

Khovanov homology, slice invariants, and exotic \mathbb{R}^4

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1 Introduction

In [6] Khovanov defines an invariant of oriented knots and links called the *Khovanov homology* which takes on the form of a graded homology theory. Khovanov homology is constructed by *categorifying* the Jones polynomial; this relationship between Khovanov homology and the Jones polynomial is captured by the fact that the graded Euler characteristic of the Khovanov homology of an oriented link L is equal to the unnormalized Jones polynomial of L . To define the Khovanov homology, we first construct a bigraded chain complex associated to an oriented link diagram D , which we call the *Khovanov complex*. We then take the homology of this complex to get Khovanov's invariant.

This thesis is divided into three main parts. First we outline the construction of Khovanov homology in detail, following [2], [6] and [14]. We then define a powerful slice invariant of knots developed by Rasmussen in [12] called the *s-invariant*, which relies on a homology theory introduced by Lee in [9] called *Lee homology*, a variant of Khovanov homology. We then conclude with a proof of the existence of exotic \mathbb{R}^4 by proving a theorem from [5] which says that the existence of such a manifold relies on the existence of a knot which is topologically slice but not smoothly slice. We then apply the *s-invariant* together with a theorem of Freedman in [3] to find examples of these knots.

2 Khovanov homology

2.1 Categorification and the Jones polynomial

To begin, we recall some basic notions from knot theory. We then briefly discuss the concept of categorification within the context of the Jones polynomial.

Definition 2.1. A *knot* is an isotopy class of embeddings of the circle S^1 into three-dimensional Euclidean space \mathbb{R}^3 . The simplest knot is the *unknot*, which is just a copy of S^1 in \mathbb{R}^3 . A *link* is a finite union of disjoint knots in \mathbb{R}^3 . Hence we can view a knot as a link with one component. The *unlink* is a finite union of disjoint copies of the unknot.

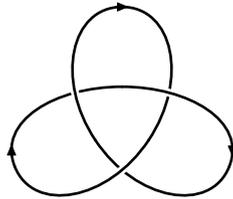


Figure 2.1. An oriented knot diagram depicting the trefoil, the simplest non-trivial knot.

A *knot or link diagram* is a planar diagram obtained by projecting the knot or link onto the plane, and then specifying *overcrossings* and *undercrossings*, where a *crossing* refers to a double point in the knot or link diagram. Each point in the knot or link diagram can have at most two points in the preimage under the projection map, so that we cannot have triple points, quadruple points, and so on. A link is said to be *oriented* if a choice of orientation is made for each component of the link (clockwise or counter-clockwise); this choice is represented by placing an arrow marking on the diagram of the link. Given an oriented knot, we call a crossing *positive* or *negative* based on the convention given in Figure 2.2.



Figure 2.2. A positive crossing and a negative crossing, respectively.

It is sometimes useful to view knots as being embedded into $S^3 = \{(a, b, c, d) \in \mathbb{R}^4 : a^2 + b^2 + c^2 + d^2 = 1\}$, the 3-sphere belonging to four-dimensional Euclidean space. Indeed, \mathbb{R}^3 is homeomorphic to S^3 minus a single point and so we can consider a knot to be embedded into either space. In most cases here we will view knots as embeddings into S^3 .

Definition 2.2. We consider two knots or two links to be *equivalent* if their knot or link diagrams are related by a sequence of *Reidemeister moves*, which are illustrated in Figure 2.3.

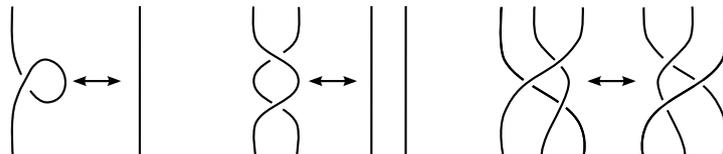


Figure 2.3. The three Reidemeister moves.

Generally, knot equivalence is defined in terms of *ambient isotopies* of \mathbb{R}^3 , i.e. continuous maps which “distort” the ambient space surrounding a knot without creating any self-intersections in the knot itself; one says that two knots are equivalent if one can be transformed into the other via an ambient isotopy. There is a classical theorem of knot theory which states that two knot diagrams which belong to the same knot can always be related by a sequence of Reidemeister moves. Hence two knots are equivalent if and only if their diagrams are related by a sequence of such moves.

Definition 2.2 allows us to speak of knot *classes*, so that two knots belong to the same knot class if and only if they are equivalent. From this point on, we shall not distinguish between a knot and its knot class. So, for instance, when we speak of the *trefoil* (Figure 2.1) we refer not only to the knot whose knot diagram is illustrated in the figure but also to all other knots which are equivalent to it. Indeed, we will see later that all properties associated to one knot in a particular knot class also hold for all other knots in that knot class. So, while there are infinitely many knots with three crossings, they all belong to the same knot class (the class represented by the trefoil, which is the simplest non-trivial knot) and so, from this point of view, there is really only one knot with three crossings.

One of the main problems in knot theory is that of determining exactly when two knots are equivalent. Given a knot with many crossings, it may be difficult to determine when such a knot is equivalent to the unknot, let alone to another non-trivial knot. One concept which can help us with this problem of knot equivalence is that of a knot invariant.

Definition 2.3. A *knot invariant* is a quantity defined in terms of a knot diagram which is invariant under the Reidemeister moves. That is, all knots within the same knot class have the same value associated to those knots by the knot invariant.

The quantity assigned to a knot by a knot invariant need not be a number: For instance, many knot invariants take the form of polynomials (known as knot polynomials) while other more complicated knot invariants can take the form of a homology theory. We mention these two examples because these are the knot invariants we will be concerned with throughout this section. In particular, we will be interested in *categorifying* a knot polynomial in order to obtain a rich but complicated knot homology theory which contains all the information carried by the knot polynomial, and much more.

In a very broad sense, *categorification* refers to the process of replacing mathematical objects defined in terms of set theory with objects defined in terms of category theory. That is, it refers to the process of finding and assigning category-theoretic analogues of set-theoretic concepts: Explicitly, we replace elements with objects, equations between elements with isomorphisms between objects, sets with categories, functions with functors, and equations between functions with natural isomorphisms between functors (see the discussion in [1] for more details). For example, let \mathcal{C} be the category whose objects are finite sets and whose morphisms are functions between those finite sets. One can check that \mathcal{C} categorifies the set of natural numbers \mathbb{N} . Indeed, each natural number gets replaced with a finite set, and the notion of equality among two numbers gets replaced with the notion of isomorphism between two sets. By categorifying, we see that each natural number n is replaced by a set with cardinality n , and each equality between natural numbers p, q is replaced by an isomorphism between sets with cardinalities p, q , respectively. Furthermore, sums and products of natural numbers in \mathbb{N} correspond respectively to the disjoint union and the Cartesian product of finite sets in \mathcal{C} . This example describes the general idea; actually applying this process of categorification is usually much more difficult than illustrated in this example.

The object which we are interested in categorifying is a knot polynomial known as the *Jones polynomial*, which is defined in terms of Kauffman's bracket polynomial. The Jones polynomial is an invariant of oriented knots and links, and it can be calculated combinatorially given the diagram of a knot or a link. We define both of those concepts as follows. (We shall describe everything in terms of links; all of the following definitions and notions clearly hold for knots as well by only considering 1-component links.)

Definition 2.4. Let L be an oriented link and D be the corresponding link diagram. Suppose D has n crossings; denote by n_+ and n_- the number of positive crossings of D and the number of negative crossings of D , respectively. The *Kauffman bracket* of D , denoted $\langle D \rangle$, is a Laurent polynomial in a variable q (so that $\langle D \rangle \in \mathbb{Z}[q^{\pm}]$) which is defined recursively as:

$$\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \rangle = \langle \begin{array}{c} \frown \\ \smile \end{array} \rangle - q \langle \begin{array}{c} \frown \\ \diagdown \\ \diagup \\ \smile \end{array} \rangle,$$

$$\langle k \text{ copies of } \bigcirc \rangle = (q + q^{-1})^k.$$

The Kauffman bracket is not a link invariant, but we can use it to form one by defining

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$$

where D, n_+ and n_- are as above. $\hat{J}(D)$ is called the *unnormalized Jones polynomial* of D . We can then define the *Jones polynomial* of the link diagram D to be

$$J(D) = \frac{\hat{J}(D)}{q + q^{-1}}.$$

Note that the usual description of the Jones polynomial involves a variable t instead of q ; indeed, our normalization of the Kauffman bracket is slightly different from the usual one, and hence we get a different formula for the Jones polynomial. But this is not a problem: Simply make the substitution $q = -t^{\frac{1}{2}}$ in order to get the usual formula for the Jones polynomial.

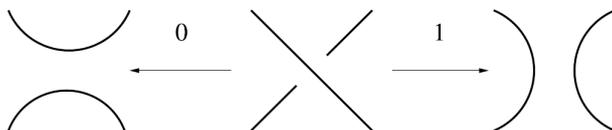


Figure 2.4. The 0-smoothing and the 1-smoothing of a crossing.

Computing the Jones polynomial traditionally can be somewhat tedious, especially for knots and links with many crossings; we would like to describe a more systematic method involving the construction of the *cube of resolutions* of a link diagram D . Given a link diagram D with n crossings, we can *smooth* each crossing in one of two ways, illustrated in Figure 2.4. We shall call the first type of smoothing the *0-smoothing*, while the second type will be called the *1-smoothing*. We call a link diagram in which every crossing has been resolved in this way a *smoothing* or a *resolution* of D . Since D has n crossings and each crossing can be resolved in one of two ways (either as a 0-smoothing or as a 1-smoothing), there are 2^n possible resolutions of D . Each smoothing α corresponds to a word consisting of n zeroes and ones, in the sense that each smoothing is an element of the set $\{0, 1\}^n$. By numbering the crossings of D as $1, \dots, n$, we can associate to the i^{th} crossing of D the i^{th} index in the smoothing $\alpha \in \{0, 1\}^n$. We can then construct a hypercube with 2^n vertices.

Now, one can observe that a resolution of a link diagram D is simply a collection of disjoint circles in the plane, since smoothing each crossing will eliminate all of the crossings in D , leaving us with at least one copy of S^1 in the plane. Given a smoothing $\alpha \in \{0, 1\}^n$ (which, note, is just a sequence of zeroes and ones indicating how each crossing is to be resolved) we can consider the associated diagram, i.e. the collection of disjoint circles in the plane obtained by applying the smoothing to D , which we denote by Γ_α . Given such a sequence α , we define r_α to be the number of 1's in α and k_α to be the number of circles in Γ_α . Applying the definition of the Kauffman bracket to these quantities, we get the following formula for the unnormalized Jones polynomial of a link diagram D :

$$\hat{J}(D) = \sum_{\alpha \in \{0, 1\}^n} (-1)^{r_\alpha + n_-} q^{r_\alpha + n_+ - 2n_-} (q + q^{-1})^{k_\alpha}.$$

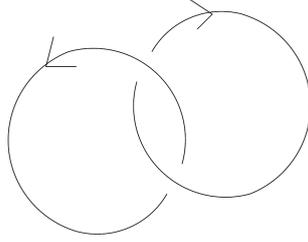


Figure 2.5. The negative Hopf link. Both crossings in the link diagram are negative (hence the name) as a result of the choice of orientation.

We solidify all of these notions through the following example. Let H be the negative Hopf link, illustrated in Figure 2.5. In this case, $n = n_- = 2$ and $n_+ = 0$. There are $2^2 = 4$ different smoothings of H , given by the set of sequences $\{00, 01, 10, 11\}$, where we number the top crossing in the diagram as crossing 1 and the other crossing as crossing 2. By applying each smoothing, we get the cube of resolutions illustrated in Figure 2.6. So we have

$$r_{00} = 0, r_{01} = r_{10} = 1, r_{11} = 2,$$

$$k_{00} = k_{11} = 2, k_{01} = k_{10} = 1.$$

Using the formula above for the unnormalized Jones polynomial of H , we can compute

$$\begin{aligned} \hat{J}(H) &= q^{-4}(q + q^{-1})^2 - 2q^{-3}(q + q^{-1}) + q^{-2}(q + q^{-1})^2 \\ &= q^{-6} + q^{-4} + q^{-2} + 1. \end{aligned}$$

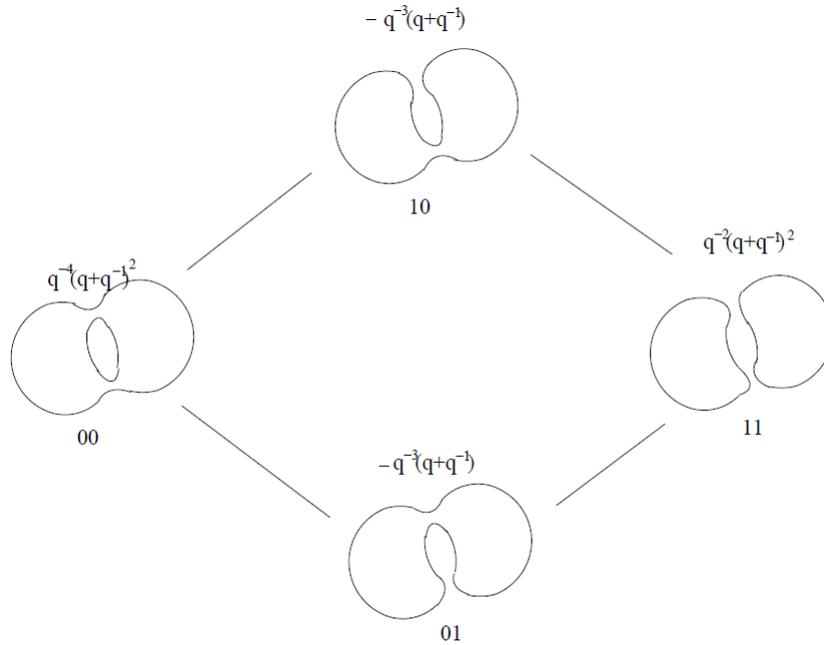


Figure 2.6. The cube of resolutions for the negative Hopf link. The factor of $(q + q^{-1})$ corresponding to each smoothing is given; we obtain the unnormalized Jones polynomial by summing over all four smoothings.

The main point here is that we can categorify the Jones polynomial in order to get a knot homology theory known as the *Khovanov homology* of knots and links. Khovanov homology categorifies the Jones polynomial in a way which is analogous to the categorification of the Euler characteristic associated to a topological space, which gives us the homology of that space. In this case, we see that a number (the Euler characteristic) gets replaced by a graded vector space (the homology) such that the graded dimension of that vector space is the number which we originally started with. Indeed, we will see later that the graded Euler characteristic of the Khovanov homology of a link diagram D is in fact the unnormalized Jones polynomial of D – in this sense, Khovanov homology *categorifies* the Jones polynomial by replacing that polynomial with a complex of graded vector spaces.

2.2 The Khovanov complex

Using the cube of resolutions introduced in the previous section, we can define the object of interest; this object will be called the *Khovanov complex* associated to a link diagram D , and we will then take its homology to obtain the invariant we are after. Before we can do this, however, we need to briefly recall some facts about finite dimensional graded vector spaces. Note that all of our vector spaces will be over the field \mathbb{Q} and so when we say “vector space” we mean “vector space over \mathbb{Q} .”

Definition 2.5. A *graded vector space* V is a vector space which can be decomposed as a direct sum of the form

$$V = \bigoplus_{i \in I} V^i$$

where each V^i is a vector space and where I is any index set. For a given $i \in I$, the elements belonging to the vector space V^i are called the *homogeneous elements* of degree i . For any $v \in V^i$, we write $\deg(v) = i$. For homogeneous elements v, w , we define $\deg(v \otimes w) = \deg(v) + \deg(w)$. The *graded dimension*, denoted $\text{qdim}(V)$, of a graded vector space V is the polynomial given by $\text{qdim}(V) = \sum_{i \in I} q^i \dim(V^i)$. We note that the graded dimension satisfies

$$\text{qdim}(V \otimes V') = \text{qdim}(V) \text{qdim}(V')$$

and

$$\text{qdim}(V \oplus V') = \text{qdim}(V) + \text{qdim}(V').$$

Furthermore, given a graded vector space V and any integer j , we can define a *shifted* graded vector space $V\{j\}$ by shifting each component of V by j so that $V\{j\}^i = V^{i-j}$.

Now, let V be the vector space $\mathbb{Q}\{1, x\}$, the \mathbb{Q} -vector space with basis elements 1 and x . Furthermore, grade the two basis elements by setting $\deg(1) = 1$ and $\deg(x) = -1$. Now consider the cube of resolutions associated to a link diagram D with n crossings. Recall that each vertex of the cube corresponds to a smoothing of D , which we call $\alpha \in \{0, 1\}^n$. To each of these smoothings, we associated a graded vector space V_α defined as

$$V_\alpha = V^{\otimes k_\alpha} \{r_\alpha + n_+ - 2n_-\}$$

where

$$r_\alpha = \text{the number of 1's in } \alpha$$

and

$$k_\alpha = \text{the number of circles in } \Gamma_\alpha$$

where Γ_α is the collection of circles in the plane obtained by completely smoothing D according to α . Now we define a vector space $C^{i,*}(D)$ as

$$C^{i,*}(D) = \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ i = r_\alpha - n_-}} V_\alpha.$$

Note that $C^{i,*}(D)$ is *bigraded*: It has an internal grading which it inherits from the grading on each vector space V_α and it has a separate grading i determined by the equation $i = r_\alpha - n_-$. Note that each vector space $C^{i,*}(D)$ is trivial outside of the range $i = -n_-, \dots, n_+$. Indeed, if $i < -n_-$ then $r_\alpha < 0$ which is not possible, since r_α is the number of 1's in the string α which obviously must be a non-negative integer. Similarly, if $i > n_+$ then $r_\alpha > n$, the number of crossings in the diagram D , which cannot hold because there are only n elements in the string α and so there cannot be more than n entries which take on a value of 1.

This is the first step towards categorifying the Jones polynomial – we have constructed one half of the chain complex. It remains to construct the differentials, at which point we can finally take homology. First we note that an element of $C^{i,j}(D)$ is said to have *homological grading* i and *q -grading* j . If v is an element of $V_\alpha \subset C^{i,j}$, then

$$i = r_\alpha - n_-$$

and

$$\begin{aligned} j &= \deg(v) + i + n_+ - n_- \\ &= \deg(v) + r_\alpha + n_+ - 2n_- \end{aligned}$$

where $\deg(v)$ is the degree of v as an element of V_α . Indeed, since 1 and x form a basis for V , we can extend the grading to elements of $V^{\otimes n}$ by setting

$$\deg(v_1 \otimes \dots \otimes v_n) = \deg(v_1) + \dots + \deg(v_n).$$

Now recall that each vertex α of the cube of resolutions corresponds to a complete resolution Γ_α of a given link diagram D . To each edge of the cube we assign a *cobordism*, i.e. an orientable surface whose boundary is a disjoint union of the circles in the smoothings at both ends of the edge (see Definition 3.16 for a more precise description). The placement of the resolutions in the cube indicates how these cobordisms should be constructed. Two resolutions Γ_α and $\Gamma_{\alpha'}$ are joined by an edge if and only if the smoothings α and α' differ in exactly one place. Denote the edge joining two such resolutions by a string of zeroes and ones with a star \star placed in the position which changes. We then turn this edge into an arrow by letting the tail correspond to $\star = 0$ and letting the head correspond to $\star = 1$.

For an arrow $\Gamma_\alpha \xrightarrow{\zeta} \Gamma_{\alpha'}$, note that the resolutions $\Gamma_\alpha, \Gamma_{\alpha'}$ are identical except within a small disk centered at the crossing which changes from a 0-smoothing to a 1-smoothing (see Figure 2.7 for an example). So each cobordism W_ζ corresponding to an arrow ζ can be taken to be the identity cobordism outside such a disk (called the *changing disk*), while inside the disk we insert a saddle cobordism. Thus each cobordism W_ζ consists of a collection of cylinders together with one “pair-of-pants” surface, where we have either one circle splitting into two or two circles fusing into one.

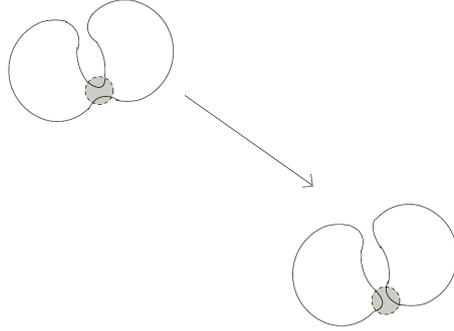


Figure 2.7. Two resolutions of the Hopf link with the changing disk identified.

Now we can replace each cobordism W_ζ between resolutions Γ_α and $\Gamma_{\alpha'}$ with a linear map d_ζ between vector spaces V_α and $V_{\alpha'}$. Since there are only two types of cobordisms (a “fusing” cobordism and a “splitting” cobordism) we only need to consider two kinds of linear maps to define d_α . We need a map $m : V \otimes V \rightarrow V$ (the fusing map) and a map $\Delta : V \rightarrow V \otimes V$ (the splitting map). We then define d_ζ to be the identity outside the changing disk and either m or Δ inside the changing disk.

We define $m : V \otimes V \rightarrow V$ by

$$\begin{aligned} m(1 \otimes 1) &= 1 \\ m(1 \otimes x) &= m(x \otimes 1) = x \\ m(x \otimes x) &= 0 \end{aligned}$$

and we define $\Delta : V \rightarrow V \otimes V$ by

$$\begin{aligned} \Delta(1) &= 1 \otimes x + x \otimes 1 \\ \Delta(x) &= x \otimes x. \end{aligned}$$

This choice of m and Δ corresponds to a *topological quantum field theory*, or TQFT for short. Hence the definition of the maps m and Δ given above is often referred to as Khovanov’s TQFT (we will see later that we can change how we define the maps m and Δ in order to obtain a different TQFT).

We can finally define our differentials $d^i : C^{i,*}(D) \rightarrow C^{i+1,*}(D)$. Recall that an arrow $\Gamma_\alpha \xrightarrow{\zeta} \Gamma_{\alpha'}$ is associated with a string of zeroes and ones together with one \star , where $\star = 0$ corresponds to the smoothing α and $\star = 1$ corresponds to the smoothing α' . With this in mind we define the *sign* of an arrow ζ to be $\text{sign}(\zeta) = (-1)^m$, where m is the number of 1’s to the left of \star in ζ . For an element $v \in V_\alpha \subset C^{i,*}(D)$ we define

$$d^i(v) = \sum_{\{\zeta: \text{Tail}(\zeta)=\alpha\}} \text{sign}(\zeta) d_\zeta(v).$$

Applying these ideas to the negative Hopf link, we get the diagram in Figure 2.8.

$$\begin{array}{ccccc}
& & V\{-3\} & & \\
& \nearrow m & \vdots & \searrow -\Delta & \\
(V \otimes V)\{-4\} & & \oplus & & (V \otimes V)\{-2\} \\
& \searrow m & \vdots & \nearrow \Delta & \\
& & V\{-3\} & & \\
0 & \longrightarrow & C^{-2,*}(D) & \xrightarrow{d} & C^{-1,*}(D) & \xrightarrow{d} & C^{0,*}(D) & \longrightarrow & 0
\end{array}$$

Figure 2.8. The cochain complex associated to the negative Hopf link.

The signs are included so that $d^{i+1} \circ d^i = 0$ – we need this property in order to form our cochain complex. It turns out that the maps d all have bigrading $(1, 0)$, so that the homological grading is increased by 1 after each application of d , while the q -grading stays the same. It also turns out that the graded Euler characteristic of this cochain complex is the unnormalized Jones polynomial of D (we refer the reader to [6, Proposition 9] for details), i.e.

$$\hat{J}(D) = \sum_i (-1)^i \text{qdim}(C^{i,*}(D)).$$

We can now take the cohomology of this cochain complex to get Khovanov’s invariant, our main object of interest.

Definition 2.6. The *Khovanov homology of the oriented link diagram D* is given by

$$KH^{*,*}(D) = H(C^{*,*}(D), d)$$

where $H(C^{*,*}(D), d)$ denotes the cohomology of the Khovanov complex $(C^{*,*}(D), d)$ of D .

Theorem 2.7. [6, Theorem 1] *If D and D' are two oriented link diagrams which are related by a sequence of Reidemeister moves, then there is an isomorphism $KH^{*,*}(D) \cong KH^{*,*}(D')$. In particular, Khovanov homology is an invariant of knots and links.*

(The proof of Theorem 2.7 is quite difficult and involves checking homotopy equivalence of chain maps in three separate cases corresponding to each of the Reidemeister moves; we omit this proof for the sake of space.) This theorem says that we can speak of *the* Khovanov homology of a link L , rather than referring to its link diagram D .

Khovanov homology has many interesting properties. For instance, it detects the unknot (see [8] for details). It is still unknown if the Jones polynomial detects the unknot – this is one of the major open problems of knot theory. One easy property of Khovanov homology which we will need later is that it is invariant under global changes of orientation. We state this result without proof:

Lemma 2.8. [6, Proposition 28] *Suppose L and L' are two oriented links related by a global change of orientation; that is, suppose L' is obtained from L by reversing the orientation of each component of L . Then $KH^{*,*}(L) \cong KH^{*,*}(L')$.*

2.3 The Khovanov homology of the trefoil

As an example, we will calculate the Khovanov homology of the left-handed trefoil knot, illustrated in Figure 2.1. Following the construction outlined in the previous section, we get the following cube of resolutions of K , the left-handed trefoil with crossings numbered as in Figure 2.9.

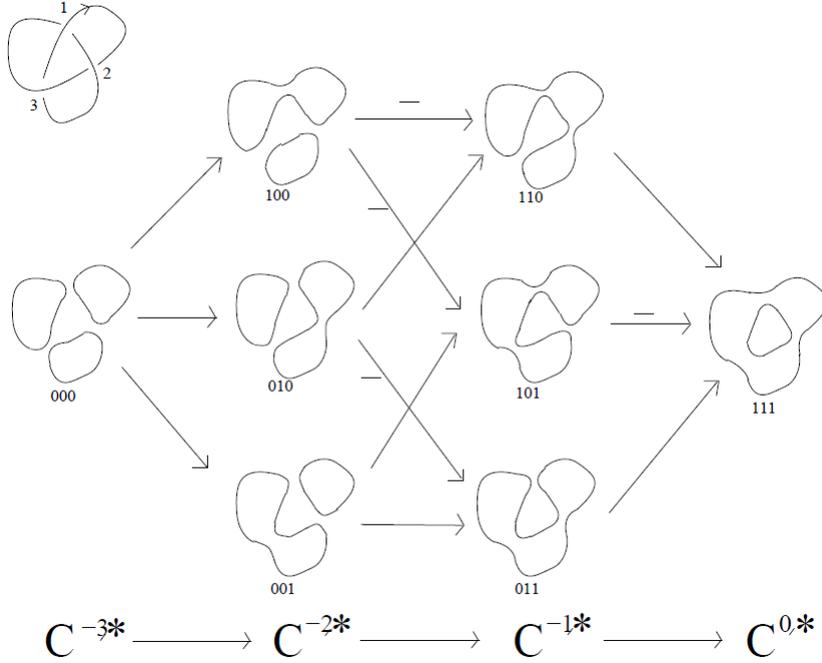


Figure 2.9. The cube of resolutions of the left-handed trefoil knot, with signs added to the maps when appropriate. The non-trivial part of the cochain complex is also given.

To calculate the homology of the cochain complex, we need to calculate $\text{Ker}(d^i)$ (the cycles) and $\text{Im}(d^{i-1})$ (the boundaries) for each homological degree i . Note that the homological degree is non-trivial only in degrees $i = -3, -2, -1, 0$. So we only need to consider homology groups with these homological gradings. We consider each case separately, keeping track of the q -degrees of the elements belonging to each homology group. We also note the dimension of each graded vector space $C^{i,*}(D)$ at each step in order to perform a series of dimension-counting arguments throughout the process.

- $i = -3$: First note that the dimension of $C^{-3,*}(D)$ is 8, since $C^{-3,*}(D)$ is isomorphic to $V \otimes V \otimes V$ and each factor in the tensor product has dimension 2. In this case, one can check that the only element of $V \otimes V \otimes V$ which belongs to the kernel of d^{-3} is $x \otimes x \otimes x$. Thus $\text{Ker}(d^{-3})$ is generated by $x \otimes x \otimes x$. Furthermore, since d^{-4} maps the trivial vector space into $C^{-3,*}(D)$, the image of this map is empty. Thus the homology group in degree -3 is equal to the kernel of d^{-3} , i.e. $H^{-3} \cong \text{span}\{x \otimes x \otimes x\}$, where by H^i we denote the homology group in homological degree i . Applying the formula for the q -degree given earlier, we see that $q(x \otimes x \otimes x) = -9$.

- $i = -2$: Note that the dimension of $C^{-2,*}(D)$ is 12, since there are three summands of the form $V \otimes V$ in the direct sum constituting $C^{-2,*}(D)$. Since the dimension of $C^{-3,*}(D)$ is 8 and the dimension of $\text{Ker}(d^{-3})$ is 1, the dimension of $\text{Im}(d^{-3})$ must be $8 - 1 = 7$. Indeed, one can check that $\text{Im}(d^{-3})$ is spanned by the elements

$$\left\{ \begin{pmatrix} 1 \otimes 1 \\ 1 \otimes 1 \\ 1 \otimes 1 \end{pmatrix}, \begin{pmatrix} 1 \otimes x \\ 1 \otimes x \\ 1 \otimes x \end{pmatrix}, \begin{pmatrix} x \otimes 1 \\ 1 \otimes x \\ x \otimes 1 \end{pmatrix}, \begin{pmatrix} x \otimes 1 \\ x \otimes 1 \\ 1 \otimes x \end{pmatrix}, \begin{pmatrix} x \otimes x \\ 0 \\ x \otimes x \end{pmatrix}, \begin{pmatrix} x \otimes x \\ x \otimes x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \otimes x \\ x \otimes x \end{pmatrix} \right\}$$

of $C^{-2,*}(D)$. Now we need to calculate $\text{Ker}(d^{-2})$; to this end, one can verify that the following elements are mapped by d^{-2} to 0:

$$\left\{ \begin{pmatrix} 1 \otimes 1 \\ 1 \otimes 1 \\ 1 \otimes 1 \end{pmatrix}, \begin{pmatrix} 1 \otimes x \\ 1 \otimes x \\ 1 \otimes x \end{pmatrix}, \begin{pmatrix} 1 \otimes x - x \otimes 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \otimes x - x \otimes 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \otimes x - x \otimes 1 \end{pmatrix}, \begin{pmatrix} x \otimes x \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \otimes x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x \otimes x \end{pmatrix} \right\}.$$

We then obtain H^{-2} by taking the quotient of $\text{Ker}(d^{-2})$ by $\text{Im}(d^{-3})$, which yields $H^{-2} \cong \text{span}\{(1 \otimes x - x \otimes 1, 0, 0)\}$. Calculating the q -degree of this generator we see that $q((1 \otimes x - x \otimes 1, 0, 0)) = -5$.

- $i = -1$: The dimension of $C^{-1,*}(D)$ is 6, since the direct sum consists of three summands each of dimension 2. Counting dimensions, we see that the dimension of $\text{Im}(d^{-2})$ must be

$$\dim(C^{-2,*}(D)) - \dim(\text{Ker}(d^{-2})) = 12 - 8 = 4.$$

One can check that the image of d^{-2} is spanned by the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ x \end{pmatrix} \right\}.$$

One can also check that $\text{Ker}(d^{-1})$ is also spanned by

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x \\ x \end{pmatrix} \right\}$$

which gives us $H^{-1} = 0$. Thus the homology is trivial in homological degree -1 .

- $i = 0$: $\dim(C^{0,*}(D)) = 4$ since there are two tensor factors of V . Furthermore, $\dim(\text{Im}(d^{-1})) = 2$ since $\dim(C^{-1,*}(D)) = 6$ and $\dim(\text{Ker}(d^{-1})) = 4$. The boundaries in homological degree 0 are spanned by the set $\{1 \otimes x + x \otimes 1, x \otimes x\}$ and the cycles are spanned by all basis elements $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ since everything in $C^{0,*}(D)$ is mapped to the trivial vector space by d^0 . Thus H^0 is isomorphic to $\text{span}\{1 \otimes 1, 1 \otimes x\}$. Applying the formula for the q -degree of these two elements, we see that $q(1 \otimes 1) = -1$ and $q(1 \otimes x) = -3$, thus completing the homology calculation.

We summarize the Khovanov homology of the trefoil in the following table.

$j \backslash i$	-3	-2	-1	0
-1				\mathbb{Q}
-3				\mathbb{Q}
-5		\mathbb{Q}		
-7				
-9	\mathbb{Q}			

Table 2.1. The Khovanov homology of the trefoil knot, with homological degree i running horizontally and q -degree j running vertically.

We omit all trivial entries from the table above. Indeed, note that all rows with even q -degree are trivial – this is a more general phenomenon which we state as a proposition:

Proposition 2.9. [6, Proposition 24] *Let L be a link. If L has an odd number of components, then $KH^{*,n}(L) = 0$ for all even integers n . If L has an even number of components, then $KH^{*,n}(L) = 0$ for all odd integers n .*

In particular, this proposition implies that all non-trivial q -degrees for a knot must be odd integers. We will use this fact repeatedly in the next section, in which we define an invariant of knots constructed in terms of these q -degrees.

3 Rasmussen’s s -invariant

3.1 Finite length filtrations and the spectral sequence

We now set up a different version of the TQFT used in the previous sections, following [9]. The underlying vector spaces are the same ones we used in constructing the cochain complex, but we now wish to alter the maps m and Δ . To this end, we define new maps $m' : V \otimes V \rightarrow V$ and $\Delta' : V \rightarrow V \otimes V$ as

$$m'(1 \otimes 1) = m'(x \otimes x) = 1$$

$$m'(1 \otimes x) = m'(x \otimes 1) = x$$

and

$$\Delta'(1) = 1 \otimes x + x \otimes 1$$

$$\Delta'(x) = x \otimes x + 1 \otimes 1.$$

Using these maps we can define a differential d' to get a cochain complex called the *Lee complex* (as opposed to the *Khovanov complex*). As before, we can identify a q -grading on elements of this complex. However, this q -grading does not behave well with respect to the differential d' – the element $\Delta'(x)$ is not homogenous since it is a sum of elements of degree -2 and 2, respectively. One can check, however, that the q -grading of every monomial in $d'(v)$ (for a homogenous element v of the Lee complex) is always greater than or equal to the q -grading of v . Thus the q -grading induces a filtration on the Lee complex of a link L . From now on we will denote the Khovanov complex of a link L by $C_{Kh}(L)$, and we will denote the Khovanov homology of L by $H_{Kh}(L)$. Similarly, we denote the Lee complex by $C_{Lee}(L)$ and we denote the cohomology of this complex by $H_{Lee}(L)$.

We now state the following theorems, both of which follow from the work of Lee in [9]. For the sake of brevity, we omit the proofs of both theorems; the proof of the first theorem is quite technical while the proof of the second involves techniques which we will not need later. (For a more detailed discussion of spectral sequences and how they arise from filtrations, see the Appendix.)

Theorem 3.1. [12, Theorem 2.1] *Let L be a link in S^3 . There exists a spectral sequence with $H_{Kh}(L)$ as its E^1 term which converges to $H_{Lee}(L)$. The terms $E^m, m \geq 1$ of this spectral sequence are invariants of the link L .*

Theorem 3.2. [9, Theorem 4.2] *The dimension of $H_{Lee}(L)$ is 2^n , where n is the number of components of the link L in S^3 .*

For the rest of this section, we specialize to the case where $n = 1$, i.e. we only consider the case where L is a knot K in S^3 . By the two theorems above we know there exists a spectral sequence associated to K with E^∞ -term isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$ (recall that all our vector spaces are defined over the field \mathbb{Q}). Each copy of \mathbb{Q} in the E^∞ -term of the spectral sequence of K has a q -grading associated to it – call these q -gradings s_{max} and s_{min} , chosen so that $s_{max} \geq s_{min}$. Since all q -gradings of a knot are odd integers (by Proposition 2.9), s_{max} and s_{min} must be odd integers. In fact, s_{max} and s_{min} are invariants of K , since the isomorphism type of the spectral sequence is itself an invariant of K . In order to formalize this idea, we require some terminology related to graded filtrations of chain complexes.

Definition 3.3. Let C be a chain complex. A *finite length filtration* of C is a sequence of subcomplexes

$$C = \mathcal{F}^n C \supset \mathcal{F}^{n+1} C \supset \mathcal{F}^{n+2} C \supset \dots \supset \mathcal{F}^m C = \{0\}$$

where $n, m \in \mathbb{Z}, n < m$. We also say that C is a *filtered chain complex* when we consider it together with a finite length filtration of C . An element $x \in C$ has *grading i* if and only if $x \in \mathcal{F}^i$ but $x \notin \mathcal{F}^{i+1}$.

Definition 3.4. Let C, C' be two filtered chain complexes. A map $f : C \rightarrow C'$ is a *filtered map of degree k* if $f(C_i) \subset C'_{i+k}$. If f is a filtered map of degree 0, we say that f *respects the filtration* associated to the complexes C, C' .

A finite length filtration $\{\mathcal{F}^i C\}$ of a chain complex C induces a filtration on homology:

$$H_*(C) = \mathcal{F}^n H_*(C) \supset \mathcal{F}^{n+1} H_*(C) \supset \mathcal{F}^{n+2} H_*(C) \supset \dots \supset \mathcal{F}^m H_*(C) = \{0\}$$

where a class $[x] \in H_*(C)$ belongs to $\mathcal{F}^i H_*(C)$ if and only if $[x]$ is represented by some x lying in \mathcal{F}^i (with respect to the finite length filtration of C). As one would expect, a filtered chain map $f : C \rightarrow C'$ of degree k induces a filtered map on homology $f_* : H_*(C) \rightarrow H_*(C')$ which is also of degree k .

3.2 The definition of the s -invariant

We can use the grading on a finite length filtration $\{\mathcal{F}^i C\}$ of a chain complex C to induce a spectral sequence with the property that the E^i -term of the spectral sequence can be identified with $\mathcal{F}^i H_*(C) / \mathcal{F}^{i+1} H_*(C)$. So the spectral sequence induced by the finite length filtration on C converges to the associated graded vector space of the induced filtration on $H_*(C)$. Now, we

can apply this construction to the case where C is $C_{Lee}(K)$ for a knot K . Let s be the grading on $H_{Lee}(K)$ induced by the q -grading on $C_{Lee}(K)$ via the spectral sequence associated to the finite length filtration of $C_{Lee}(K)$. (Note that this s is different from the “ s ” in the name of the invariant referred to in the title of this section – we will define this invariant later.) Using this idea of a kind of q -grading on homology, we can formally define s_{max} and s_{min} .

Definition 3.5. Let K be a knot in S^3 . We define

$$s_{min} = \min\{s(x) : x \in H_{Lee}(K), x \neq 0\},$$

$$s_{max} = \max\{s(x) : x \in H_{Lee}(K), x \neq 0\}.$$

The s -invariant of a knot K is equal to the average of the values of $s_{min}(K)$ and $s_{max}(K)$. Our main goal in this section is to prove the following theorem, which justifies the definition of the s -invariant.

Theorem 3.6. *Let K be a knot in S^3 . Then*

$$s_{max}(K) = s_{min}(K) + 2.$$

First we need the following result, whose proof we omit.

Lemma 3.7. [12, Lemma 3.5] *Let K be a knot in S^3 . There is a direct sum decomposition*

$$C_{Lee}(K) \cong C_{Lee}^e(K) \oplus C_{Lee}^o(K),$$

where $C_{Lee}^e(K)$ is generated by all states with q -grading congruent to 1 mod 4 and $C_{Lee}^o(K)$ is generated by all states with q -grading congruent to 3 mod 4. Furthermore, if o is an orientation on K and \bar{o} is the reverse orientation, then $\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}$ is contained in one of the two summands while $\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}$ is contained in the other summand.

Here \mathfrak{s}_o denotes the *canonical generator* of $C_{Lee}(K)$ associated to the orientation o of K , obtained by applying a 0-smoothing to each positive crossing of K and a 1-smoothing to each negative crossing of K relative to the orientation o . In this sense, \mathfrak{s}_o corresponds to the *orientation-preserving* smoothing of K , since any other choice of smoothing will result in an unoriented knot. We form these canonical generators of $H_{Lee}(L)$ as follows: Given an oriented link diagram D with an orientation o , we let D_o be the oriented resolution of D described above with respect to o . Now, we divide the circles in D_o into two collections A and B as follows. A circle belongs to A (resp. B) if it has the counter-clockwise orientation and it is separated from infinity by an even (resp. odd) number of circles, or if it has the clockwise orientation and is separated from infinity by an odd (resp. even) number of circles. Label the circles in A with a 1 (corresponding to the basis element with degree +1) and label those in B with an x (corresponding to the basis element with degree -1). In this way, we have labeled each circle in D_o and we now have a generator of $H_{Lee}(L)$ – we denote this state by \mathfrak{s}_o . Note that, by Theorem 3.2, there are two such generators for the Lee homology of a knot K .

To solidify these notions, we present the following example. Consider the link L pictured in Figure 3.1 (a), known as the *Borromean rings*. Since L has three components, by Theorem 3.2 the dimension of $H_{Lee}(L)$ is $2^3 = 8$, and so there are 8 canonical generators of the Lee homology

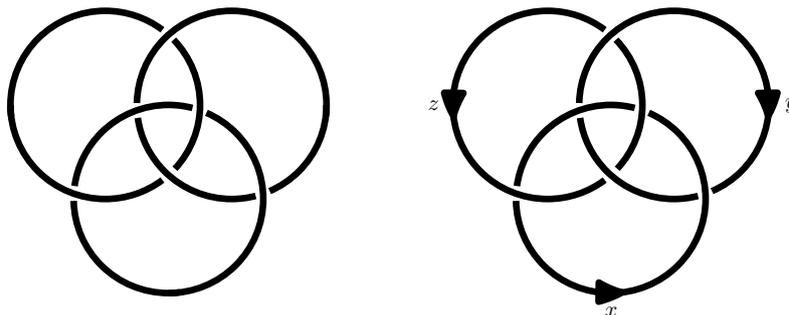


Figure 3.1 (a). The Borromean rings.

(b). The oriented Borromean rings, with components labeled.

of the Borromean rings, where the set of canonical generators is in bijective correspondence with the set of orientations of L . Now consider the oriented copy (which we will also call L) of the Borromean rings in Figure 3.1 (b); call its orientation o . To determine the canonical generator \mathfrak{s}_o corresponding to o , we smooth each crossing in L in the orientation-preserving way: If the crossing is positive (resp. negative), we apply a 0-smoothing (resp. 1-smoothing). In the example pictured above, components x and z are oriented positively (counter-clockwise) while component y is oriented negatively (clockwise). According to this choice of orientation, we get the following oriented resolution of L (pictured in Figure 3.2).

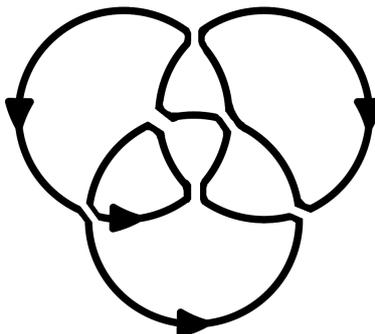


Figure 3.2. The oriented resolution of the Borromean rings, according to the orientation given in Figure 3.1 (b).

In this case, we get three disjoint circles in the plane. One such circle is oriented negatively (clockwise) and is separated from infinity by zero circles; this circle belongs in the collection B described earlier. The other two circles are oriented positively (counter-clockwise); the outer circle is separated from infinity by zero circles and so we place it in collection A , while the inner circle is separated from infinity by one circle and so we place it in B . We label the circle in A with a 1 and we label the circles in B each with an x , and we get one of the canonical generators of the Lee homology of the Borromean rings. The other generators are obtained in a similar fashion.

The proof of Lemma 3.7 is not difficult, but involves ideas from [9] which are not mentioned here. The main point is that we can split the differentials m' and Δ' in Lee's TQFT each as a sum of a differential which preserves the q -grading and a differential which raises the q -grading by 4. Rasmussen proves this result for an n -component link L , in which case $C_{Lee}^e(K)$ is generated by

states with q -grading congruent to $n \pmod 4$ and $C_{Lee}^o(K)$ is generated by states with q -grading congruent to $(2+n) \pmod 4$. Since we are only concerned with knots here, we have specialized to the case $n = 1$.

Corollary 3.8. $s(\mathfrak{s}_o) = s(\mathfrak{s}_{\bar{o}}) = s_{min}(K)$.

Proof. We know from observations made in the previous section that $H_{Kh}(K)$ and $H_{Lee}(K)$ are invariant under global changes of orientation. Thus the q -grading associated to the state \mathfrak{s}_o is equal to that of $\mathfrak{s}_{\bar{o}}$ since the orientations o and \bar{o} of K are related by a global change of orientation (namely, by changing the orientation of the only component of K). By definition, the q -grading on \mathfrak{s}_o induces a grading on $H_{Lee}(K)$ which is minimal when considered over all possible non-zero cycles in $H_{Lee}(K)$. To see this, suppose not. Then there is some non-zero $[x] \in H_{Lee}(K)$ with $s([x]) < s(\mathfrak{s}_o)$; denote by x the element of $C_{Lee}(K)$ which has $[x]$ as a representative cycle in homology. By the lemma, we can write x as a decomposition $y + z$, where $y \in C_{Lee}^e(K)$ and $z \in C_{Lee}^o(K)$. Since the rank of $H_{Lee}(K)$ is always equal to 2, $[\mathfrak{s}_o]$ and $[\mathfrak{s}_{\bar{o}}]$ are the only generators of $H_{Lee}(K)$, and so $[y]$ and $[z]$ can be written as scalar multiples of $\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}$ and $\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}$, respectively. So

$$s([x]) = s((\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}) + (\mathfrak{s}_o - \mathfrak{s}_{\bar{o}})) = s(2\mathfrak{s}_o) = s(\mathfrak{s}_o),$$

which is a contradiction, since we supposed that $s([x]) < s(\mathfrak{s}_o)$. Thus it must be that $s(\mathfrak{s}_o) = s(\mathfrak{s}_{\bar{o}}) = s_{min}(K)$. \square

Corollary 3.9. $s_{max}(K) > s_{min}(K)$.

Proof. By Lemma 3.7, $C_{Lee}(K)$ decomposes into the direct sum $C_{Lee}(K) \cong C_{Lee}^e(K) \oplus C_{Lee}^o(K)$. Hence we can decompose the spectral sequence associated to $C_{Lee}(K)$ as a direct sum into two components. Since $H_{Lee}(K) \cong \mathbb{Q} \oplus \mathbb{Q}$, the homology of each summand is isomorphic to \mathbb{Q} and so each summand accounts for one of the two surviving terms in the spectral sequence. By definition, the two summands $C_{Lee}^e(K)$ and $C_{Lee}^o(K)$ are supported in different q -gradings, since they are generated by all states with q -grading congruent to $1 \pmod 4$ and to $3 \pmod 4$, respectively. Thus one of $s_{max}(K)$, $s_{min}(K)$ is greater than the other; we chose $s_{max}(K)$ and $s_{min}(K)$ so that $s_{max}(K) \geq s_{min}(K)$ and so we have strict inequality. \square

We need one more tool to prove Theorem 3.6: a short exact sequence whose existence is guaranteed by the following lemma, which we will also state without proof. Briefly, the proof involves obtaining a short exact sequence for $C_{Lee}(K_1 \# K_2)$ by twisting one of the knots so that there is a new crossing in the region where the connected sum of the knots is formed. One then applies the differential from Lee's TQFT to this new crossing to obtain a map from $C_{Lee}(K_1 \sqcup K_2)$ to $C_{Lee}(K_1 \# K_2^r)$, where K^r denotes the knot K with its orientation reversed. We proceed in this manner to obtain a long exact sequence in homology, which we can split into two short exact sequences using the fact that the dimensions of $H_{Lee}(K_1 \# K_2)$ and $H_{Lee}(K_1 \# K_2^r)$ equal 2, while the dimension of $H_{Lee}(K_1 \sqcup K_2) \cong H_{Lee}(K_1) \otimes H_{Lee}(K_2)$ is 4. Combining all of these facts we get the following:

Lemma 3.10. [12, Lemma 3.8] *Let K_1, K_2 be oriented knots in S^3 . There exists a short exact sequence*

$$0 \rightarrow H_{Lee}(K_1 \# K_2) \xrightarrow{p_*} H_{Lee}(K_1) \otimes H_{Lee}(K_2) \xrightarrow{\partial} H_{Lee}(K_1 \# K_2^r) \rightarrow 0$$

where K_2^r is the reverse of K_2 , i.e. K_2 with its orientation reversed. The maps p_* and ∂ are filtered maps of q -degree -1 .

Now we can give a proof of Theorem 3.6, which says that $s_{max}(K)$ and $s_{min}(K)$ are related by the equation $s_{max}(K) = s_{min}(K) + 2$.

Proof. Consider the short exact sequence of the previous lemma in which we let K_1 be a knot K and we let K_2 be U , the unknot. Orient both K and U . K has two canonical generators which we denote by \mathfrak{s}_a and \mathfrak{s}_b according to their label near the point where K and U are joined by the connected sum. We also denote the canonical generators of U by \mathbf{a} and \mathbf{b} . Suppose, without loss of generality, that $s(\mathfrak{s}_a - \mathfrak{s}_b) = s_{max}(K)$ – if not, then by Lemma 3.7 we would have $s(\mathfrak{s}_a + \mathfrak{s}_b) = s_{max}(K)$, as each canonical generator corresponds to one of the two orientations of K . So if $s(\mathfrak{s}_a + \mathfrak{s}_b) = s_{max}(K)$, we can simply reverse the orientation of K (since the orientations on K and U are assigned arbitrarily) to get $s(\mathfrak{s}_a - \mathfrak{s}_b) = s_{max}(K)$. Applying the map ∂ in the short exact sequence of Lemma 3.10 to $(\mathfrak{s}_a - \mathfrak{s}_b) \otimes \mathbf{a} \in H_{Lee}(K) \otimes H_{Lee}(U)$ we see that $\partial((\mathfrak{s}_a - \mathfrak{s}_b) \otimes \mathbf{a}) = \mathfrak{s}_a$, since $H_{Lee}(K \# U^r) \cong H_{Lee}(K)$. ∂ is a filtered map of degree -1 , which means that the q -degree of the image of an element $[x]$ under ∂ is bounded below by $q([x]) - 1$. Hence we have $s((\mathfrak{s}_a - \mathfrak{s}_b) \otimes \mathbf{a}) \leq s(\mathfrak{s}_a) + 1$. Applying Corollary 3.8, we can conclude that $s_{max}(K) - 1 \leq s_{min}(K) + 1$, i.e. $s_{max}(K) \leq s_{min}(K) + 2$. But $s_{max}(K) \neq s_{min}(K)$ by Corollary 3.9 and so, since $s_{max}(K), s_{min}(K)$ are odd integers, we must have $s_{max}(K) = s_{min}(K) + 2$. \square

Theorem 3.6 justifies the definition of the s -invariant, which we can finally state.

Definition 3.11. Let K be a knot in S^3 . Then Rasmussen’s invariant, denoted by s , is defined to be

$$s(K) = s_{max}(K) - 1 = s_{min}(K) + 1 = \frac{s_{max}(K) + s_{min}(K)}{2}.$$

Note that since $s_{max}(K)$ and $s_{min}(K)$ are always odd integers, $s(K)$ is always an even integer.

3.3 Properties of the s -invariant

We now list properties of the s -invariant which we will need later. Most of the properties will be stated without proof, as the techniques employed are not used later on. The following two results tell us that the s -invariant behaves well with respect to reversing the orientation of a knot and taking the mirror image of a knot.

Lemma 3.12. *Let K be an oriented knot in S^3 and let K^r be K with its orientation reversed. Then $s(K) = s(K^r)$.*

Proof. This property follows immediately from an observation we made earlier: $C_{Kh}(K)$ and $C_{Lee}(K)$ are both invariant under the operation of global orientation reversal. Since changing the orientation of a knot is equivalent to applying a global reversal of orientation, $C_{Lee}(K) \cong C_{Lee}(K^r)$ and so the q -gradings associated to these complexes induce the same values of s_{max} and s_{min} on $H_{Lee}(K)$ and $H_{Lee}(K^r)$, thus implying $s(K) = s(K^r)$. \square

Proposition 3.13. [12, Proposition 3.10] *Let K be a knot, and denote by \bar{K} the mirror image of K . Then the following properties hold:*

$$\begin{aligned} s_{max}(\bar{K}) &= -s_{min}(K), \\ s_{min}(\bar{K}) &= -s_{max}(K), \\ s(\bar{K}) &= -s(K). \end{aligned}$$

The main idea behind this proposition is that the filtered complex $C_{Lee}(\bar{K})$ is isomorphic to $(C_{Lee}(K))^*$, the dual of the filtered complex of K . Given a filtration of $C_{Lee}(K)$ we can define a dual filtration, where the submodule $\mathcal{F}^{-i}C^*$ in the dual filtration consists of all those elements in the dual space which are orthogonal to (i.e. have inner product equal to zero with) every element in the original submodule \mathcal{F}^iC . Using Lee's TQFT, we can define an isomorphism $r : (V, m', \Delta') \rightarrow (V^*, \Delta'^*, m'^*)$ which maps v_{\pm} to v_{\mp}^* . One can then construct the desired isomorphism of chain complexes using r , which gives us the properties listed above. The general behaviour of spectral sequences arising from dual filtered complexes is encompassed by the following result, which is needed in the proof of Proposition 3.13.

Lemma 3.14. [12, Lemma 3.11] *Let C and C' be dual filtered complexes over a field. Then their associated spectral sequences E^i and E'^i are dual in the sense that E^i is isomorphic to $(E'^i)^*$.*

The s -invariant also behaves nicely with respect to taking connected sums of knots:

Proposition 3.15. [12, Proposition 3.12] *Let K_1, K_2 be knots in S^3 . Then*

$$s(K_1 \# K_2) = s(K_1) + s(K_2).$$

3.4 The slice genus

We are now in a position to prove that the s -invariant gives a lower bound on the slice genus. This result is one of the main highlights of Rasmussen's theory, and it can be used to give a quick proof of the Milnor conjecture. But first we need a few notions, which we define in general for links.

Definition 3.16. Let L_0 and L_1 be two links in \mathbb{R}^3 . A *cobordism* from L_0 to L_1 is a smooth, oriented, compact surface S embedded in $\mathbb{R}^3 \times [0, 1]$ such that $S \cap (\mathbb{R}^3 \times \{0\}) = L_0$ and $S \cap (\mathbb{R}^3 \times \{1\}) = L_1$.

Given a cobordism of links, our goal is to construct a map on the level of chain complexes which induces a map on homology. If we have a cobordism S between two links L_0 and L_1 , we can decompose S into a series of elementary cobordisms, where by *elementary* we mean a cobordism which bounds two links which are related by a Morse move or by a Reidemeister move. We omit all of the details; briefly, there is one Reidemeister-type cobordism for each of the ordinary Reidemeister moves and one for each of their respective inverses, and there is one Morse-type cobordism for each of the three Morse moves, where by a *Morse move* we mean a move which adds a 0-handle, a 1-handle or a 2-handle to L_i for $i \in [0, 1]$ (see [12] for a more detailed discussion). So we need only construct a map associated to each of these elementary cobordisms and then compose all of them to get our desired map.

An elementary cobordism corresponding to the i^{th} Reidemeister move or its inverse yields a map $\rho'_i : C_{Lee}(L_0) \rightarrow C_{Lee}(L_1)$, which we can use to induce an isomorphism on homology. We then define ϕ'_S to be the induced map on homology ρ'_{i*} ; this is a filtered map of degree 0. The case where the elementary cobordism corresponds to a Morse move is a bit more complicated, but in the end we have a map ϕ'_S on homology which is filtered of degree 1 in the case of a 0-handle or 2-handle addition, or of degree -1 in the case that the cobordism corresponds to the addition of a 1-handle.

Thus, given such a cobordism S , we can decompose it into a union of elementary cobordisms $S_i, i = 1, \dots, k$ so that $S = S_1 \cup \dots \cup S_k$ for some $k \in \mathbb{N}$. Each S_i corresponds to a map on the level of chain complexes, which induces a map on homology ϕ'_{S_k} . We then take $\phi'_S : H_{Lee}(L_0) \rightarrow H_{Lee}(L_1)$ to be the composition $\phi'_{S_k} \circ \dots \circ \phi'_{S_1}$. By construction, ϕ'_S is a filtered map of degree $\chi(S)$, the Euler characteristic of S . Moreover, this construction is well-defined, as the map ϕ'_S does not depend on the choice of decomposition of S . With this construction, we can prove the main theorem of this section.

Theorem 3.17. *Let K be a knot in S^3 . Then*

$$|s(K)| \leq 2g_*(K)$$

where $g_*(K)$ is the smooth slice genus of K , i.e. the minimal genus of a smoothly embedded, orientable surface in B^4 which bounds K .

Proof. Let $K \subset S^3$ be a knot which bounds an oriented surface of genus g lying in B^4 . We can remove a small disk from this surface to get an orientable, connected cobordism of Euler characteristic $-2g$ between K and the unknot U , where the cobordism lies in $\mathbb{R}^3 \times [0, 1]$. Let $[x] \in H_{Lee}(K) - \{0\}$ be a cycle such that $s(x)$ is maximal (such a cycle must exist because the set of cycles in $H_{Lee}(K)$ is always non-empty, and hence there must be one that maximizes $s(x)$). Then $\phi'_S(x)$ is a non-zero element of $H_{Lee}(U)$. ϕ'_S is a filtered map of degree $-2g$ and so

$$s(\phi'_S(x)) \geq s(x) - 2g.$$

Now, one can easily check that $H_{Kh}(U)$ is supported in two q -gradings, those being 1 and -1 . Thus the maximal q -grading induced on $H_{Lee}(U)$ by $C_{Lee}(U)$ is 1 and so $s_{max}(U) = 1$. We also have

$$s(\phi'_S(x)) \leq 1.$$

Combining the two inequalities above yields the inequality $s(x) \leq 2g + 1$; this implies $s_{max}(K) \leq 2g + 1$ and thus

$$s(K) \leq 2g$$

since $[x]$ was chosen such that $s(x)$ is maximal. Now we can apply the same argument to \bar{K} to get the inequality $s(\bar{K}) \leq 2g$. We can then apply Proposition 3.13 to get $-s(K) \leq 2g$, which implies

$$s(K) \geq -2g.$$

So we have $-2g \leq s(K) \leq 2g$ and thus $|s(K)| \leq 2g$. This argument works for any oriented surface with boundary K and hence we can apply it to the surface of minimal genus to get $|s(K)| \leq 2g_*(K)$. \square

We can use Theorem 3.17 together with the properties listed earlier to prove a fact about the smooth concordance group of knots in S^3 . Recall that two knots K_1 and K_2 in S^3 are *smoothly concordant* if there exists a smooth embedding $f : S^1 \times [0, 1] \rightarrow S^3 \times [0, 1]$ such that the boundary of the image of f is $K_1 \times \{0\} \sqcup K_2 \times \{1\}$, which is isotopic to a disjoint union of K_1 and $-K_2$ (where by $-K$ we mean a copy of K with the opposite orientation). Concordance is an equivalence relation and so we can consider the equivalence classes of all knots in S^3 . The set of all such equivalence classes forms an abelian group, denoted \mathcal{C} , under the operation of taking connected sums of knots. The s -invariant gives us a homomorphism from this group to the group of integers.

Theorem 3.18. *The function $s : \mathcal{C} \rightarrow \mathbb{Z}$ mapping a knot $K \in \mathcal{C}$ to $s(K) \in \mathbb{Z}$ is a group homomorphism, where \mathcal{C} is the smooth concordance group of knots in S^3 .*

Proof. Recall that a knot K is *smoothly slice* if K is smoothly concordant to the unknot (see Section 4.1 for an equivalent definition of sliceness). It is a fact that if two knots K_1, K_2 are concordant, then the knot $K_1 \# \overline{K_2^r}$ is slice and so $g_*(K_1 \# \overline{K_2^r}) = 0$ since a knot is smoothly slice if and only if its smooth slice genus is zero. So by the previous theorem, we must have $s(K_1 \# \overline{K_2^r}) = 0$. By Proposition 3.15, this implies that $s(K_1) + s(\overline{K_2^r}) = 0$. By Proposition 3.13, $s(\overline{K_2^r}) = -s(K_2^r)$ and so an application of Lemma 3.12 yields $s(\overline{K_2^r}) = -s(K_2)$. Thus we have $s(K_1) - s(K_2) = 0$, i.e. $s(K_1) = s(K_2)$. Thus s is a well-defined map from \mathcal{C} to \mathbb{Z} , since if two knots lie in the same concordance class then they have the same s -invariant. The fact that this map is a group homomorphism follows immediately from Proposition 3.15. \square

We can also use the s -invariant to give a quick proof of the Milnor conjecture, which was originally proved in [10] using techniques from gauge theory. We will say that a knot K is *positive* if it has a knot diagram D with all crossings positive. Since all crossings in D are positive, the canonical generator \mathfrak{s}_o corresponds to the resolution in which we apply a 0-smoothing to each of the crossings in D . Thus the state \mathfrak{s}_o lies in the extreme corner of the cube of resolutions of K ; \mathfrak{s}_o has homological degree zero. Since there are no generators in $C_{Lee}(K)$ with homological degree -1, the only class homologous to \mathfrak{s}_o is \mathfrak{s}_o itself, so by definition we get $s_{min}(K) = s([\mathfrak{s}_o]) = q(\mathfrak{s}_o)$.

The minimal q -grading in homological degree zero occurs when each circle in the resolution corresponding to the canonical smoothing is labeled with the basis element x . If D has n crossings (all of which are positive) and there are k circles in the oriented resolution of D , then using the formula for the q -grading given in Section 2.2 we get

$$\begin{aligned} q(\mathfrak{s}_o) &= \deg(\mathfrak{s}_o) + i(\mathfrak{s}_o) + n_+ - n_- \\ &= -k + 0 + n - 0 \\ &= -k + n \end{aligned}$$

and so we have

$$s(K) = -k + n + 1.$$

Seifert's algorithm for constructing a Seifert surface for K gives us a surface S with Euler characteristic $k - n$. Combining all of these observations with Theorem 3.17 we get

$$2g(K) \leq 2g(S) = n - k + 1 = s(K) \leq 2g_*(K) \leq 2g(K)$$

which implies that the inequalities above must all be equalities. We have proved the following:

Theorem 3.19. *If K is a positive knot, then*

$$s(K) = 2g_*(K) = 2g(K)$$

where $g(K)$ denotes the ordinary genus of K .

Now let K be the (p, q) -torus knot, where p and q are coprime integers. One can show using a braid presentation of K that $s(K) = (p - 1)(q - 1)$ using the formula for the s -invariant of positive knots. By Theorem 3.19 we get the value for the smooth slice genus of K , easily proving the following famous result, with which we conclude this section.

Corollary 3.20. (The Milnor Conjecture) *The smooth slice genus of the (p, q) -torus knot is*

$$\frac{(p - 1)(q - 1)}{2}.$$

It should be clear now that the s -invariant is at least somewhat special: It is defined combinatorially, yet it tells us something about results involving smoothness. This property will be made even more prominent in the next section when we prove the existence of *exotic* \mathbb{R}^4 . We conclude this section by noting, for the sake of completeness, the following fact about the s -invariant: If K is an alternating knot, then $s(K) = \sigma(K)$, where σ denotes the usual signature of a knot. The proof of this fact follows from the work of Lee in [9], which builds on a conjecture made by Bar-Natan in [2]. So, for alternating knots, s yields no new information.

4 Slice knots and exotic \mathbb{R}^4

In this section we define the notions of sliceness of a knot in the topological category and in the smooth category, and we prove a theorem which states that if we are in possession of a topologically slice knot that is not smoothly slice, then we can construct a 4-manifold which is homeomorphic but not diffeomorphic to \mathbb{R}^4 . Then, using the s -invariant defined in the previous section, we show that such a knot does exist, allowing us to easily prove the existence of an exotic \mathbb{R}^4 .

4.1 Slice knots

Definition 4.1. Let X, Y be topological manifolds of dimensions n, m respectively such that $n < m$. A topological embedding $\phi : X \rightarrow Y$ is *flat* if it can be extended to a topological embedding $\Phi : X \times D^{m-n} \rightarrow Y$.

Using flat topological and smooth embeddings, we can consider knots in S^3 which bound “nicely embedded” disks D^2 in D^4 . The following definition formalizes this idea.

Definition 4.2. A knot $K \subset \partial D^4$ is *smoothly (resp. topologically) slice* if there is a copy of the closed disk D^2 smoothly (resp. flat topologically) embedded in D^4 such that the boundary of D^2 is K . Such a disk D^2 is called a *slice disk for K* .

Note that if a knot is smoothly slice, then it must be topologically slice since a smooth embedding is clearly also a continuous embedding. The converse, however, is not true: There exist topologically slice knots which are *not* smoothly slice; this fact is highly non-trivial, and we will later show the existence of such knots in order to prove the main result of this section. Furthermore, note that the requirement that our topological embeddings be flat is necessary: if we drop this condition, then any knot $K \subset S^3$ can be shown to be topologically slice by “coning” K to a single point and observing that K bounds a disk D^2 embedded in D^4 such that D^2 is “locally knotted” at the cone point. We will also need the following construction from geometric topology.

Definition 4.3. Let k, n be non-negative integers such that $k \leq n$. An *n -dimensional k -handle* h is a copy of $D^k \times D^{n-k}$ attached to the boundary of an n -manifold X along $\partial D^k \times D^{n-k}$ by an embedding $\phi : \partial D^k \times D^{n-k} \rightarrow \partial X$.

This definition is general; for our purposes we only consider the case $n = 4, k = 2$. In this case we refer to h simply as a 2-handle.

4.2 A proof of the existence of exotic \mathbb{R}^4

Definition 4.4. An *exotic* \mathbb{R}^4 is a smooth manifold which is homeomorphic but not diffeomorphic to \mathbb{R}^4 . An exotic \mathbb{R}^4 is called *small* if it can be smoothly embedded as an open subset of \mathbb{R}^4 ; otherwise we call it a *large* exotic \mathbb{R}^4 .

The existence of a small exotic \mathbb{R}^4 is (relatively) easily established (see, for instance, Theorem 9.3.1 in [5]). The existence of a large exotic \mathbb{R}^4 , on the other hand, is not easy to prove. The construction of a large exotic \mathbb{R}^4 often makes use of techniques from gauge theory and is generally too complicated to construct directly. However, we can show that a large exotic \mathbb{R}^4 must exist using the machinery developed in previous sections. First we prove the following lemma.

Lemma 4.5. *Let K be a knot in S^3 and let X_K be D^4 with a 2-handle attached along K with framing coefficient 0. Then X_K has a smooth embedding in \mathbb{R}^4 if and only if K is smoothly slice.*

(Here, the *framing coefficient* of K is the linking number $lk(K, K')$ where K' is a parallel copy of K with orientation parallel to that of K . Requiring the framing coefficient to be 0 guarantees that X_K is indeed a submanifold of \mathbb{R}^4 .)

Proof. Suppose there is a smooth embedding $\phi : X_K \rightarrow S^4$ such that $\phi(X_K)$ is mapped to one hemisphere of S^4 . Since the 2-handle in X_K is attached along K with framing 0, the core of the 2-handle is a copy of D^2 . Since X_K is smoothly embedded in S^4 by assumption, the same must hold for D^2 in $S^4 - \text{int}(\phi(X_K))$, which is homeomorphic to D^4 since X_K is mapped to one hemisphere of S^4 which we can identify with D^4 . Indeed, $\partial D^2 = K$ and so D^2 is a slice disk for K in D^4 and thus K is smoothly slice.

Conversely, suppose K is smoothly slice. Then there is a smooth embedding of D^2 into D^4 such that $\partial D^2 = K$. By the tubular neighbourhood theorem (see, for instance, related results found in [4] and [5]) we can construct a tubular neighbourhood $D^2 \times D^2$ of D^2 which is smoothly embedded in D^4 . Since $S^4 \cong D^4 \cup_{\partial D^4} \overline{D^4}$, where $\overline{D^4}$ is D^4 with the opposite orientation, we can attach another copy of D^4 in this manner to our copy of D^4 with the tubular neighbourhood to get $X_K \subset S^4$ (the tubular neighbourhood becomes the 2-handle of X_K). X_K is smoothly embedded in S^4 by construction since $X_K \hookrightarrow X_K \cup_{\partial D^4} \overline{D^4} \hookrightarrow S^4$ is a composition of smooth embeddings and so we have a smooth embedding $\phi : X_K \rightarrow S^4$. \square

Lemma 4.6. *Let K be a knot in S^3 and let X_K be as in the statement of Lemma 4.5. Then X_K has a flat topological embedding in \mathbb{R}^4 if and only if K is topologically slice.*

Proof. The proof of Lemma 4.6 is exactly the same as that of Lemma 4.5, except we replace all instances of “smooth” with “flat topological” and “smoothly slice” with “topologically slice”. We also need to apply a topological version of the tubular neighbourhood theorem in the construction of X_K to ensure that the tubular neighbourhood can be flat topologically embedded in D^4 , which yields a flat topological embedding of X_K into \mathbb{R}^4 , as before. \square

We will also need the following theorem, which is a general result about the behaviour of smooth structures on 4-manifolds. The proof of the theorem shows existence of smooth structures on a certain class of 4-manifolds by directly constructing such a structure through techniques arising from 4-manifold theory. We omit the proof, as the methods involved are beyond the scope of this thesis.

Theorem 4.7. [4, Theorem 8.2] *Any connected, non-compact 4-manifold X admits a smooth structure.*

With this theorem in our possession, we can now state and prove the main result of this section. This result is presented as Exercise 9.4.23 in [5].

Theorem 4.8. *Let $K \subset S^3$ be a topologically slice knot which is not smoothly slice, let X_K be as in the statement of Lemma 4.5, and let \mathcal{R} be the set of all smoothings of \mathbb{R}^4 up to orientation-preserving diffeomorphism. Then X_K embeds smoothly in some $R \in \mathcal{R}$, which is necessarily a large exotic \mathbb{R}^4 .*

Proof. Since K is topologically slice, by Lemma 4.6 we know that there exists a flat topological embedding ϕ of X_K into \mathbb{R}^4 . We must show that $\mathbb{R}^4 - \text{int}(\phi(X_K))$ satisfies the hypotheses of Theorem 4.7. It is clear that $\mathbb{R}^4 - \text{int}(\phi(X_K))$ is a 4-manifold, since ϕ is a topological embedding and so its image $\phi(X_K)$ is a topological submanifold of \mathbb{R}^4 , which implies that $\mathbb{R}^4 - \text{int}(\phi(X_K))$ is a 4-dimensional submanifold of \mathbb{R}^4 and thus a 4-manifold in its own right. Furthermore, $\mathbb{R}^4 - \text{int}(\phi(X_K))$ is not compact, since if it were then it would be closed and bounded by the Heine-Borel theorem – if this were the case then $\text{int}(\phi(X_K))$ would be unbounded since it is the complement of a bounded subset of \mathbb{R}^4 . But this is a contradiction since X_K is compact by construction and hence so is its image under ϕ , since ϕ is an embedding, and thus $\phi(X_K)$ is bounded and hence so is $\text{int}(\phi(X_K))$. Finally, $\mathbb{R}^4 - \text{int}(\phi(X_K))$ is connected: To see this, view $\phi(X_K)$ as a closed subset of the connected 4-manifold S^4 and consider $S^4 - \phi(X_K)$. We claim that $S^4 - \phi(X_K)$ is path-connected. Since S^4 is path-connected, we can join any two points in $S^4 - \phi(X_K)$ by a path γ lying entirely in S^4 . If γ intersects $\phi(X_K)$, which is topologically embedded in S^4 , we can perturb γ so that its intersection with $\phi(X_K)$ is empty since $\phi(X_K)$ is a closed subset of a compact set S^4 and thus $\phi(X_K)$ must be compact when considered as a subset of S^4 . Indeed, since a path γ is 1-dimensional, we can in general always perturb γ around a copy of D^n topologically embedded in S^n for any $n > 1$ by applying a topological version of transversality as in [4, Theorem 9.5A]. So we can remove $\phi(X_K)$ from S^4 while retaining path-connectedness. Hence $S^4 - \phi(X_K)$ is path-connected and is therefore connected. We then remove a point (which does not disconnect a 4-manifold) from $S^4 - \phi(X_K)$ to conclude that $\mathbb{R}^4 - \phi(X_K)$ is connected, which tells us that $\mathbb{R}^4 - \text{int}(\phi(X_K))$ is connected since adding in the boundary points of $\phi(X_K)$ certainly cannot disconnect $\mathbb{R}^4 - \phi(X_K)$ (a fact which is guaranteed because of the flatness condition imposed on the topologically slice disk D^2).

So, by Theorem 4.7, the manifold $\mathbb{R}^4 - \text{int}(\phi(X_K))$ can be smoothed. Note that we can induce a smooth structure on X_K by treating it as the abstract manifold $D^2 \times D^2$, but also note that such a smooth structure is not the same as the one it would inherit as a subset of \mathbb{R}^4 . Now, since X_K is a smooth manifold, it induces a smooth structure on its boundary ∂X_K . Similarly the smooth structure on $\mathbb{R}^4 - \text{int}(\phi(X_K))$ induces a smooth structure on its boundary $\partial(\mathbb{R}^4 - \text{int}(\phi(X_K)))$. Both boundaries are 3-manifolds and so, by the existence and uniqueness of smoothings on 3-manifolds, the homeomorphism $\phi|_{\partial X_K}: \partial X_K \rightarrow \partial(\mathbb{R}^4 - \text{int}(\phi(X_K)))$ is isotopic to a diffeomorphism and so the smooth structures on X_K and $\mathbb{R}^4 - \text{int}(\phi(X_K))$ fit together (by gluing them along their respective boundaries via the diffeomorphism $\phi|_{\partial X_K}$) to give a smoothing R of \mathbb{R}^4 . We claim that this smoothing R of \mathbb{R}^4 is exotic, i.e. R , when viewed as a manifold, is homeomorphic but not diffeomorphic to \mathbb{R}^4 in the sense that the smooth structure on R is inconsistent with that of \mathbb{R}^4 . Indeed, if R were not exotic then there would exist a diffeomorphism $\psi: R \rightarrow \mathbb{R}^4$. The restriction of ψ to X_K would yield a smooth embedding of X_K into \mathbb{R}^4 . Then by Lemma 4.5 K would have to be smoothly slice; this contradicts our initial assumption that K

is not smoothly slice. Thus no such diffeomorphism exists and hence R cannot be diffeomorphic to \mathbb{R}^4 and yet, at the same time, R is homeomorphic to \mathbb{R}^4 since X_K embeds topologically into \mathbb{R}^4 by assumption and so the manifold formed by fitting X_K and $\mathbb{R}^4 - \text{int}(\phi(X_K))$ together must be homeomorphic to \mathbb{R}^4 . Now it remains to be shown that R is a large exotic \mathbb{R}^4 . Again, suppose not: Then R can be smoothly embedded as an open subset of \mathbb{R}^4 and hence so can X_K since the latter is smoothly embedded in R . But since K is not smoothly slice, this smooth embedding yields a contradiction (again, applying Lemma 4.5). Therefore R is a large exotic \mathbb{R}^4 . \square

We can now use the s -invariant to show that a knot which is topologically slice but not smoothly slice does indeed exist. To this end we use the following theorem of Freedman.

Theorem 4.9. [3, Theorem 1 (0)] *A knot K with Alexander polynomial $\Delta_K(t) = 1$ is topologically slice.*

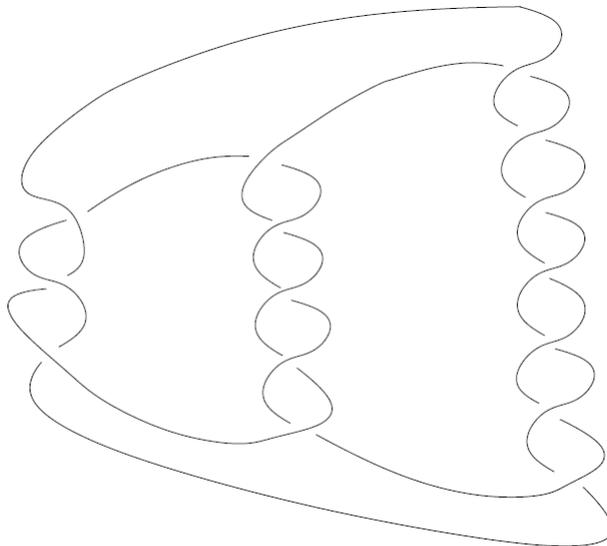


Figure 4.1. The $(-3, 5, 7)$ -pretzel knot – the knot we’ve been looking for.

So now all we need to do is find a knot K such that $\Delta_K(t) = 1$ and $s(K) \neq 0$, as a knot is smoothly slice if and only if its slice genus $g_*(K)$ is zero. If $s(K) \neq 0$ then we have an obstruction to K being smoothly slice, since $|s(K)| \leq 2g_*(K)$ by Theorem 3.17. The simplest example is the $(-3, 5, 7)$ -pretzel knot K (illustrated in Figure 4.1), which is known to have Alexander polynomial equal to 1. We will show that $s(K) \neq 0$ using the following result, which requires a few notions from the theory of braids (which, for the sake of brevity, we will not introduce here).

Proposition 4.10. [13, Proposition 1.F] *Let K be a knot which can be represented as the closure of a braid β of the form*

$$(w_1 \sigma_{j_1} w_1^{-1})(w_2 \sigma_{j_2} w_2^{-1}) \dots (w_b \sigma_{j_b} w_b^{-1})$$

where $\sigma_{j_i}, i = 1, \dots, b$ are the standard generators of the corresponding braid group, and where the w_i are braid words. Suppose, moreover, that β has k strands. Then

$$s(K) = b - k + 1.$$

(The proof of this proposition invokes an inequality known as the *Slice-Bennequin inequality*, which was originally proved using gauge theory. In [13] Shumakovitch provides a more immediate proof of the Slice-Bennequin inequality by applying properties of the s -invariant to the writhe $w(D)$ of a knot diagram D .)

A knot satisfying the hypotheses of Proposition 4.10 is said to be *quasipositive*. The $(-3, 5, 7)$ -pretzel knot can be shown to be quasipositive (in fact, it is what is known as *strongly* quasipositive, but quasipositivity is sufficient for our purposes) as the closure of the braid

$$\beta = \sigma_1 \sigma_2 \sigma_{2,4} \sigma_{3,6} \sigma_{1,4} \sigma_5 \sigma_{2,5}$$

illustrated in Figure 4.2, which consists of $k = 6$ strands and $b = 7$ factors of the form $w\sigma_i w^{-1}$, as seen in its explicit presentation.

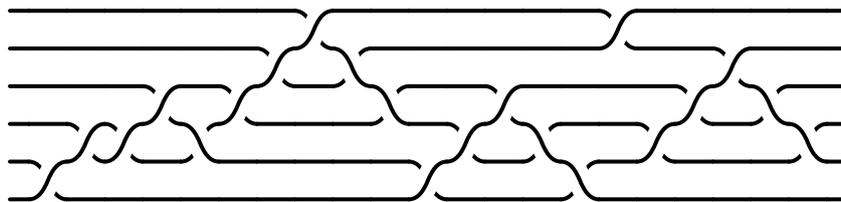


Figure 4.2. The braid β whose closure is the $(-3, 5, 7)$ -pretzel knot.

Thus, applying Proposition 4.10, we see that

$$s(K) = 7 - 6 + 1 = 2 \neq 0$$

for K the $(-3, 5, 7)$ -pretzel knot and hence $g_*(K) \neq 0$. Thus K is not smoothly slice. Combining these results with Theorem 4.8, we can finally show the existence of exotic \mathbb{R}^4 .

Corollary 4.11. *There exists a large exotic \mathbb{R}^4 .*

Note that the $(-3,5,7)$ -pretzel knot is actually not so special: It turns out that there are at least 82 knots with up to 16 crossings that have Alexander polynomial 1 and a non-zero value for the s -invariant. See [13] for a full list of these knots. Regardless, we have a Khovanov homology proof of the existence of exotic \mathbb{R}^4 , relying only on concepts developed in this paper.

5 Appendix: Spectral sequences

The purpose of this appendix is to give the reader an idea of where the spectral sequence in Section 3.1 comes from, and how the filtration induced by the q -grading gives rise to that spectral sequence. We follow the introductory treatment given in [11].

Definition 5.1. A *differential bigraded module* over a ring R is a collection of R -modules $\{E^{p,q}\}$, where p, q are integers, together with an R -linear mapping $d : E^{*,*} \rightarrow E^{*,*}$, called the differential, of bidegree $(s, 1 - s)$ or $(-s, s - 1)$ for some integer s , satisfying $d^2 = 0$. The *homology* of a differential bigraded module is given by

$$H^{p,q}(E^{*,*}, d) = \text{Ker}(d : E^{p,q} \rightarrow E^{p+s,q-s+1}) / \text{Im}(d : E^{p-s,q+s-1} \rightarrow E^{p,q}).$$

Definition 5.2. A *spectral sequence* is a collection of differential bigraded R -modules $\{E_r^{*,*}, d_r\}$ where r is a positive integer, the differentials are either all of bidegree $(-r, r - 1)$ (for a spectral sequence of *homological type*) or all of bidegree $(r, 1 - r)$ (for a spectral sequence of *cohomological type*), and for all p, q, r , $E_{r+1}^{p,q}$ is isomorphic to $H^{p,q}(E_r^{*,*}, d_r)$.

Note that the spectral sequence used in Section 3 is of cohomological type. Next, we fix some terminology. Let $Z_r = \text{Ker}(d_r)$, $B_r = \text{Im}(d_r)$. Suppose we are given the differential bigraded module $E_2^{*,*}$ (spectral sequences usually start with the bigraded module corresponding to $r = 2$). We say that an element of $E_2^{*,*}$ which lies in Z_r *survives to the r^{th} stage*, having been in the kernel of the previous $r - 2$ differentials. The submodule B_r of $E_2^{*,*}$ is the set of elements which are *boundaries by the r^{th} stage*. The bigraded module $E_r^{*,*}$ is called the E_r -*term* or the E_r -*page* of the spectral sequence.

Let $Z_\infty = \bigcap_{n \in \mathbb{N}} Z_n$ be the submodule of $E_2^{*,*}$ consisting of elements that *survive forever*, i.e. elements which are cycles at every stage. We also consider the submodule $B_\infty = \bigcup_{n \in \mathbb{N}} B_n$ consisting of the elements which *eventually bound*. We claim that $B_\infty \subset Z_\infty$. One can see from this from the following tower of inclusions, the construction of which follows simply by considering the definitions of the submodules B_n and Z_n :

$$B_2 \subset B_3 \subset \cdots \subset B_n \subset \cdots \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_3 \subset Z_2 \subset E_2.$$

Thus $E_\infty = Z_\infty / B_\infty$ is the bigraded module which remains after the computation of an infinite sequence of successive homologies. However, sometimes we may only need a finite number of such computations: We say that a spectral sequence *collapses at the N^{th} term* if $d_r = 0$ for all $r \geq N$. In this case, the computation ends at a finite stage and we have $E_\infty = E_N$.

Definition 5.3. A *filtration* F^* on an R -module A is a family of submodules $\{F^p A\}$ for $p \in \mathbb{Z}$ such that

$$\begin{aligned} \cdots \subset F^{p+1} A \subset F^p A \subset F^{p-1} A \subset \cdots \subset A \quad (\text{the filtration is } \textit{decreasing}), \text{ or} \\ \cdots \subset F^{p-1} A \subset F^p A \subset F^{p+1} A \subset \cdots \subset A \quad (\text{the filtration is } \textit{increasing}). \end{aligned}$$

Given a filtration F^* , we can collapse a filtered module to its *associated graded module* $E_0^*(A)$, given by

$$E_0^p(A) = \begin{cases} F^p A / F^{p+1} A & \text{if } F^* \text{ is decreasing} \\ F^p A / F^{p-1} A & \text{if } F^* \text{ is increasing} \end{cases}.$$

If H^* is a graded R -module with a filtration, we can consider the filtration on each degree by letting $F^p H^n = F^p H^n \cap H^n$. Thus the associated graded module becomes bigraded by defining

$$E_0^{p,q}(H^*, F) = \begin{cases} F^p H^{p+q} / F^{p+1} H^{p+q} & \text{if } F^* \text{ is decreasing} \\ F^p H^{p+q} / F^{p-1} H^{p+q} & \text{if } F^* \text{ is increasing} \end{cases}.$$

Definition 5.4. A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to *converge* to H^* , a graded R -module, if there is a filtration F^* on H^* such that

$$E_\infty^{p,q} \cong E_0^{p,q}(H^*, F)$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence.

Definition 5.5. An R -module A is a *filtered differential graded module* if

(1) A is a direct sum of submodules, i.e.

$$A = \bigoplus_{n \in \mathbb{N}} A^n;$$

(2) There is an R -linear mapping $d : A \rightarrow A$ of degree $+1$ (so that $d : A^n \rightarrow A^{n+1}$) or of degree -1 (so that $d : A^n \rightarrow A^{n-1}$) satisfying $d^2 = 0$; and

(3) A has a filtration F^* and the differential d *respects the filtration*, i.e. $d : F^p A \rightarrow F^p A$.

Now we can state (without proof) the main theorem needed to justify the developments in Section 3. To this end, we suppose A is a filtered differential graded module with a differential d of degree $+1$ and with a decreasing filtration F^* (so we are working in the setting of cohomology).

Theorem 5.6. *Each filtered differential graded module (A, d, F^*) determines a spectral sequence $\{E_r^{*,*}, d_r\}$, $r = 1, 2, \dots$ with d_r of bidegree $(r, 1 - r)$ and with $E_1^{p,q} \cong H^{p+q}(F^p A / F^{p+1} A)$. Suppose further that the filtration is bounded (i.e. it is of finite length, as in Definition 3.3). Then the spectral sequence converges to $H(A, d)$; that is, we have*

$$E_\infty^{p,q} \cong F^p H^{p+q}(A, d) / F^{p+1} H^{p+q}(A, d).$$

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