valent graphs with decorations at the vertices, such as over/under information, and a choice of pairs of opposite edges for each vertices, possibly orientation on the resulting paths, and a set of allowed moves. In this formalism, the graphs need not be planar, and crossings of the edges are virtual crossings from the previous point of view.

2.1 The planarity problem

A *signed Gauss diagram* is an object which represents a flat virtual knot or link. Given an oriented flat link diagram, each crossing is numbered, and the strand going to the right when a crossing is oriented up is marked positive, while the other is negative. The sequence of numbers and signed obtained by going once around each component by following the orientation and choosing arbitrary base points is called a signed Gauss paragraph for the link.

As was mentioned in the introduction, the original planarity criteria was for intersection sequences, which would correspond to unsigned Gauss diagrams. While multiple proofs in that setting exists, for the purpose of virtual knot theory, the planarity of a virtual link diagram depends on its signed Gauss diagram.

Carter's algorithm produces a minimal genus surface on which a Gauss paragraph can be realized as a collection of immersed curves. To a signed Gauss paragraph, one associates a cell complex, where the 0-cells are the crossings, with 1-cells for each pair crossings which appear as subsequent labels in the paragraph up to cyclic permutation of each word, and 2-cells pasted to any path obtained by traveling around the graph either by always turning left at crossings until each edge is adjacent to two distinct cycles.

The algorithm above can be interpreted at the level of Gauss diagrams. In Figure 14, the left part depicts cycles found around a classical crossing while applying the algorithm to a flat link diagram. The bijection between the link diagram and the Gauss diagram extends to the colouring of the paths onto which the 2-cells are to be pasted around the chord of the Gauss diagram on the right.

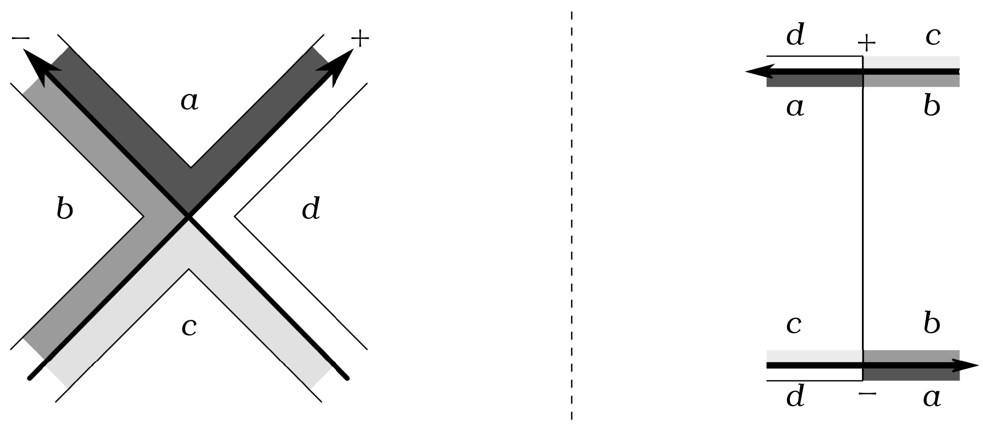

Figure 14: Bi-colouring of a Gauss diagram.

Moreover, to a Gauss diagram G, with $cr(G)$ crossings, and a $b(G)$ number of colours for the bi-colouring, one can draw the corresponding virtual link on a surface of genus $g(G)$, called the Carter genus of G where

$$g(G) = (cr(G) - b(G) + 2)/2.$$

The minimal Carter genus over all diagrams of a link is called its *Kuperberg genus*, in reference to [32].

Definition 4. *An abstract link is a pair (K, Σ) consisting of a surface Σ made of disks containing classical crossings, linked by bands which are* *coherently oriented, such that every band contains a part of the knot diagram K.*

Given a virtual link diagram K, obtain a canonical abstract link diagram, the pair (K, C_K) by gluing parts of surfaces around the diagram as illustrated in Figure 15. For virtual crossings, it does not matter which part of the surface is drawn to cross over as one is not concerned with the embedding of the surface C_K in space. The boundary of C_K is a disjoint collection of circles. As in Carter's algorithm, glue disks along those boundaries. By a result in [28], this yields a compact oriented surface of minimal genus that supports K.

Definition 5. *Given a virtual link diagram L, the band surface associated to the abstract link interpretation of L is denoted C_L. The Carter surface for L is $\hat{C}_L$.*

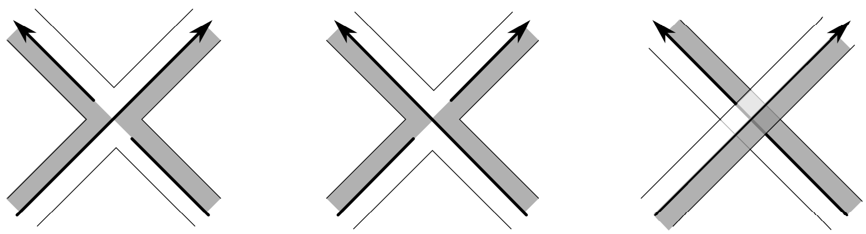

Figure 15: Construction of the canonical abstract link diagram.

In [9], a simple criteria was given to determine if a signed Gauss diagram with one core circle corresponds to a classical knot diagram. It combines the index of the crossing, computed as the sum of signs in one half of the smoothed diagram, with the sum of the signs in the intersection of halves of the diagrams obtained by smoothing two distinct chords. The numbers are then arranged in a grid called the *incidence matrix*. A diagram is planar if and only if that matrix vanishes. In the sequel, [10], the criteria is refined and applied to unsigned Gauss paragraphs. That is, to diagrams of free links.

Unfortunately, both of these planarity criteria are sensitive to Reidermeister moves.

2.1.1 Index and virtual linking numbers

Let $D_1, D_2 \subset L$ be two components of a virtual link diagram D. The *virtual linking numbers* of D_1 and D_2 are $\text{vlk}(D_1, D_2)$, the sum of the signs of the crossings where D_1 goes over D_2, and $\mathrm{vlk}(D_2, D_1)$, the sum of the signs of the crossings where D_1 goes under D_2. Generally, those numbers are different. Then, the *index* of a crossing p in a knot is defined to be $I(p) := \mathrm{vlk}(D_p^+, D_p^-) - \mathrm{vlk}(D_p^-, D_p^+)$ where D_p^+ and D_p^- are as in Figure 18, and $w(p)$ is the sign of the crossing.

The classical linking number of two components is defined to be half the sum of their virtual linking numbers. These virtual linking numbers are similar to the definition of the incidence matrix in [9].

2.2 Parities for virtual knots

In the introduction, the index of the crossing of a knot was used to compute decorations taking value 0 or 1, called the parity of the crossing. Parities are families of maps, one for each diagram of a knot, sometimes decorated by the diagram they are computed on.

Definition 6 ([38]). *Let ψ be a map assigning 0 or 1 to the classical crossings of some diagrams D and D' of a virtual knot. Then, ψ is a parity if the following hold.*
(Ψ1) If D and D' are related by any Reidemeister move, and $c \in D$ is a crossing which is not involved in the move, then $\psi_D(c) = \psi_{D'}(c')$, where $c' \in D'$ is the crossing corresponding to c.
(Ψ2) If $c_1, c_2 \in D$ can be removed by a RM2, then $\psi_D(c_1) = \psi_D(c_2)$.
(Ψ3) If crossings $c_1, c_2, c_3 \in D$ can participate in a RM3 mapping them to $c_1', c_2', c_3' \in D'$ respectively, then $\psi_D(c_i) = \psi_{D'}(c_i')$ for $i = 1, 2, 3$, and $\psi_D(c_1) + \psi_D(c_2) + \psi_D(c_3) \neq 1$.

Moreover, [38] takes the approach that knot diagrams are objects in a category corresponding to the knot class, and that the Reidemeister moves and detour moves are the morphisms between those objects. As such, projections are functorial maps between the knot categories, and are in bijective correspondence with the weak parities which generate them. In that paper three main weak parities are defined:

1. the *trivial parity* for which any crossing in any diagram is odd;
2. the *null parity*, doing the opposite by calling every crossing even;
3. and finally, the *homotopical parity*.

Given a Gauss diagram G, a diagram obtained by erasing some or all of the arrows of G is called a *subdiagram* of G. The process of creating the subdiagram consisting of exactly the even crossings of G according to some parity f is called projecting G with respect to f. That diagram is denoted $P_f(G)$, and P_f is a parity projection map. *Stable* projections is denoted $P^{\infty}(G)$ and is the result of iterating this map until the resulting diagram has only even crossings.

The effect of removing an arrow from a Gauss diagram is seen at the knot diagram level in Figure 7

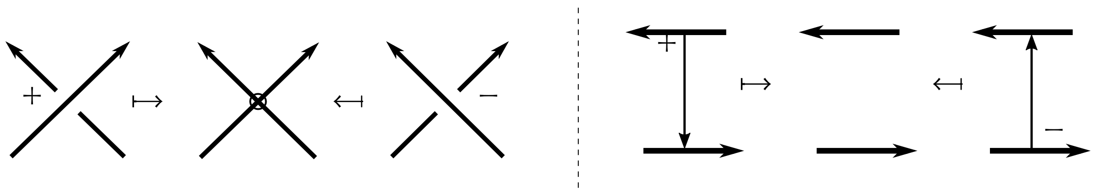

Figure 16: Projecting classical crossings to virtual crossings.

The fundamental property of projections is presented in Lemma 7. Several special cases are proved in papers, but the general technique is shown in Section 8 of [7].

Lemma 7. *Let G and G' be Gauss diagram that are related by a sequence of Reidemeister moves, and f be a parity. Then $P_f(G)$ is Reidemeister equivalent to $P_f(G')$.*

Multiple important results are proved using parities and projections. Amongst them, the most notable are:

1. Any non-trivial parity is null on classical knot diagrams.
2. The minimal number of classical crossings for a virtual knot can be realised on a minimal genus surface.
3. The minimal bridge number of a virtual knot can be realised on a minimal genus surface.

The first statement comes from [38], while the latter two appear and are proved in [34].

2.2.1 Gaussian parities

Gaussian parities are the basic examples of parity. Let $n \in \mathbb{N}$, and $c \in K$ for some knot diagram K. Then,

$$p_n(c) = \begin{cases} 0 & \text{if } I(c) = 0 \bmod n \text{ and,} \\ 1 & \text{otherwise.} \end{cases}$$

The *absolute Gaussian parity* of a crossing is 0 if and only if that crossing has index 0. Its associated stable projection maps virtual knots to almost classical knots and acts trivially on AC knots. It can be used to lift invariants defined on AC knots to all virtual knots [25]. In particular this means that an alternative definition of an almost classical knot is one which admits a diagram where all the crossings have index 0.

Gaussian parities are invariant under crossing changes, but not under either forbidden moves. A particular case is p_2 which is also invariant under both virtualizations.

2.2.2 Homotopical and homological parities

This section assumes some familiarity with basic concepts in algebraic topology, including the fundamental group and homology

The homotopical parity is a weak parity defined as follows. Given an oriented knot diagram D of a virtual knot K on a surface Σ, such that the genus of Σ is the minimal genus of surfaces on which K can be represented, crossing $c \in D$ is called even if and only if there exists $k \in \mathbb{N}$ such that $[D_c^+] = [D]^k \in \pi_1(\Sigma, c)$, where D_c^+ denotes one of the two curves in Σ obtained by smoothing K at c, chosen as in Figure 18.

Theorem 8 (Projection to classical knots, [38])**.** *The stable projection with respect to the homotopical parity of any virtual knot K is a classical knot.*

This parity can be defined for flat knots since $[D_c^+][D_c^-] = [D]$, hence $[D_c^-] = [D]^{1-k}$, and the choice of a component of the smoothed link was inconsequential.

It is important that the surface be of minimal genus for the virtual knot concerned. It is easy to see that by taking the example of a representative of the trefoil given by the braid $\sigma_1^3 \in B_2$, drawn in a square with opposite sides identified to create a torus. Then, every crossing of this knot is odd with respect to the homotopical parity.

However, one can weaken the homotopical parity by calling crossings even if and only if there exists a rational number q such that $[D_c^+] = q[D] \in H_1(\Sigma, \mathbb{Q})$, where $H_1(\Sigma, \mathbb{Q})$ is the first homology group with rational coefficients. In that case, the crossings of the trefoil are even when drawn winding around a torus in the way described previously. Call this function the *homological parity*. The next theorem shows that this satisfies the weak parity axioms.

Theorem 9. *Let D be a knot diagram on a closed compact orientable surface Σ of of minimal genus for the virtual knot represented by D.*

Then, if h_D denotes the homological parity, it respects the weak parity axiom over the Reidemeister moves that keep the knot on Σ.

Proof. Reidemeister moves do not change the homology class of the knot, hence they do not affect the homological parity of crossings that do not participate in them. This verifies (Ψ_1) from Definition 6.

For an isolated crossing a, either $[D_a^+] = [D]$, or D_a^+ is a contractible loop on the surface, since for the genus of Σ to be minimal, it need be obtainable by Carter's algorithm, which would paste a disk in the region delimited by a loop created via RM1, hence $h_D(a) = 0$. This is property that holds for many parities.

For crossings b, c removable by a RM2, $[D_b^+] = [D_c^+]$ if smoothing both crossings with respect to orientation yields the diagram that would be obtained by performing a RM2. Otherwise, smoothing both crossings created a three component link, and $[D_b^+] = [D_c^-]$, but the crossings still satisfy $h_D(b) = h_D(c)$, hence (Ψ_2) holds.

For a RM3 move, involving crossings c_1, c_2, c_3 in D, bringing them to $c_1', c_2', c_3' \in D'$, it suffices that the parity depends only on the flat knot, and thus that the equivalent of the RM3 can be applied to the diagram after a smoothing for any configuration of orientation. To check that it is impossible that exactly two of the crossings c_1, c_2, c_3 involved in some RM3 be even, assume that both c_1 and c_2 are even. Then, since there exists signs ϵ_1, ϵ_2 and ϵ_3 such that $[D_{c_1}^{\epsilon_1}] + [D_{c_2}^{\epsilon_2}] + [D_{c_3}^{\epsilon_3}] = [D]$, it must be that c_3 is also even. This is easier to see looking at the preimages of the halves in the Gauss diagram of the knot. This shows that the third part of the definition of a parity, (Ψ_3) is satifies, and finishes the proof. QED

Question 1. *How does the projection of a knot with respect to the homological parity depend on the Carter genus of the diagram?*

Question 2. *Can the homological parity be defined for virtual links?*

2.2.3 Cohomological parity

In this, for two virtual knot diagrams, $K_1 =_v K_2$ means that there exists a finite sequence of extended Reidemeister moves bringing K_1 to K_2. The same symbol is used throughout to denote a knot diagram in the disk with virtual crossings, and a knot diagram on a surface with the same GD. The notation introduced in Definition 5 is used extensively.

In [34], V. O. Manturov defines a parity, called cohomological for knots in a surface Σ of fixed genus g, with respect to some simple closed curve

$\gamma \in \Sigma$. Let K be a knot diagram such that $\hat{C}_K = \Sigma$, but such that there exists $K' =_v K$ with a Carter surface of lower genus. The parity is constructed by first selecting a simple closed curve $\gamma \subset \Sigma$ which is non-separating, but for which there is a diagram $\tilde{K}$ which is obtained from K by Reidemeister moves on the surface, and which does not intersect γ.

Full details for the claims made in [34] are provided here, upon the request of M. Chrisman.

Proposition 10. *If γ is a non-separating simple closed curve in $\hat{C}_K$, then it must intersect K.*

Proof. First, since γ is not separating, it must be that $g(\hat{C}_K) > 0$. By construction, $\hat{C}_K$ has the minimal genus such that K can be realized. So it admits no destabilization away from K, and in particular, no simple closed curve is disjoint from K. As a curve, one says K fills Σ if and only $\Sigma \setminus K$ is a disjoint union of disks. By the above, it is equivalent to saying that any non-separating simple closed curve on Σ intersects K. QED

Proposition 11. *If $g(\hat{C}_K)$ is not the Kuperberg genus of the virtual knot represented by K, then there exists a sequence of Reidemeister moves in $\hat{C}_K$ to a representative $\tilde{K}$ of lower genus.*

Proof. Consider K as a knot diagram. It suffices to check how each Reidemeister move influences the band surface associated to the knot diagram. As previously, stabilizing and destabilizing RM2s are the only moves which can affect the genus of the Carter surfaces of the respective diagrams. Then, there must be a pair of crossings of K that can be cancelled by a sequence of Reidemeister moves which concludes with a destabilizing RM2, but no stabilizing one.

It suffices to take any sequence of moves which cancels the mentioned pair, and replace any stabilizing RM2 by a similar looking RM2. This is a parity projection, hence it preserves the rest of the sequence, possibly changing some RM3 moves to mixed moves or a RM3-like operation where all crossings are virtual. These moves can be seen in Figure 6. The result is a sequence of diagrams with a fixed Carter genus, except for the very last one that lowers the genus. QED

In particular, when choosing γ to go around the handle created by a sequence of essential virtual Reidemeister moves and RM2, the knot can be changed to avoid it, and the cohomological parity with respect to γ behaves as desired. A crossing c in K is called odd for the cohomological parity if and only if there is a component of the link obtained by smoothing

K at c with respect to the orientation which intersects γ an odd number of times. Let the projection associated to that parity be denoted by P_γ.

Proposition 12. *Let γ, K and $\tilde{K}$ be as above. Then, $P_\gamma(K) =_v \tilde{K}$.*

Proof. The general case is that γ is located close to a single extraneous clasp. Without loss of generality, $\gamma \cap K$ consists of exactly two points. Then, in the Gauss diagram, one can denote the intersection by two open circles, and there are exactly three cases of smoothings. Either one smooths one of the crossings of the clasp, or a crossing with a chord which intersects them, or one which does not intersect the clasp.

A crossing is removed by P_γ if and only if it is forming the clasp. To check this, consider the GD associated to K. The intersection of γ with K can be recorded as some distinguished points on the GD, close to one part of the clasp. Smoothing any crossing which isn't in the clasp either yields some half of the diagram which contains none of the distinguished points, or both of them. The latter happens exactly when the chord intersects the clasp's crossings. Hence, $P_\gamma(K) =_v K =_v \tilde{K}$.

Notice that there could be multiple extraneous clasps intersecting γ, and in that case the projection removes them all at the same time. QED

2.3 Almost classical links

A knot which admits a diagram where every crossing has index zero is called *almost classical*, or AC, in analogy to classical knots having this property. The term initially was used for virtual knots admitting diagrams which are evenly intersticed [29]. However, those have since been renamed checkerboard colourable knots, freeing up the term for a smaller class of knots.

For AC knot diagrams, the Alexander numbering can be extended to regions of the Carter surface, such that adjacent regions separated by an arc of the diagram oriented up are numbered λ on the left and $\lambda + 1$ on the right, for some $\lambda \in \mathbb{N}$, for all arcs in the diagram. This follows from smoothing each crossing of a numberable diagram with respect to orientation and noticing that each component inherits a unique label.

From this, one can define *almost classical oriented links* to be virtual links which admit a diagram D on a closed orientable surface such that $\Sigma \setminus D$ is numberable as above. This definition is derived from the definition of a checkerboard colourable virtual link in [8].

Unfortunately, the notion of index does not extend so well to virtual links. Some crude parities have been defined for virtual links. For example, calling all the mixed crossings odd, then computing the index of the internal crossings of each component by treating them as stand-alone knots. This is a weak parity since any situation where arcs from three different components interact in such a way that one could perform a RM3, all three crossings are odd. If the three arcs come from exactly two different components of the link, then there will be two mixed, hence two odd crossings. The third crossing may be even or odd depending on the rest of the link.

3 Braids

Although they are conceptually simple,
the braid groups [...] will undoubtedly continue
to supply us with surprises and fascination...
- P. Dehornoy, I. Dynnikov, D. Rolfsen, and B. Wiest, [16].

Braid groups were defined in Section 1.2. The goal of this section is to recall known parities for braids, and introduce new ones, while discussing their applications.

3.1 Braid diagrams

3.1.1 Carter surfaces

Most of the construction of Carter surfaces can be translated to work on braids. However, since strands have endpoints, some of the boundary components are not closed. In [14], this was solved by first drawing the braids as connecting the boundaries of a cylinder, and then adding handles as needed to support the virtual crossings. However, this creates some loss of information as there are non-classical braids that can be drawn on thecylinder, and formally nothing stopping the strands from cyclically permuting at the boundary. A solution is to force the braid diagram to live in a square. Then, the boundaries of the band surface are made to follow the boundary of the disk when they encounter it. Gluing the vertical edges of the square gives the construction in [14], while glueing the horizontal edges together and filling the resulting boundaries with disks yields the standard closure of the braid in a surface.

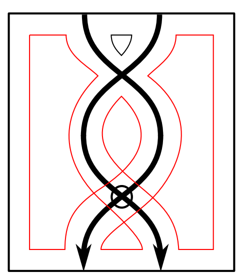

Figure 17: A virtual braid on its band surface.

3.1.2 Braid closure

Let $\mathbb{S}^3$ be the one point compactification of $\mathbb{R}^3$. Then, a knot with a regular projection to the xy-plane in $\mathbb{R}^3$ is said to be in braid form if it winds monotonically around the z-axis. Such a diagram can be "cut" by any half-line from the origin not intersecting a crossing to create a braid diagram whose closure is the knot. The converse construction is to start with a classical braid diagram β in a square, identify its top and bottom edges to create an annulus. Embed that annulus in $\mathbb{R}^3$ such that it is linked with the z-axis, and resolve the crossings. The resulting link is called the standard closure of β and is denoted $\hat{\beta}$. In that case, the z-axis is called the *braid closure axis*, and once $\mathbb{R}^3$ is compactified, that axis becomes an unknot in $\mathbb{S}^3$.

Abstractly, virtual braid diagrams can be closed to obtain virtual link diagrams by connecting the top ith endpoint to the bottom ith endpoint, in a manner which does not create extraneous crossings. A number of operations can be applied to change a braid while preserving its closure.

1. Conjugation: given $\alpha, \beta \in vB_n$, $\widehat{\alpha\beta\alpha^{-1}} = \hat{\beta}$.
2. Stabilization: $\widehat{\beta\sigma_n} = \widehat{\beta\sigma_n^{-1}} = \widehat{\beta\tau_n} = \hat{\beta}$.

This list is not exhaustive.

3.1.3 Bound for the virtual crossing number

In [6], it is conjectured that there exists an upper bound on the virtual crossing number depending on the classical crossing number.

Given a Gauss diagram G with c crossings representing a virtual link L, it is always possible to construct a link diagram which would yield that exact diagram. In fact, that link diagrams can even be the closure of a braid in vB_{2c}. For each chord in G, draw an appropriate positive or negative crossing in the braid, such that the nth crossing is between the $2n-1$st strand and the $2n$th strand. Then, add virtual crossings such that the closure of the braid would encounter the crossing in the order dictated by G. This is quite similar to the approach to creating link diagrams from Gauss words presented in [27].

This yields an upper bound on the number of virtual crossings needed to represent a virtual link. Without loss of generality, c is the minimal number of classical crossing in any diagram of L.

Assuming that virtual crossings are always used in the most economical way possible, the maximal length of a sequence of virtual crossings created using the technique above is $2c^2 - c$, since the first strand from the right needs to cross (with virtual crossings) at most $2c - 1$ strands, then the second one, at most $2c - 2$, and so on.

This gives a positive answer to the conjecture mentioned above.

In [33], Manturov constructed a family of knots whose virtual crossing number grows quadratically with respect to their classical crossing number. By the bound above, it is the fastest possible growth rate.

3.2 Braid parities

Much like for links, it is not clear how to define the index of crossings in a braid. This immediately rules out the possibility of defining Gaussian parities for braids.

3.2.1 Knots in the solid torus

In [19], the notion of *type* for a crossing is defined. Starting with an oriented link $K \cap F \subset \mathbb{S}^3$ where K is a classical knot and F is an unknot, a crossing c of K is of type $w(K_c^+, F)$ mod 2 where K_c^+ is the half of the knot obtained by smoothing K at c with respect to orientation and selecting the component where the understrand is smoothed to the overstrand, and $w(K_1, K_2)$ is the sum of the signs of the crossings between the two knots. The assignment of a type is a parity for the knot in the solid torus $S^3 \setminus F$. One can consider the addition of a linked yet unknotted component to a knot to be a parity, albeit a non-canonical one. As such, the projection it generates fails to have the functorial behaviour that is expected, and it is non-trivial on classical knots, which, in a way would contradict the ideas of [25].

However, the choice of the addition of an unknotted linked component to a knot is quite natural when that knot is in braid form, and the accessory component to the parity is chosen to be the braid closure axis, as defined in Section 3.1.2. Alternatively, that parity has been generalized to virtual knots in certain cases under the name cohomological parity in [34], and as discussed in Section 2.2.3.

Combining these two ideas, one obtains a parity for virtual braids, called toroidal. Let $\beta \in vB_n$ be a braid which closes up to a knot. A classical crossing $c \in \beta$ is even with respect to the *toroidal parity*, denoted $t(c) = 0$

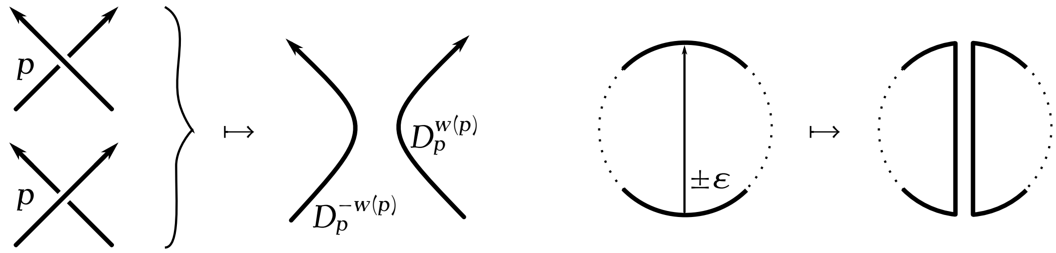

Figure 18: Oriented smoothings.

if and only if both halves of $\hat{\beta}_c$, have an even linking number about the braid axis. Otherwise, $t(c) = 1$. Notice that if n is odd, then this parity is trivial and all the crossings are odd.

Proposition 13. *The toroidal parity is invariant under conjugation, but not stabilization.*

Proof. By definition, the toroidal parity is an invariant of knots in the thickened torus. QED

Question 3. *What is the structure of the set of braids $\beta \in vB_n$ such $t(c) = 0$ for all $c \in \beta$?*

3.3 Braids and almost classicality

From the point of view that geometric braids are a special case of tangle, the disk-band surface of a braid diagram B was constructed in Section 2.1 to have the support of a boundary annulus, causing its closure by disk to be an orientable surface with S^1 as its boundary. In the case of a classical knot, such a surface admits an Alexander numbering. An obvious result states that a braid in B_n can always be numbered using all of $0, 1, ..., n$. This is illustrated by Figure 19

3.3.1 Braid presentations of AC knots

As for knots and links, arcs of a braid diagram consist of subsets of the strands that are delimited by classical crossings. Again, an Alexander numbering of a braid is a choice of integer label for each arc such that the value of the label changes by $+1$ when traversing a crossing towards right (since it is assumed crossings are oriented down), and by -1 otherwise.

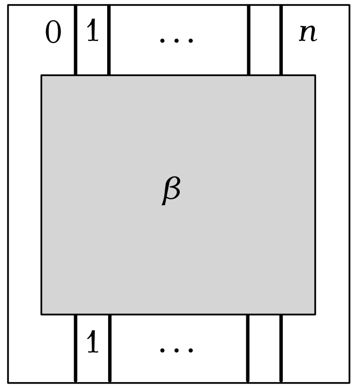

Figure 19: A fully numberable braid diagram if β is classical.

Definition 14. *A braid $\beta \in vB_n$ is called* partially numberable *if its arcs admit a consistent Alexander numbering at each crossing. The initial labels on each strand are possibly independent of each other.*

The stronger requirement that those numbers extend to the region of the surface supporting a geometric diagram of β corresponds to β being *totally numberable* or almost classical.

The *braid index* of a link L is the minimal n such that there exists $\beta \in vB_n$ with $\hat{\beta}$ representing L. The crossing number of the link is the minimal number of classical crossings in a diagram of it.

Proposition 15. *Any almost classical (AC) knot K can be represented by a partially numberable braid. Said braid can be made to obtain either the braid index of K or its crossing number.*

Proof. It suffices to construct the expected braids. In the first case, apply the absolute Gaussian parity projection to a diagram with the minimal braid index in order to obtain a partially numberable diagram. The numbering comes from the fact that the closure is an Alexander numberable knot, due to having only crossings with index 0.

For the second claim, use the construction in Section 3.1.3. QED

3.3.2 Failure of numberable n-braids to form a group

The main problem arises when trying to multiply two partially numberable braids together. If a braid $\alpha \in vB_n$ did not arise from an AC knot or link, it is possible that even if α is partially numberable α^2 is not. For example, let $\alpha = \sigma_1\tau_1 \in vB_2$, as in Figure 17.

This problem comes from the fact that the closure is not AC since the arcs on the top and bottom of the braid have different numbers. One should then choose an appropriate definition of a subgroup of vB_n which contains B_n and depends on the colourability of its elements. A possible choice is the set of totally numberable braids. It follows quickly by their definition and the structure of the supporting surface for braids that the numbering of the ith strand from the left at both the top and the bottom of the braid needs to be i (up to a global translation of the indices). This is illustrated on the left of Figure 19, since the numbering of the region changes by $+1$ when crossing an arc positively, which would be, for a braid diagram from left to right.

Theorem 16. *Any almost classical abstract braid diagram has genus 0.*

Proof. Assume that the ith strand at the top of the braid $\beta \in vB_n$ has Alexander number i for all $i = 1, 2, ..., n$. Then, since every crossing satisfies the numbering condition, it means that the first classical crossing of β is between the kth and $(k+1)$st strands such that the kth strand goes left to right. After that crossing, the ith strand is still numbered i. By a similar argument, the following classical crossing can also be realized without virtual crossings. In fact, if there was to be a virtual crossing which cannot be pushed down nor cancelled with another crossing, the following classical crossings would fail to be numberable, or, if it was located after every classical crossing, then the numbers along the bottom of the braid would not be sequential, and the numbering would fail to extend to the regions of the Carter surface bounded by the braid diagram. QED

This answers a question asked by B. Audoux.

3.3.3 Checkerboard colourable braids

Definition 17. *A virtual braid is called checkerboard colourable if it has a representative which admits a full Alexander numbering modulo 2. Denote this set by cB_n.*

As a set, cB_n is generated by the sets $\{\sigma_i\}_{i=1}^{n-1}$, and $\{\nu_{i,j}\}_{i<j, i-j \equiv_2 0}$ where the σ_i are the same generators as above, and $\nu_{i,j}$ is a virtual braid corresponding the permutation of the ith and jth strands, fixing the rest of the braid.

Similar to B_n having the normal subgroup P_n of pure classical braids, the subset $cB_n \cap vP_n$ is denoted cP_n, and called the checkerboard colourable pure braids.

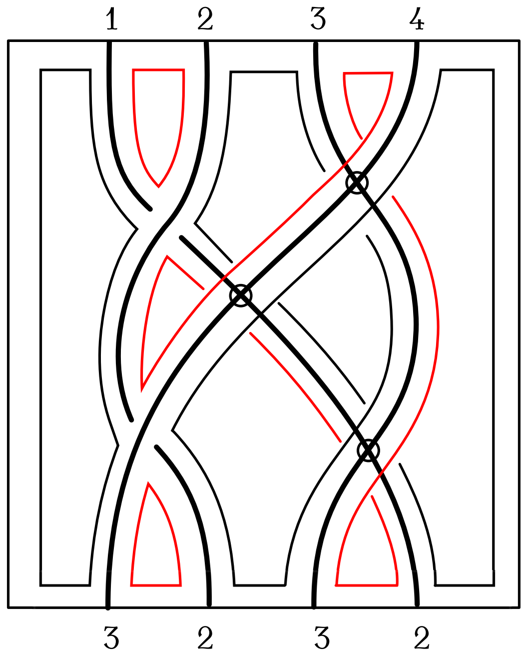

Figure 20: A checkerboard colourable braid which fails to be AC.

Theorem 18. *Let $n > 2$. Then, cB_n is a proper subgroup of vB_n which itself properly contains B_n, and cP_n is a normal subgroup of cB_n.*

Proof. It suffices to find an element of cB_n which isn't in B_n. Let $n > 2$.

Fix $\beta := \nu_{1,3} = \tau_1\tau_2\tau_1$, then $\beta \in cB_n$. However, since it contains no classical crossings, and is not the identity braid, $\beta \notin B_n$.

Similarly, τ_1, as a braid in vB_n fails to have a checkerboard colouring, hence cB_n is a proper subgroup of vB_n.

To show that cP_n is normal subgroup, it suffices to recall that vP_n is normal in vB_n. Then, for every $\alpha \in vP_n$, and $\gamma \in cB_n$, $\gamma\alpha\gamma^{-1} \in cB_n$, since α is, in particular in cB_n, which is closed under multiplication and inverses. But since α is also vP_n, which is normal in vB_n, $\gamma\alpha\gamma^{-1} \in vP_n$. Hence $\gamma\alpha\gamma^{-1} \in cP_n$, making this a normal subgroup of cB_n. QED

A more complex example is drawn, with its checkerboard colouring in Figure 20. Taking that diagram and trying to number it as one would an AC braid, the numbers at the top of the diagram are forced to be as shown. Then, following each strand, number the arcs by adding $+1$ if the strand is going right at the crossing and -1 if it is going left. This yields the final numbering of $3, 2, 3, 2$ at the bottom of the braid, hence this braid is not classical. Unlike for links where a diagram not being numberable is not enough to conclude that no numberable diagram exists, for virtual braids the numbers at the bottom are invariants under all allowed moves.

3.3.4 A checkerboard parity

One could hope that since cB_n is a proper subgroup of vB_n, there may be a parity for virtual braids whose associated projection is onto cB_n, and which acts as the identity on that subgroup. Consider the following candidate for such a parity.

Let β is vB_n. Mark the even numbered strands at the top of the braid. Let $c \in \beta$ be a classical crossing; $\kappa(c) = 1$ if and only if exactly one of the strands crossing at c is marked. Otherwise, $\kappa(c) = 0$.

Theorem 19. *The function κ satisfies the parity axioms.*

Proof. Since this is defined for braids, there are no isolated crossings. The only cases of RM2 moves are $\sigma_i\sigma_i^{-1}$ and $\sigma_i^{-1}\sigma_i$. In both cases, it is immediate that the crossings are between the same two strands and hence κ takes the same value on both of them. There are eight cases of possible markings for RM3 moves, but in all of them, it suffices to check that in fact the value of κ depends only on the strands that are crossed together, and not on the order of the crossings. Moreover, if either all or none of the strands taking part in the move are marked, all the crossings are even. If one one or two strands are marked, only the crossing between the strands that are both marked or unmarked is even, and the others are odd. In particular, this means κ is not a weak parity, since it is impossible that all three crossings in a RM3 be odd. QED

Notice that this parity is non-trivial on classical braids, and is precisely the opposite of the kind of functions that would generate a projection to cB_n. Ideas from linear algebra tell us that if there is some way to make braid groups into vector spaces (see for example the representations featured in [3]) then this projection admits an orthogonal complementary projection which is the identity in cB_n.

Question 4. *Does there exist a parity f on virtual braid diagrams such that $P_f(\beta)$ is checkerboard colourable for all $\beta \in vB_n$, and $P_f(\alpha) = \alpha$ whenever $\alpha \in cB_n$?*

It is possible that the technique used in [35] can be modified to yield the desired parity.

4 String Links

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Our results involve a mixture of topology, algebra, and
                                          combinatorics.
                          - N. Habegger, and X.-S. Lin, [23].
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String links are a special type of tangles which do not have closed components, and which connect ordered points on an interval to the point of the same order on another interval. They are the generalization of long knots to links which were first introduced, at least in the classical context, by Habegger and Lin, [23]. Virtual string links, like virtual links, have both classical and virtual crossings. The set of classical string links on n strands is represented by such tangle diagrams up to classical Reidemeister moves of GDs which remain bounded by the lines supporting the endpoints of the strands. It forms a monoid under concatenation denoted by SL_n. String link diagrams in $I \times I$ are oriented from the top boundary to the bottom one. Concatenation is the unique binary operation on SL_n mapping two tangle diagrams to a single one by contracting them by a factor 1/2 in the vertical direction an stacking them in the unit square, identifying the bottom boundary of the top tangle to the top boundary of the bottom tangle. This operation is not commutative, and not invertible. It does have an identity element, the trivial string link consisting of n vertical strands.

As one might expect, string links are easily represented as Gauss diagrams on n core intervals with a finite number of signed arrows [2]. Like the virtual pure braids, there is no need to indicate a permutation induced by the link. Unlike braids, the chords of the Gauss diagram of a virtual string link need not be horizontal, and may even go from one core interval to itself. Such a chord corresponds to a *self-crossing*. Crossings between different strands of a string link are called *mixed*. The set of those Gauss diagrams up to Reidemeister moves is called vSL_n. Analogously to other virtual objects, virtual string links admit diagrams with classical and virtual crossings.

Define quotients of vSL_n by the relations introduced in Section 1.2. Allowing the OC move on vSL_n gives the welded string link monoid denoted wSL_n. Another possible quotient of vSL_n is fSL_n, the flat virtual string links. It is obtained by allowing the CC move. A classification of flat virtual pure tangles, which includes fSL_n has been done in [13]. Notice that since crossing change is an unknotting operation for classical links, classical string links are flat equivalent to the identity string link.

In [2], many other quotients of the virtual string link monoid are defined.

For example, vSL_n^v, virtual string links where crossings within any given strand are allowed to be made virtual or classical at will. Such a move is almost impossible to code from the point of view of elementary tangles, hence it is preferable to see this from the point of view of GD where mixed crossings are visually different from self crossings.

4.1 Almost classical string links

It suffices to construct a surface which adequately – that is, minimally – supports a virtual string link to be able to define almost classicality for elements of vSL_n. Unlike the braid cases, there exist non-classical virtual string links which admit an Alexander numbering. For example, take any long AC knot in vSL_1. For classical knots, the choice of a basepoint to pull to infinity is immaterial. It it possible to move it by pulling the first strand crossing the knot on the left over (or under) the whole diagram if the knot encounters that strand by going under (respectively over) it. However for virtual knots, the Kishino twists [24] shows that the choice of basepoint can radically change the nature of the long knot obtained. The Kishino twist T is a non-trivial long virtual knot whose closure is the unknot.

To construct a string link from a link, one needs to choose a basepoint for each component and then an ordering for those points.There are such choices on classical links which yield string links that do not admit an Alexander numbering.

The construction of a Carter surface is entirely analogous to the braid case, and the concatenation of elements in the monoid extends to glueing part of their Carter surfaces together, and again, the existence long AC knots allows for the definition of a non-classical AC subset of vSL_n, denoted $acSL_n$. To show those are not equivalent to SL_n for all n, it suffices to take a long, non-classical AC knot as an element of $acSL_1$, and iterate the use of the standard embedding map from vSL_k to vSL_{k+1} which adds a vertical strand to the right of a string link diagram. This is an obvious generalization of the embedding of B_k in B_{k+1}.

4.1.1 Checkerboard colourable string links, cSL_n

Using the construction of a band surface for virtual string links, which includes, as it did for braids, an annulus "frame", one can define the sub-monoid $cSL_n \subset SL_n \subset vSL_n$, consisting of those string links which admit a checkerboard colouring. Because of the requirement that string links be pure tangles, meaning that the i-th string of the diagram $D \subset I \times I$ connect $(i/(n+1), 1)$ to $(i/(n+1), 0)$.

Question 5. *Is there a parity projection whose image is $acSL_n$? cSL_n?*

The results in this section so far, when combined with some observations from [2] can be summarized in a single statement:

Proposition 20. *Let each of the following maps be the inclusion induced by the identity map at the level of Gauss diagrams. Then, $P_n \to SL_n \to cSL_n \to acSL_n \to vSL_n$ are injective but not surjective maps.*

4.2 Other string link parities

Since string links are a generalisation of pure braids, defining parities for them has all the same problems that were discussed in the previous section about the absence of an index. The toroidal parity easily extend to string links, but since they never induce a permutation of the strands, it is the trivial parity, which calls every crossing even. As for the checkerboard parity, it extends by setting self-crossings to be even.

4.2.1 A non-unique definition of index for string links

As vSL_n contains braids, a natural notion of closure of string links to ordered links arises. However, like all tangles, there are other ways to close them. Let $\lambda \in vSL_n$, ignoring the orientation of strands, define the *stitched closure* $\tilde{\lambda}$ to be the knot obtained by first adding an unknotted strand to the right of λ if n is odd, then glueing the top of the $2i - 1$th strand to the top of the $2i$th strand, and the bottom of the the $2i$th strand to the bottom of the $2i + 1$st strand. Finally, connect the bottoms of the first and last strand.

One can then compute the index of crossings in $\tilde{\lambda}$ and define the index of a crossing in λ to be the index of the corresponding crossing in the closure. Of course, any other choice of closure gives an equally valid index. However, this technique does not commute with multiplication in vSL_n. It remains to show that this index generates the expected parities.

Theorem 21. *Let $\lambda \in vSL_n$. Then, any parity f arising from the index of chords in $\tilde{\lambda}$ lifts to a function f^* which assigns 0 or 1 to crossings of λ and respects the parity axioms.*

Proof. Let c_0 be an isolated crossing in λ. Then, $\tilde{c}_0$ is the corresponding crossing in $\tilde{\lambda}$ and it is also isolated since the stitching is not adding crossings, nor cutting the strands. Actually, this exact same argument works for every other Reidemeister move, since they are local. In fact, the only subtle point is that when closing a string link, there are more moves that can be done. QED

Notice that a string link with no crossing with non-zero index need not be AC as according to the definition coming from numbering the surface. For example, the braid in Figure 17 closes to the unknot, but it is not AC as a string link.

Previously, there were only two non-trivial parities for string links, both of which defined from lifting parities from the standard closure of the string link to a link. In [25], they define a link parity by calling any crossing which is between two different components odd (and in general, it is called *mixed*), and any self-crossing even. The induced projection maps vSL_n to vSL_1^n, where vSL_1 is equivalent to the theory of long knots, and the power denoted the disjoint union of such knots. The other parity for virtual links is from [26], and is constructed in many steps.

4.2.2 A dynamical parity

In [2], the authors ask a number of questions. The first of them,

Does SL_n embed in vSL_n or wSL_n?

is partially answered by the following theorem.

Theorem 22. *If L and L' are two classical string links which are equivalent as elements of vSL_n. Then, they are equivalent as elements of SL_n*

Proof. Let $L = L_1 \to L_2 \to \dots \to L_k = L'$ be a sequence of string link diagrams where L_{i+1} is obtained from L_i by a single Reidemeister move. Using cohomological parity with respect to every handle, project back to a classical string link at each step. The cohomological parity does not change the string link type since handles can be avoided by the string link, due to the assumption that it is classical. QED

The implied oddness of the crossings which would change the Carter genus of the diagram is a kind of parity, which is deemed *dynamical* since it depends on the sequence of diagrams.

4.3 An application of virtual linking numbers

So far, the "stringy-ness" of the elements of the various monoids has proved to be a contrasting characteristic from the closure of these elements. Let $uwSL_n$ be the set of virtual string links on n strands where, atop the standard Reidemeister moves, both forbidden moves are allowed.

4.3.1 Cobordism

Combinatorially, virtual link cobordism is an equivalence relation generated by births (creation of unknotted components disjoint from the link), deaths (erasing an unknot disjoint from the rest of link), and saddle moves (see figure 21) such that a sequence of these moves and Reidemeister moves maps components to themselves set-wise, and the initial and final number of components are equal.

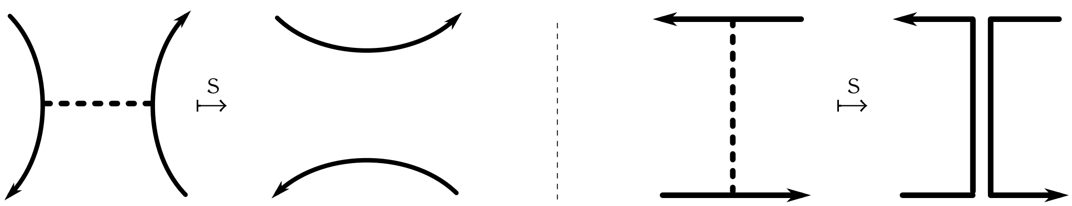

Figure 21: Saddle move (S).

This apparently complex theory has a simple classification theorem. Moreover the idea of cobordism is very topological, and the name itself refers to the equivalence relation being realized as the disjoint union of two links forming the boundary of an orientable surface. For virtual string links, the appropriate topological interpretation is a bit more abstract.

Theorem 23 ([12]). *Equivalences classes of links up to virtual link cobordisms are completely classified by pairwise virtual linking numbers.*

As defined in Section 2.1.1, to any pair of components, there is a pair of virtual linking numbers. The definition of virtual linking number extends naturally to the strands of virtual string links.

4.3.2 Structure of the the unwelded link monoid

Compare the classification of virtual links up to cobordism to this theorem about unwelded links.

Theorem 24 ([5]). *Any unwelded link is isotopic to the closure of an element of* uwP_n.

Both of these results can be strengthened to apply to virtual or unwelded string links.

Theorem 25. *Let* L_1 *and* L_2 *be virtual string link diagrams. Then, the following are equivalent*

1. L_1 *is cobordant to* L_2*,*
2. L_1 *is unwelded equivalent to* L_2*,*
3. *The pairwise virtual linking numbers the components of* L_1 *equal those of* L_2*.*

Proof. All three statements are proved by putting the Gauss diagrams of the string links in a standard form. Using either cobordisms, in Figure 22 or forbidden moves, in Figure 23, any chord can commute with any other chord.

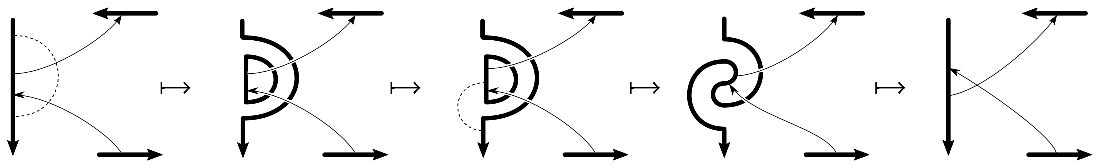

Figure 22: Commuting crossings with a cobordims

For cobordisms, the signs of the arrows in Figure 22 is suppressed since it is irrelevant.

In the unwelded link case, two arrow feet and two arrow heads already commute. Only one mixed commutation is illustrated, but all other cases follow similarly by using an appropriate sequence of moves.

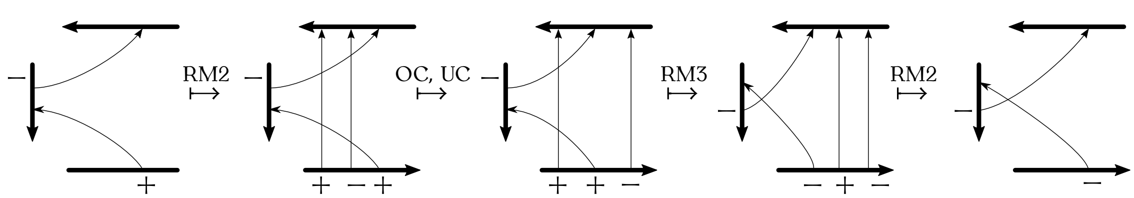

Figure 23: Commuting crossings using both forbidden moves.

The first step towards the standard form is to delete any self crossing by isolating it and performing a RM1. The second step is to collect the arrows where the first component goes over the second at the top of the string link, then those where the first component goes over the third, and so on. The ith such layer consists of arrows with feet on the ith component.

At that point, arrows can be further commuted such that each group of crossing i over j consists of arrows all having the same sign. That sign, times the number of such crossings is $\text{vlk}(C_i, C_j)$, where C_i is the ith component of the string link.

In particular, that standard form is a pure braid. QED

This theorem implies both of the classification results for link cobordism and unwelded links that are cited above, simply by choosing a basepoint for each component of those links, and making the link into a string link in such a way that the basepoints correspond to the points which would be identified together to form the standard closure. Moreover, the permutability of the arrows on the Gauss diagram makes the choice of basepoints immaterial.

Applying a theorem from [39], Corollary 26 follows.

Corollary 26. *Any unwelded string link admits a monotone representative, hence $uvSL_n$ is isomorphic to $\mathbb{Z}^{2n(n-1)}$.*

Conclusion

> New parities can be used practically in all problems, where parities were previously applied.
> - V. O. Manturov, [35]

By now, it should be clear that the applications of parities are as varied as their constructions can be. From a local, combinatorial assignment of a binary value, the global structure of classes of knotted objects was revealed. For convenience, the open problems featured in the main body of the text are repeated here.

Question 1. *How does the projection of a virtual knot with respect to the homological parity depend on the Carter genus of the diagram?*

Question 2. *Can the homological parity be defined for virtual links?*

Question 3. *What is the structure of the set of braids $\beta \in vB_n$ such $t(c) = 0$ for all $c \in \beta$? Here, t is the toroidal parity defined in Section 3.2.1.*

Question 4. *Does there exist a parity f on virtual braid diagrams such that $P_f(\beta)$ or $P_f^{\infty}(\beta)$ is checkerboard colourable for all $\beta \in vB_n$, and $P_f(\alpha) = \alpha$ whenever $\alpha \in cB_n$?*

Question 5. *Is there a parity g defined for virtual string links such that its projection P_g has stable image equal to $acSL_n$? cSL_n?*

Aside from those precise lines of research, parities that were defined in this thesis can be used in other way. Some research has already been done towards creating *parity sensitive invariants*. Invariants of knotted objects are functions which are, as expected, invariant under the Reidemeister moves, and sometimes, under some of the other moves that have been mentioned. As such, is there are two ways to use parity to modify an invariant. The first way is to consider the sequence of values taken by the diagrams obtained by applying a parity projection repetitively. The second depends greatly on the way the invariant is computed, but in some cases, it is possible to treat odd and even crossings in two different ways. In [30], Gaussian parities are used to refine some virtual knot invariants.

As by the approach of Habberger and Lin, classical string links are the building blocks of classical knots. The same relation exists for virtual objects. Hence, defining and refining invariants for virtual string links is a fundamental, yet barely studied approach to virtual knot theory.

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