From Classical to Unwelded - An Examination of Four Knot Classes

FROM CLASSICAL TO UNWELDED - AN EXAMINATION OF FOUR KNOT CLASSES

BY

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To my father, the late Richard Parchimowicz, without whose guidance as a young man I would never have found my way to academia.

Abstract

This thesis is an introduction to virtual knots and the forbidden moves, and the closely related classes of welded and unwelded knots. Extensions of the Jones polynomial and the knot group to the various knot types are considered. We also examine the operation of connected sum for virtual and welded knots, and we review the proof that every virtual knot can be untied using the forbidden moves.

Acknowledgements

My utmost gratitude goes to my thesis advisor, Dr. Hans Boden for allowing me the chance to work with him. Four years ago, when I was an undergraduate Kinesiology student, it was Dr. Boden's introductory course in Topology, his expertise, kindness, and above all patience as an instructor that helped me to realize my potential as a Mathematics student. At the Masters level, he has remained a kind and patient advisor, and has also provided me with meticulous feedback and revisions to this thesis. I believe that one of the best parts of this 2-year program was the opportunity to meet with Dr. Boden on a regular basis and just chat about knot theory. Without his constant support and influence this thesis would not have been possible.

I would also like to express my gratitude to my sources of moral support. To my girlfriend of many years Lindsay Kueh, without whose moral support I would have given up years ago. As a math major herself, she would always support my efforts in undergraduate math courses and helped me to succeed. She had also gently persuaded me to pursue a Masters degree and is one of the key figures in my life who helped me to make the tough decision to switch to math. Even now she continues to motivate me. She has been a true friend.

Lastly, I would like to acknowledge my family members, Renata, Andrea, Patrick and Anne-Marie Parchimowicz for creating an uplifting and positive environment. Their constant moral support during an otherwise difficult time was invaluable.

Notation and abbreviations

`Basic Notation`

K_i - denotes both, a knot (in $\mathbb{R}^3$) as well as a knot diagram

$\bigcirc$ - the knot type of the knot with no crossings, the unknot

3_1 - the classical trefoil

3_1^* - the virtual trefoil

`Equivalence`

$\sim_C$ - Classical Isotopy (equivalence under the three Reidemeister moves)

$\sim_V$ - Virtual Isotopy (equivalence under the Reidemeister and virtual moves)

$\sim_W$ - Welded Isotopy (equivalence under the Reidemeister, virtual moves, and F_t)

$\sim_U$ - Unwelded Isotopy (equivalence under all of the above, as well as F_h)

`Spaces`

$\mathscr{D}_C$ - the set of all knot diagrams with no virtual crossings

$\mathscr{D}_G$ - the set of all Gauss diagrams

$\mathscr{D}_V$ - the set of all virtual knot diagrams

$\mathscr{C} = \mathscr{D}_C / \sim_C$ - the class of classical knot types

$\mathscr{V} = \mathscr{D}_V / \sim_V$ - the class of virtual knot types

$\mathscr{W} = \mathscr{D}_W / \sim_W$ - the class of welded knot types

$\mathscr{U} = \mathscr{D}_U / \sim_U$ - the class of unwelded knot types

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Chapter 1

Introduction

In 1996, Louis Kauffman initiated the study of virtual knots [16], and since then many different aspects of virtual knots have been explored by various authors. For example, in their study of finite type invariants of knots [11], Goussarov, Polyak and Viro showed how virtual knots arise naturally when considering Gauss diagrams, and they established an important result that implies that classical knot theory embeds faithfully into the theory of virtual knots. It follows that any invariant of virtual knots restricts to an invariant of classical knots.

While virtual knots share many properties and features established first in the classical setting, there are also some notable differences. For example, the forbidden moves, which are semi-virtual analogues of the third Reidemeister move, change the virtual knot type in a drastic way. Indeed, upon allowing both forbidden moves, every virtual knot diagram can be untied [15, 27]. This thesis gives an examination of virtual knots, the forbidden moves, and two invariants: the Jones polynomial and the knot group.

Throughout this thesis, the word "knot" may refer to either a *knot diagram*, or a *knot type*, an equivalence class of knot diagrams depending on its context. We consider four knot classes, namely *classical*, *virtual*, *welded* and *unwelded*.

Chapter 2 is an introduction to the four knot classes. It begins with the definition of classical knots, the idea of being untied, or trivial and their notion of equivalence. We define virtual knots by way of Gauss diagrams, and virtual knots give an extension of the classical knots roughly analogous to the way $\mathbb{C}$ is an extension of $\mathbb{R}$. Interestingly, the classical knots embed into the virtual knots, [11] and we present a proof of this fact. We then introduce the forbidden moves, F_t and F_h, and two additional classes of knots, the welded knots and the unwelded knots.

Chapter 3 introduces the Jones polynomial $V_K(t)$ by way of the bracket polynomial and how the algorithm for computing the Jones polynomial extends to virtual knot diagrams. We present elementary examples of virtual knots K with $V_K(t) = 1$, effectively showing that the Jones polynomial does not detect the virtual unknot. We introduce the *n-stranded Jones polynomial* and take a collection of such polynomials to define a more powerful knot invariant. We conclude the chapter by showing that the Jones polynomial does not extend to a well-defined invariant of welded knots.

Chapter 4 concerns the knot group definition via Wirtinger presentations. This construction extends naturally to Gauss diagrams and hence any virtual knot. Classical knot groups have *deficiency* 1 and satisfy a number of other constraints, while virtual knot groups form a much richer class. Indeed, the *group realization* result of Se-Goo Kim [21] shows that every group with a Wirtinger presentation of deficiency 0 or 1 arises as the knot group of some virtual knot. We present several examples to illustrate this point.

Chapter 5 introduces the connected sum of classical knots. A beautiful result in classical knot theory is the prime decomposition theorem, saying that every knot factors as a connected sum of *prime* knots under knot decomposition, with uniqueness of factors up to permutation. Unfortunately, this property does not extend to virtual knots, and the reason is that the connected sum depends on the knot diagram and the choice of basepoints. We present a series of examples illustrating the pathologies that arise from the connected sum of virtual knots.

Chapter 6 concerns the forbidden moves and their effect on the knot group. We see that F_t leaves the knot group fixed, while F_h generally alters it. We then summarize the relationships between the four knot classes. For example, we strengthen the result in Chapter 2 and show that classical knots embed into welded knots, and we provide a proof that every virtual knot can be untied using the two forbidden moves. This implies that every unwelded knot is equivalent to the unknot. We conclude with a discussion of open questions and give additional references for the curious reader.

Chapter 2

Virtual Knots

2.1 Classical Knots

By a **Classical Knot** (or **Knot** if not otherwise specified) K we mean a simple closed polygonal curve in $\mathbb{R}^3$. The ordered set of vertices of the knot $(p_0, p_1, ..., p_{n-1})$ uniquely defines the knot, namely it is the union of the closed line segments $\cup_i [p_i, p_{i+1}]$ where i is an integer modulo n. We obtain an **Oriented Knot** by appropriately ordering its vertices. We can represent this information by assigning the knot an arrow indicating the direction of travel. An (oriented) **n-component Link** is the disjoint union of n non-intersecting (oriented) knots.

Since we are mainly interested in the topology of knots, we can replace the line segments with smooth curves *isotopic* to these line segments called **Strands**. Care must be taken to assure that the strands do not pass through each other, which is why we require the curves to be isotopic (a family of continuous *bijections*) and not simply homotopic to the initial strands.

A **Knot Diagram** is the image under a **Regular Knot Projection** $\pi : K \to P$ of the knot onto any plane $P \subset \mathbb{R}^3$. We will use the terms *Link Diagram* and *Link Projection* interchangably if we are currently working with links. A knot projection is called **Regular** if the pre-image of any point in the plane contains at most 2 points (i.e. $\#(\pi^{-1}(x)) \leq 2 \ \forall \ x \in P$). A knot diagram for an oriented knot inherits an orientation. If a knot does not have a regular projection, then there is a knot isotopic to it which does.

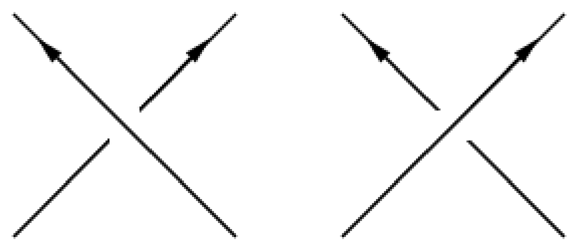


Figure 2.1: A left, and right handed crossing

In order to avoid ambiguity, we replace each double point with an over or under-crossing carefully representing the strand which passes above the other. Each crossing will be given a **Left-handed** or **Right-handed** designation as in Figure 2.1.

2.2 Knot Equivalence

In order to study knots and decide whether knots are "the same," we need to define some notion of equivalence. Intuitively, we say that two knots are equivalent if they can be continuously deformed into each other without allowing the strands to pass through each other. We say such equivalent knots K_1, K_2 are **Ambient Isotopic**, and that they are of the same **Knot Type**.

If there is a projection π for a knot K such that the knot diagram associated to K and π has no crossings, then we say that $K = \bigcirc$ is **Untie d** or **The Unknot**. A challenging task in knot theory is determining whether or not a given knot is equivalent to $\bigcirc$ and in general, the equivalence of any two arbitrarily complicated knots. There are knot diagrams for $\bigcirc$ with an arbitrarily high number of crossings, such as a rubber band with n half-twists. Intuitively, we may be able to 'pull apart'

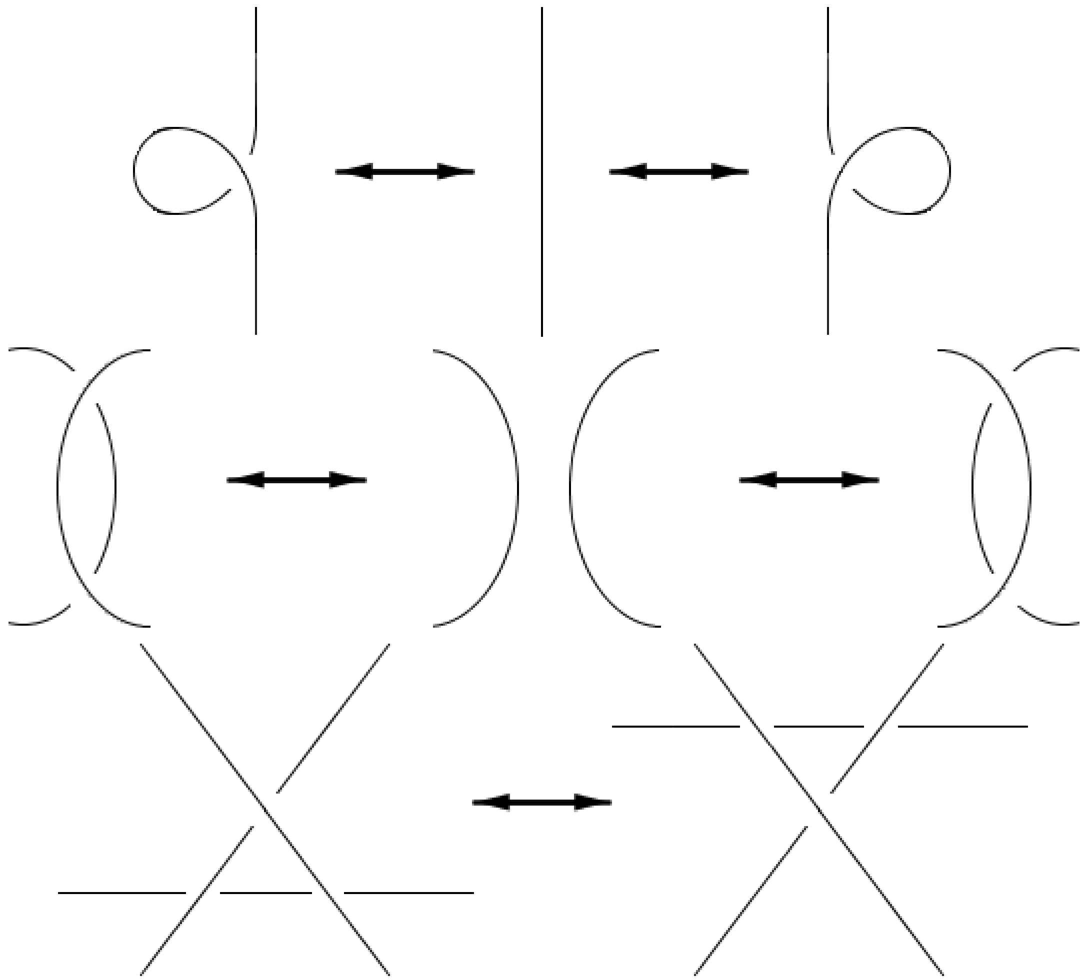


Figure 2.2: The Reidemeister moves

knots and simplify them immensely, in this case we simply untwist the rubber band and reveal ◯. This intuition translates into 3 operations which can be performed on knot diagrams (see for instance, [29]):

Theorem 2.1. *K_1 and K_2 are Ambient Isotopic (ie. Equivalent) if and only if their knot diagrams are related to each other by a sequence of* ***Reidemeister Moves****. (Figure 2.2) Isotopy defines an equivalence relation, which we denote by $K_1 \sim_C K_2$.*

These moves provide necessary and sufficient conditions to check for equivalence, and hence they play a pivotal role in defining knot invariants. The problem is that if we cannot find such a sequence, we can't rule out the possibility that we aren't being clever enough. Namely, if we designed computer software to repeatedly perform these moves, if $K_1 \not\sim_C K_2$, the program would never terminate. And even if $K_1 \sim_C K_2$, there is no telling how long this algorithm would need to run. Ideally, knot theorists wish to find a process which terminates even if the knots are not equivalent, or at least some upper bound on how long the program would need to run.

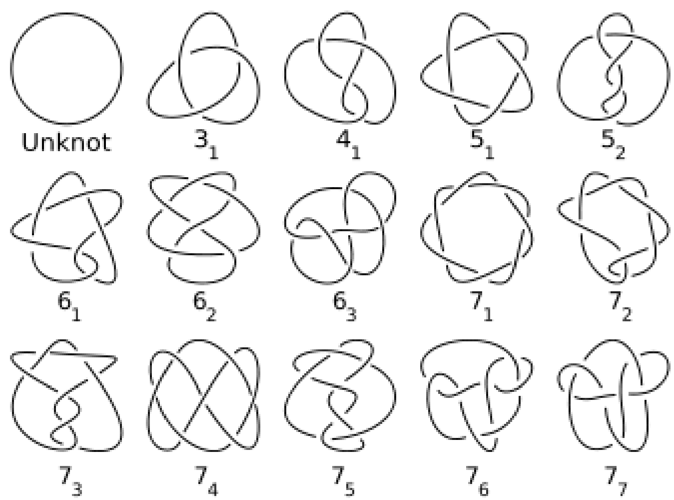


Figure 2.3: Table of knots with at most 7 crossings [25]

2.3 Gauss Codes

From a given knot diagram K, we can label it in a special way, producing something called a **Gauss Code**. We do so with the following algorithm:

1. Choose any basepoint.
2. Trace along the knot following the appropriate orientation and label the first crossing as Crossing 1.

3. Take note of whether the crossing is over (O) or under (U) with respect to the incoming strand.

4. Take note of the local orientation (±) of the crossing

5. Continue tracing the knot and labelling each subsequent crossing with the next consecutive integer, repeating (2)-(4).

6. Proceed in this way until you have returned to the basepoint.

We will call a specific crossing along with designated over/under-crossing information and orientation a **Marker** (ie. O1- is a marker). Since each crossing involves an overstrand and an understrand, crossing i should appear in our Gauss code exactly twice (O$i\pm$ and U$i\pm$), and so if our knot has n crossings then our Gauss code will have precisely $2n$ markers. It is easy to see that for a given labelling of crossings, the Gauss code is unique (up to a cyclic permutation of some markers). For this reason, we will propose that every Gauss code should begin with the Marker O1±.

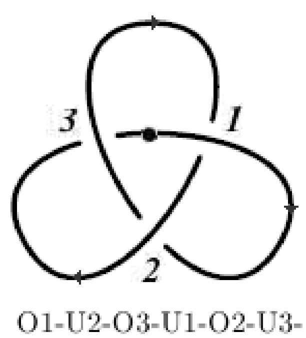


Figure 2.4: The Gauss code for the left-handed trefoil

What is less obvious is that from a given Gauss code, we can recover the knot diagram corresponding to it, and that this is well-defined. In other words, no matter how we choose to do it, we will always get a knot diagram representative from a unique knot type. Suppose our Gauss code has $2n$ labels. Draw n crossings and label them, 1 to n. Beside each number is either a + or -, and given this we can assign each crossing the correct orientation. Then connect the strands accordingly, as dictated by the ordering of the labels in the Gauss code.

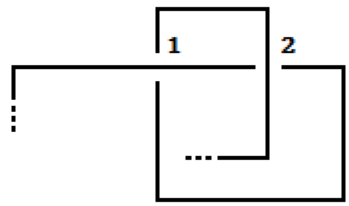


Figure 2.5: O1+U2+U1+O2+ gives rise to this knot diagram

An interesting thing occurs: Not every conceivable Gauss code corresponds to a knot that we can draw in the plane. O1+U2+U1+O2+ is such a Gauss code (Figure 2.5). In order to connect the first and last crossing, we need to introduce another crossing (which would change the Gauss code). For this reason, we allow ourselves to momentarily leave the plane in our diagrams, and create crossings which are *not really there*. We will call such crossings **Virtual Crossings** and represent them as in Figure 2.6.

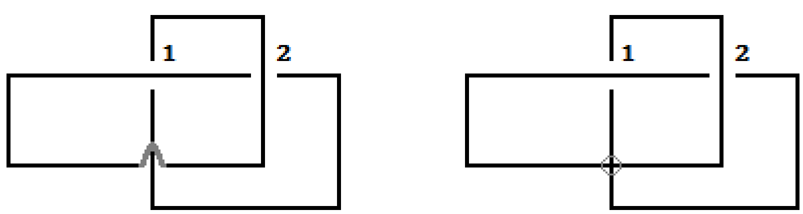
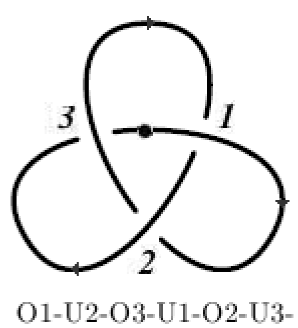


Figure 2.6: A virtual crossing

2.4 Gauss Diagrams

In this section, we introduce Gauss diagrams which encode the crossings of a knot diagram in a pictorial and combinatorial way.

Every Gauss code corresponds to a **Gauss Diagram**. We begin with a circle with $2n$ chosen points. Selecting any vertex as our starting point, we write each label of our Gauss code on a point in counterclockwise order. We then connect the points corresponding to the same crossing with a line segment. To make things a bit cleaner, we can assign each line segment an arrow, pointing in the direction of the under-crossing.



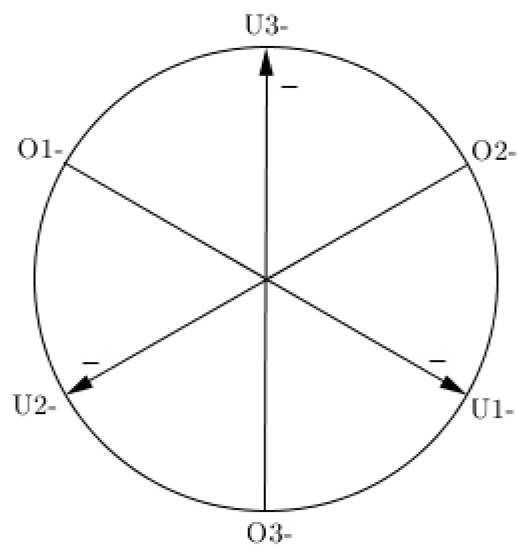


Figure 2.7: The trefoil and its corresponding Gauss diagram

It is necessarily true that every knot diagram has a corresponding Gauss diagram. In fact, upon translating the Reidemeister moves into moves on Gauss diagrams, one obtains a corresponding notion of equivalence. The moves transform into those shown in Figure 2.8, where the diagrams imply that anything not shown is identical on both sides.

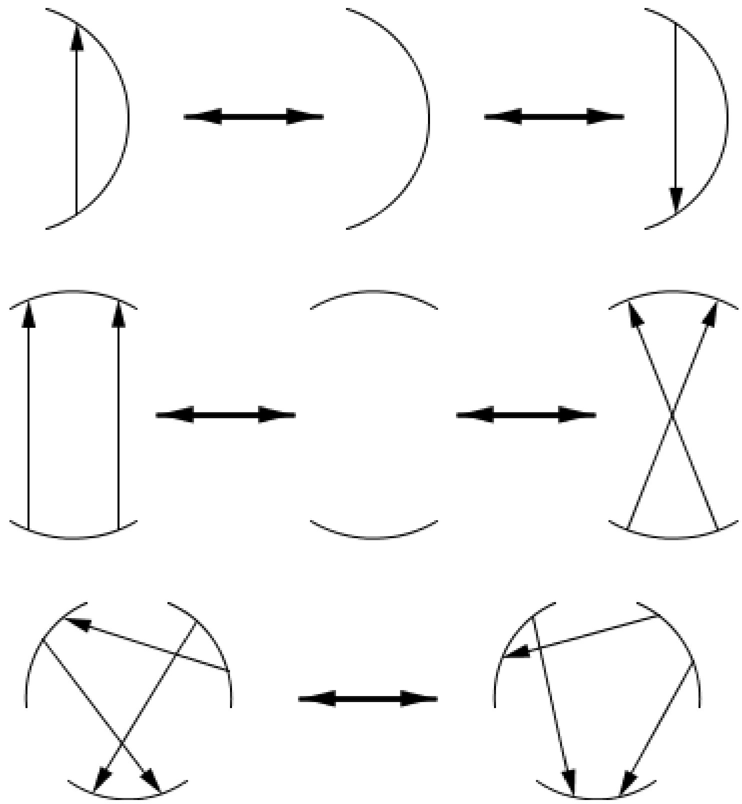


Figure 2.8: The Reidemeister moves induce these actions on Gauss diagrams

2.5 Virtual Knots

We wish for two Gauss diagrams to represent the same virtual knot if and only if the diagrams are related by a sequence of GR-moves. Hence we define a **Virtual Knot** to be an equivalence class of Gauss diagrams under the Gaussian Reidemeister moves. We will call such a sequence a **Gaussian Isotopy** or **Virtual Isotopy**. To each Gauss diagram, we associate a **Virtual Knot Diagram** by attempting to draw a knot diagram, and filling in virtual crossings (Figure 2.6) as necessary to complete the diagram.

We can now define a new set of allowable operations on virtual knot diagrams (ie. those involving virtual crossings). There are three standard **Virtual Moves** and one **Semi-Virtual Move** which commonly appear in virtual knot theory papers. We will refer to these four operations collectively as the V-moves. These are chosen because they closely resemble the classical Reidemeister moves, but involve virtual crossings (Figure 2.9). These operations do not change things on the level of Gauss diagrams, since they do not alter the number of crossings, their orientations or their order of traversal.

Remark 2.2. *While Gauss diagrams and virtual knot diagrams are technically different objects, they are closely associated. It is not true that every Gauss diagram represents a unique virtual knot diagram associated to it, as virtual crossing information is not encoded anywhere in the Gauss diagram. However, there is a bijection from the equivalence classes of Gauss diagrams onto the equivalence classes of virtual knot diagrams. In this way, we can use Gauss diagrams and virtual knot diagrams interchangeably, and we can simply choose representatives of each when convenient.*

Two equivalent classical knot diagrams, K_1, K_2 will be related by a sequence of R-moves, and hence GR-moves. However we can regard any classical knot as a virtual knot, by simply considering the equivalence class of its representative Gauss diagram. Under this new interpretation, for $K_1 \sim_C K_2$, we may be able to find a sequence of Gauss diagrams $K_1, V_1, ...V_n, K_2$, related by GR-moves, relating two classical knots, whose intermediate Gauss diagrams specify virtual knots. For example, $\bigcirc$ can be transformed into U1+U2-O1+O2- by using GR-II. This is a Gauss diagram representing the virtual trefoil 3_1^*, a knot which cannot be drawn without using a virtual crossing. The concern is then that two inequivalent classical knot diagrams may become equivalent in the larger space of virtual knot diagrams. This is not the case, as we shall see in Theorem 2.4.

We state a fundamental result of F. Waldhausen [[5], Theorem 3.15], necessary for the proof of Theorem 2.4:

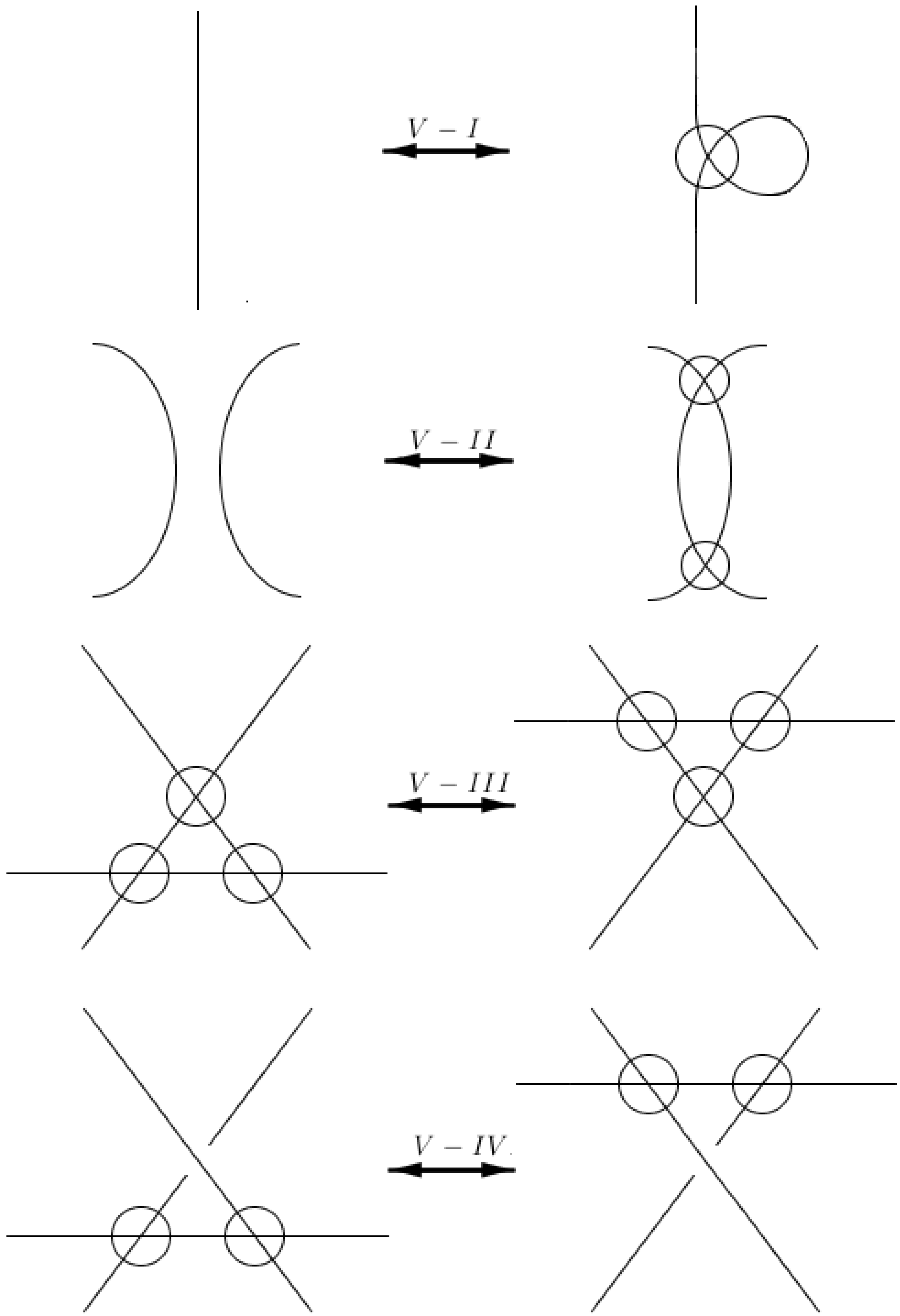


Figure 2.9: The three virtual moves and semi-virtual move (Bottom)

Theorem 2.3. *Two knots, K_1 and K_2 in S^3 with peripheral systems $(G_i, \mu_i, \lambda_i), i = 1, 2$ represent the same knot type if there is an isomorphism of groups $\phi : G_1 \to G_2$ with the property that $\phi_{\mu_1} = \mu_2$ and $\phi_{\lambda_1} = \lambda_2$.*

Theorem 2.4 ([11])**.** $K_1 \sim_C K_2 \iff K_1 \sim_V K_2$*, Two classical knot diagrams K_1 and K_2 are virtually isotopic if and only if they are equivalent as classical knots.*

Proof. The *Group System* of a knot is defined to be a triple (G, μ, λ), where G is its knot group (we will introduce this in Section 4) paired with its peripheral structure (see for instance, [16]). This entails a choice of meridian μ and longitude λ for the knot. The knot group extends to the world of virtual knots, since we can compute the knot group and peripheral structure for any virtual knot. Since the Group System is a complete invariant by Theorem 2.4, the result is clear. □

As a consequence, we see that the class of classical knots embeds in the class of virtual knots. As a result, it makes sense to ask the following question:

Problem 2.5. *Which invariants of classical knots extend in a sensible way to invariants of virtual knots?*

We will examine this question for some knot invariants in the upcoming chapters.

2.6 The Forbidden Moves

We conclude the Chapter by introducing some notation which will help to distinguish between various knot types. We will also define two new types of knot classes, namely the welded and unwelded knots. This is summarized in the Notation section (Page vi) for future reference.

- Let $\mathscr{D}_C$ be the set of all knot diagrams with no virtual crossings
- Let $\mathscr{D}_V$ be the set of all virtual knot diagrams

- Let $\sim_C$ denote **Classical Isotopy**, that is, equivalence under the three Reidemeister moves. If we consider the set of equivalence classes of knot diagrams under the Reidemeister moves, we obtain the **Class of Classical Knots**, $\mathscr{C} = \mathscr{D}_C / \sim_C$.

- Let $\sim_V$ denote **Virtual Isotopy**, that is, equivalence under the Reidemeister and virtual moves. The **Class of Virtual Knots** is the set of equivalence classes of $\mathscr{V} = \mathscr{D}_V / \sim_V$.

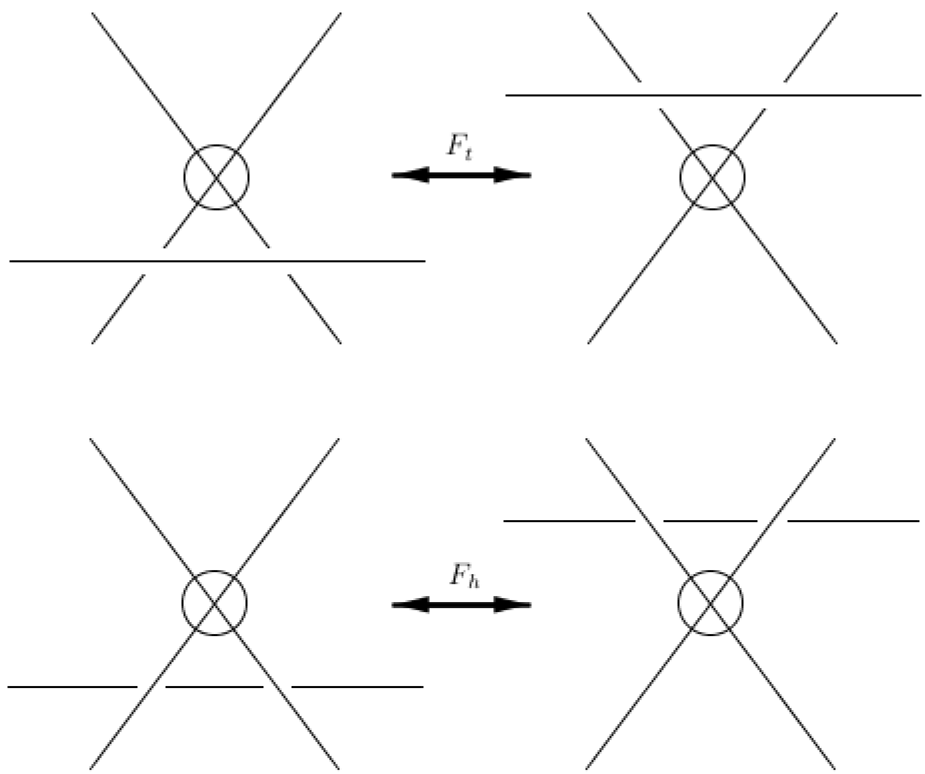


Figure 2.10: The forbidden moves F_t and F_h

There are two additional operations on virtual knot diagrams that closely resemble the virtual moves (Figure 2.10). One might expect that these moves are allowable, given that they are semi-virtual versions of Reidemeister-III.

These so-called forbidden moves are non-trivial operations on the level of Gauss diagrams (Figure 2.11), each one interchanges the order of two crossings. Indeed, F_h interchanges two adjacent arrow heads, and F_t interchanges two adjacent arrow tails.

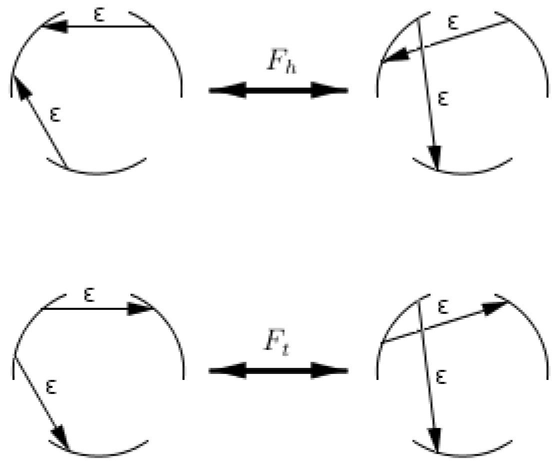


Figure 2.11: The effect of the forbidden moves on Gauss diagrams

At first glance the forbidden moves may seem harmless, but they change the knot type in a catastrophic way. To illustrate this, we explain how to untie the trefoil by using the forbidden moves (Figure 2.12). We will give a more general argument in Section 6.2 showing that this can be done for any virtual knot diagram.

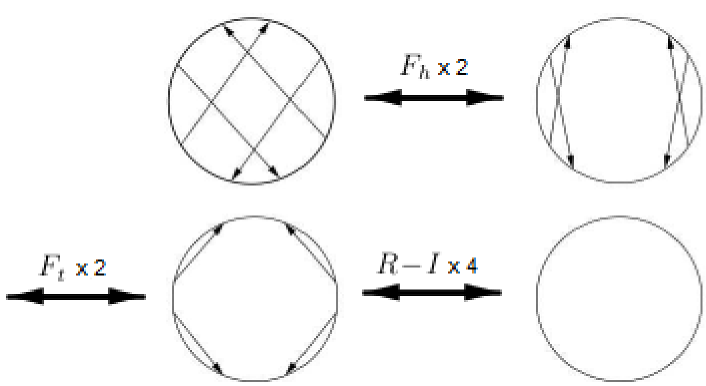


Figure 2.12: Untying the trefoil

If in addition to the Reidemeister and virtual moves, we allow the forbidden move F_t then we form a class of knots known as the **Welded Knots**. If we additionally allow the second forbidden move, then we form a class of knots called the **Unwelded Knots**.

- Let $\sim_W$ denote **Welded Isotopy**, that is, equivalence under the Reidemeister and virtual moves, and F_t. The **Class of Welded Knots** is the set of equivalence classes $\mathscr{W} = \mathscr{D}_W / \sim_W$
- Let $\sim_U$ denote **Unwelded Isotopy**, that is, equivalence under all of the above, as well as F_h. The **Class of Unwelded Knots** is the set of equivalence classes $\mathscr{U} = \mathscr{D}_U / \sim_U$

Chapter 3

The Jones Polynomial

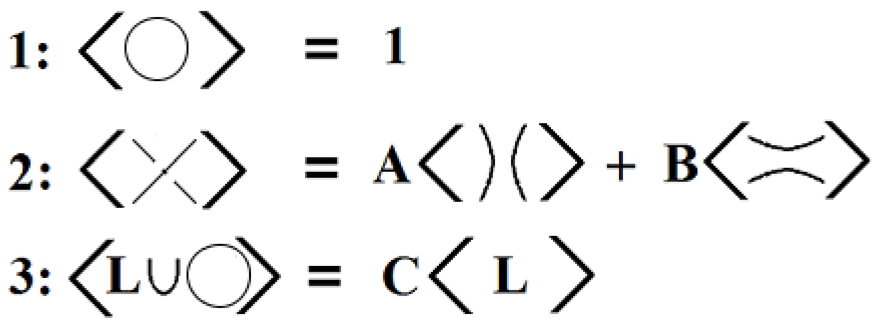
Informally, a knot invariant is like a ruler for measuring knots. Rulers, while capable of distinguishing many things are not perfect: upon measuring two people and realizing that they have different heights, it is clear that they must be different people. However, it is possible to find two different people with the same height. We would say the ruler has *failed to distinguish* these individuals. If we wanted to tell these individuals apart, we would need to look at a different set of criteria, perhaps using a different measuring device such as a weight scale. Analogously, knot theorists are actively producing new and improved knot invariants which can distinguish more and more knots.

A **Classical Knot Invariant** is a function g mapping $\mathscr{D}_C$ into some other space, which is well defined on $\mathscr{C}$. It is a function on knot diagrams which evaluates to the same thing on equivalent knots, that is, if g is a knot invariant, then $K_1 \sim_C K_2 \implies g(K_1) = g(K_2)$. Put yet another way, this means that g must be the same for any two knots which differ by an application of Reidemeister moves. Hence, it suffices to check such invariance if we wish to show a given function is in fact a knot invariant. The goal of many knot theorists is to find a computable knot invariant that faithfully distinguishes all knots.

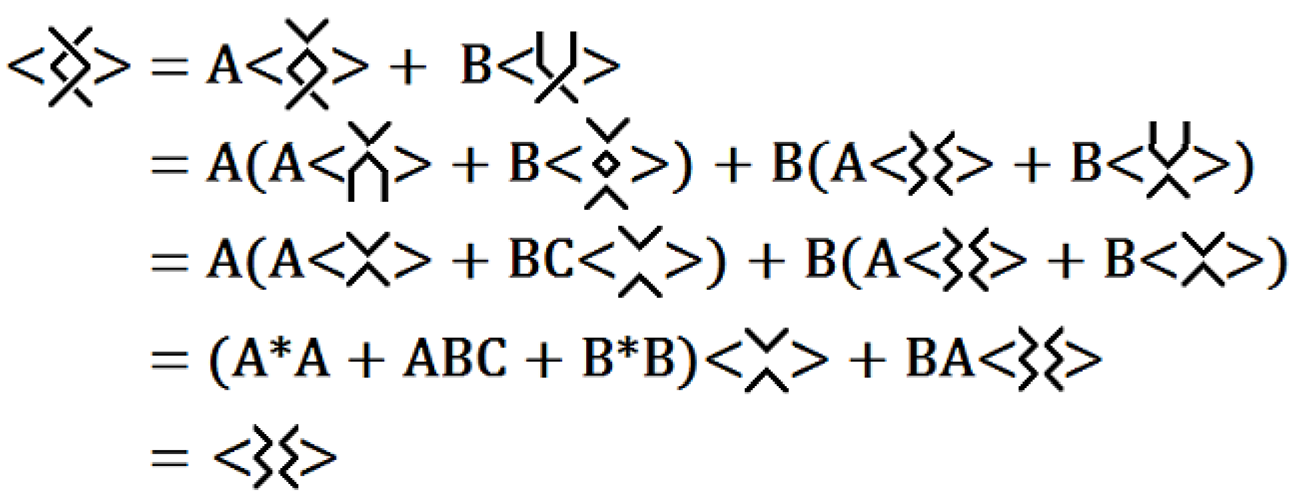
We can make analogous definitions for **Virtual** and **Welded Knot Invariants**, by simply demanding that be well defined under the appropriate operations (ie. the Reidemeister moves, virtual moves, and F_h for welded knots). We will see in Section 6.2 that every unwelded knot is trivial, and hence we do not define invariants for unwelded knots.

3.1 Classical Knots

One invariant of particular interest is the so called **Bracket Polynomial**, which we will denote by $< L >$ for a given link diagram L. $< >$ is anything which satisfies the following Skein Relations:

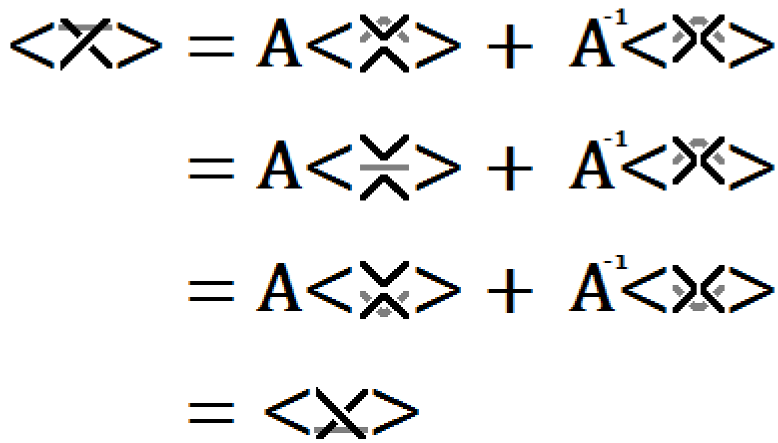


In order for $< >$ to be a knot invariant, it's a necessary condition to remain unchanged under the three Reidemeister moves. Beginning with R-II:

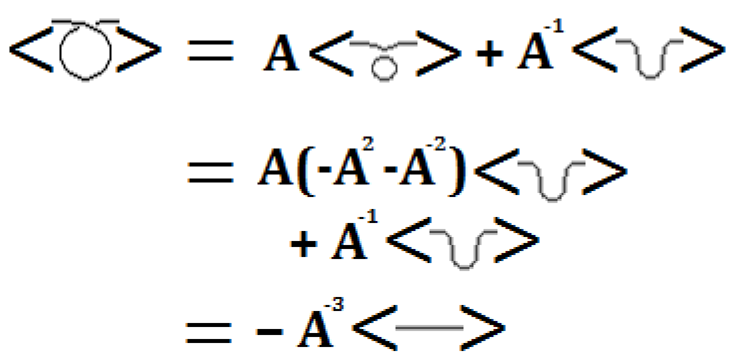


In order for the last equality to hold, we need to make an appropriate choice for A, B and C. The necessary choice is to let $A = B^{-1}$ and then $(A^2 + C + A^-2)$ must vanish, so $C = -A^2 - A^{-2}$.

Once we've established that the bracket is R-II invariant, R-III invariance follows rather nicely:



Unfortunately our polynomial is not quite a knot invariant, as it fails preservation under R-I. Upon each application of R-I, we obtain an extra coefficient of $-A^{\pm 3}$:



We can normalize the Bracket to avoid this problem however. Associate to each crossing an integer +1 if the crossing has positive orientation, and -1 if it has negative orientation. We define the **Writhe** of a link diagram, $w(L)$ to be the sum of these integers. The $w(L)$ is left fixed by R-II since each application of R-II either adds two crossings of opposite orientation, or removes two such crossings. Likewise it is fixed by R-III because upon its application we fail to change the number of crossings or the local orientations of any of the crossings.

R-I however, introduces or removes an additional crossing. This can be positively or negatively oriented, and so R-I always changes the writhe by ± 1. We can then define the **Normalized X-Polynomial** by $X_L(A) = (-A^3)^{-w(L)} < L >$.

Theorem 3.1. $X(\ )$ *is a knot invariant.*

Proof. R-II and R-III invariance follow from the fact that both $w(\ )$ and $< >$ are invariant under R-II and R-III. For R-I, suppose we have two link diagrams, L and L', differing only by an application of a single R-I move (Figure 3.1), which produces a positively oriented crossing. That is, $w(L') = w(L) + 1$.

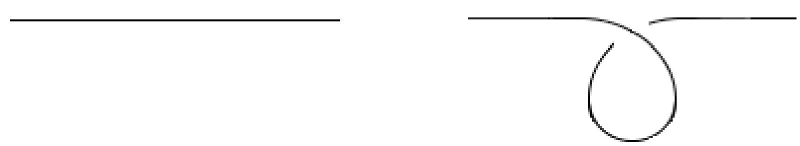


Figure 3.1: L and L'

Hence:

$$\begin{aligned} X_{L'}(A) &= (-A^3)^{-w(L')} < L' > \\ &= (-A^3)^{-w(L)+1} < L' > \\ &= (-A^3)^{-w(L)+1}(-A^3 < L >) \\ &= (-A^3)^{-w(L)} < L > \\ &= X_L(A) \end{aligned}$$

□

The **Jones Polynomial** $V_K(t)$ is a knot invariant which assigns to each knot or link diagram a Laurent polynomial. It can be obtained from the X-polynomial by substituting $A = t^{-\frac{1}{4}}$.

We can compute the Jones polynomial for a given knot diagram by creating a **Resolving Tree**. We systematically apply the Skein Relations until the knot diagram is a disjoint union of trivial knots or links, to find $< K >$. Then we normalize to obtain $X_K(A)$, and finally make the required substitution. A rather easy example to compute is that of the unknot, $V(\bigcirc) = 1$.

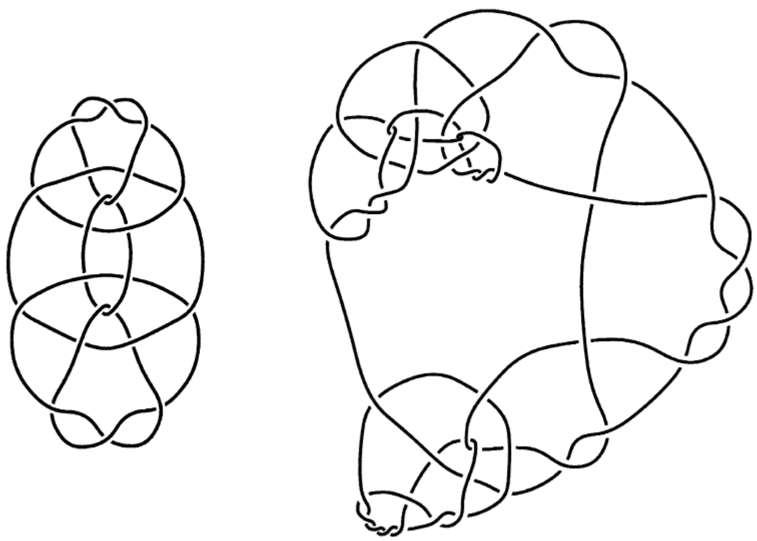


Figure 3.2: A 4 and 5-component link with trivial Jones polynomial [9]

An open problem is whether or not the Jones polynomial completely distinguishes $\bigcirc$ from other $K \in \mathscr{C}$: is there some sufficiently complex knot, whose Jones polynomial is that of $\bigcirc$?

Problem 3.2. *(Conjecture) The Jones polynomial can detect* $\bigcirc$.

This conjecture has stood up to numerous tests that pushed the limits of computing power. In 1996, Dasbach and Hougardy showed (without the use a computer) that a knot with trivial Jones polynomial must have at least 18 crossings [6].

While this is currently unknown, numerous facts about the Jones polynomial have been discovered. One interesting fact is that there are infinite families of *links* like the ones in Figure 3.2 whose Jones (and bracket) polynomial is that of the corresponding unlink (see for instance, [9] for more information).

3.2 Extensions to Other Knot Classes

We can extend the X-polynomial (and hence the Jones polynomial) to virtual knots in a natural way. We simply apply the skein relations to the virtual knot diagram as we did in the classical case, smoothing every (classical) crossing. The difference is that the resolving tree will have components involving virtual crossings, and not simply concentric and/or disjoint circles. To show this is indeed an invariant of virtual knots, we must check the three virtual and semi-virtual moves. The proofs are analogous to the cases for the classical Reidemeister moves and we leave them to the interested reader.

Lemma 3.3. *Let $K \in \mathscr{D}_V$ with no Classical crossings. Then $K \sim_V \bigcirc$.*

Proof. Since it has no crossings, the Gauss diagram of K is the empty Gauss diagram. Hence $K \sim_V \bigcirc$. □

This simple fact shows that we simply count the number of components remaining after smoothing every (classical) crossing, since they are each equivalent to $\bigcirc$. In this way we can compute an extension of the Jones polynomial, $V(K)$ for various virtual knots.

Example 3.4. *Contrary to the classical case, the simplest example of a non-trivial virtual knot has two crossings. Let K_1 be the knot with Gauss code O1-O2-U1-U2- (Figure 3.3). Its Jones polynomial is $V(K_1) = -t^{-5/2} + t^{-3/2} + t^{-1}$, distinguishing K from $\bigcirc$.*

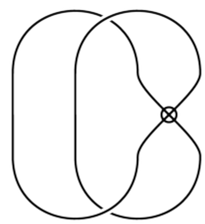


Figure 3.3: A two-crossing, non-trivial virtual knot

However, this extension is far from perfect. There are many K with $V(K) = 1$.

Example 3.5. *Consider the virtual knot K_2 with Gauss code O1-O2-U1-O3+U2-U3+ (Figure 3.4). $V(K_2) = 1$.*

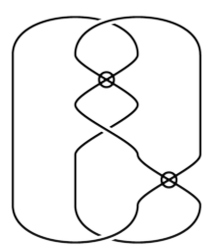


Figure 3.4: A non-trivial knot with trivial Jones polynomial

We can define another knot invariant based on the concept behind the Jones polynomial. Take any knot, and add a parallel copy of the knot. Then add sufficiently many twists to make the linking number of the two knots zero as in Figure 3.5. We call such an operation **2-Stranding** and denote the resulting link, the **Untwisted 2-Strand Cable** of a knot K by $T^2(K)$. Analogously, we can define **n-Stranding** to be the same operation with n parallel copies of K, twisted in a way to make the components have pairwise linking number 0. Let $T^n(K)$ denote the **Untwisted n-Strand Cable** of a knot K.

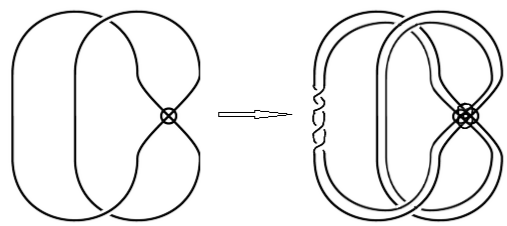


Figure 3.5: A knot K and its untwisted 2-strand cable

The **n-Stranded Jones Polynomial** of a knot K is defined as $V^n(K) := V(T^n(K))$. This is an invariant of the original knot K since the association $K \to T^n(K)$ is an invariant of K. That is, if we change a virtual knot K_1 into K_2 by any of the V-moves, the corresponding change $T^n(K_1)$ to $T^n(K_2)$ can be expressed as a sequence of V-moves. There are numerous cases to be checked and we leave this to the interested reader.

We can also define more powerful knot invariants by taking increasing collections of n-stranded Jones polynomials up to n. Let $V^n_*(K) = (V^1(K), V^2(K), ..., V^n(K))$.

Example 3.6. *In Examples 3.4 and 3.5 above:*
$V^2(K_1) = -t^{-15/2} + t^{-13/2} + t^{-11/2} + t^{-9/2} - t^{-7/2} - 2t^{-5/2} - t^{-1/2}$,
$V^2(K_2) = t^{-9/2} + t^{-7/2} - t^{-5/2} - 2t^{-3/2} - 2t^{-1/2} + t^{3/2}$,
distinguishing K_2 *from* $\bigcirc$.

We end the chapter with an example that shows the Jones polynomial is not well-defined as an invariant of welded knots. As a result, neither is $V^n_*(K)$.

Example 3.7. *Let* K_3 *be the knot with Gauss code O1-O2-U1-U2- (Figure 3.6).* $V(K_3) = -t^{-\frac{5}{2}} + t^{-\frac{3}{2}} + t^{-1} \neq 1$. *However it is easy to see that it is trivial as a welded knot. Hence,* V *is not an invariant of welded knots, and must change under the forbidden move* F_t.

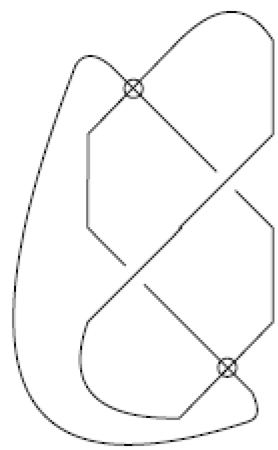


Figure 3.6: A trivial welded knot with non-trivial Jones polynomial

Chapter 4

The Knot Group

In this chapter, we discuss a remarkable property of virtual knots. They are extension of the classical knots with respect to knot groups. It is known that every classical knot has a corresponding knot group with a Wirtinger presentation. However the converse need not hold: given any group with a Wirtinger presentation, there is no guarantee that it arises as a knot group. For virtual knots however, things are much more elegant. There is not only an affirmative answer, but an algorithm for constructing the required virtual knot diagram.

4.1 Wirtinger Presentations

We would like to associate a group to a knot diagram K, classically realized as the fundamental group of the knot complement, $\pi_1(\mathbb{R}^3 \setminus K)$. An elegant way to do this is by producing its *Wirtinger Presentation.* We can assign a group to a given oriented knot diagram as follows:

Imagine cutting a Gauss diagram D at each arrow head (under-crossing). This will produce a series of disconnected *arcs.* Label each arc $x_1 ... x_n$, these will serve as the generators of the knot group.

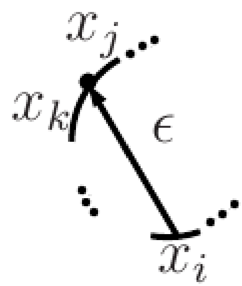


Figure 4.1: The Wirtinger presentation

We will obtain one relation from each arrow (knot crossing). Suppose the sign of an arrow is ϵ, its tail lies on an arc labeled x_i, and its head touches the final point of arc x_j and the initial point of x_k (Figure 4.1). Then we assign to this crossing the relation $x_k = x_i^{-\epsilon} x_j x_i^{\epsilon}$. We denote the group associated to the Gauss diagram D by π_D. If we take a representative virtual knot diagram K for the Gauss diagram and compute its Wirtinger presentation we would obtain the *Virtual Knot Group* π_K. (See for instance, [16] for details).

The group represented by this Wirtinger presentation is a knot invariant. Though the presentation may change slightly, the group does not change up to isomorphism (Figure 4.2) as we now explain.

- Upon application of GR-I, we introduce one new generator x' and one new relation $x' = x^{-1}xx$ and so $x' = x$.

- Upon application of GR-II, we introduce a new generator y' and two relations, $y' = xyx^{-1}$ and $y = x^{-1}yx$, one of which is redundant. But xyx^{-1} is a word in the original Wirtinger presentation, and so we aren't changing the corresponding group by adding y' as a generator.

- The GR-III case must also be checked. It is similar, though slightly more complicated than the first two cases and is left for the interested reader.

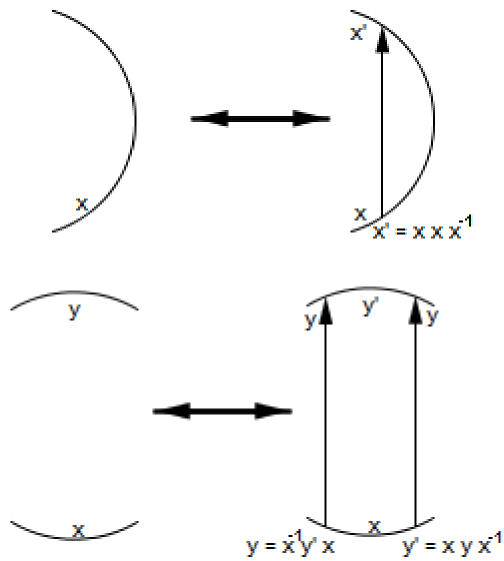


Figure 4.2: The knot group is invariant under GR-I and GR-II

We say the **Deficiency of a Group Presentation** is equal to the number of generators minus the number of relations. We can then define the **Deficiency of a Group** to be the maximum over all Group presentations. Classical knots give rise to knot groups with deficiency 1, since the final relation is always realized as a consequence of the other $n-1$ relations. This begs an interesting question: Which groups of deficiency 1 can be realized as knot groups? There are constraints, such as residual finiteness (see for instance, [13]) and we will see a specific example of a group, $G_{(2,3)}$, that is not the knot group of a classical knot.

4.2 Group Realization

Se-Goo Kim explored this question and found a solution to the group realization problem.

Theorem 4.1. *Every Group G with a Wirtinger presentation of deficiency 0 or 1 can be realized as the virtual knot group of a virtual knot.*

The proof is given in three steps in [21]. Rather than prove his result explicitly, we will consider a special case as an example which will aid in illustrating why this is true, and how we can find such a virtual knot. We will follow an algorithm proposed in [1].

Algorithm 4.2. *Suppose we have a group* $\mathscr{G} =< x_1, x_2 \mid w^{-1}x_1w = x_2 >$ *where* $w = x_{i_1}^{\pm\epsilon}x_{i_2}^{\pm\epsilon}...x_{i_n}^{\pm\epsilon}$ *is any word in the group. We can change this into a natural Wirtinger presentation of a knot diagram by defining new generators,* s_i *and relations that preserve the group structure, using the following steps:*

- *Let* $s_1 = x_1$
- *Let* $s_{i+1} = x_{i_{n-i}}^{\mp\epsilon}s_ix_{i_{n-i}}^{\pm\epsilon}$ *for* $i \leq n$
- *Let* $s_{j+1} = x_{j_{i-n}}^{\pm\epsilon}s_jx_{j_{i-n}}^{\mp\epsilon}$ *for* $n < j \leq 2n$

This set of transformations will put the group into a form which is precisely the Wirtinger presentation of a virtual knot diagram, and working in reverse we can then draw either a Gauss diagram or a virtual knot diagram.

Example 4.3. *The Baumslag-Solitar group with* $p = 2, q = 3$ *is given by* $G_{(2,3)} =< a, x \mid x^{-1}a^2x = a^3 >$. *Baumslag showed that the group* $G_{(2,3)}$ *is not residually finite [4], and the work of Hempel showed that every knot group is residually finite [13]. Alternatively, Kawauchi proved that* $G_{(2,3)}$ *is not a 3-manifold group (see [20], p. 194). Hence this group does not arise as a classical knot group.*

If we let $y = xa^{-1}$, *then we see that* $G_{(2,3)} \cong< a, x, y \mid x^{-1}y^2x = y^3, y = xa^{-1} >$. *Some manipulation yields that* $G_{(2,3)} \cong< x, y \mid (y^{-1}x)^{-2}x(y^{-1}x)^2 >$, *with* $w = y^{-1}x^2$ *as above.*

Following the instructions, Let $s_1, ..., s_8$ *be generators, and define the following relations:*

- $s_1 = x$
- $s_2 = ys_1y^{-1}$
- $s_3 = x^{-1}s_2x$
- $s_4 = ys_3y^{-1}$
- $s_5 = x^{-1}s_4x \ (= y)$
- $s_6 = xs_5x^{-1}$
- $s_7 = y^{-1}s_6y$
- $s_8 = xs_7x^{-1}$

We obtain that $G_{(2,3)} \cong < x, y, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 | s_1 = x, s_2 = ys_1y^{-1}, s_3 = x^{-1}s_2x, s_4 = ys_3y^{-1}, s_5 = x^{-1}s_4x, s_6 = xs_5x^{-1}, s_7 = y^{-1}s_6y, s_8 = xs_7x^{-1} >$. *This gives rise to the virtual knot diagram and Gauss diagram given in Figure 4.3.*

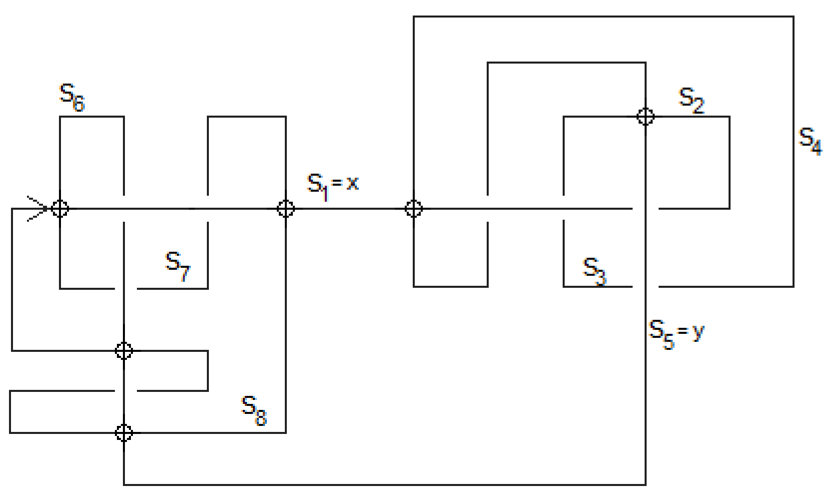


Figure 4.3: A virtual knot diagram with knot group $G_{(2,3)}$

Chapter 5

The Connected Sum of Knots

In this chapter, we describe the connected sum and explore its extension to virtual knots. Various problems arise and several examples are given to illustrate precisely what goes wrong.

5.1 The Classical Case

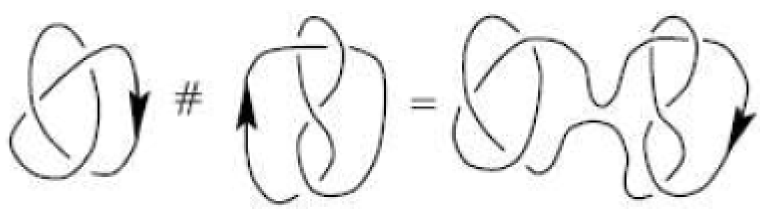


Figure 5.1: The connected sum

Given any two knots, K_1 and K_2, we can form their **Connected Sum** $K_1 \# K_2$ by deleting a small arc from each knot (which does not contain a crossing) and connecting the four endpoints in such a way as to not introduce any new crossings in the corresponding knot diagram.

We can also work in reverse and **Decompose** a knot. If we can inscribe a sphere

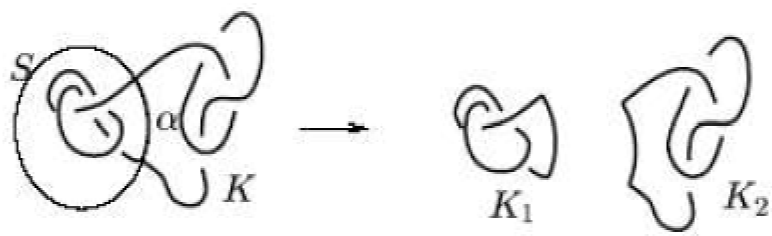


Figure 5.2: Knot decomposition

S around some part of K, such that the knot and circle only intersect twice, then we have found a way to decompose the knot into the "inner knot" K_1 and the "outer knot" K_2. In this way, $K = K_1 \# K_2$. (See Figure 5.1). A classical knot is called **Prime** if for any decomposition as a connected sum, one of the factors is $\bigcirc$.

Theorem 5.1 (Prime Decomposition Theorem [25]). *Every knot can be decomposed as the connected sum of non-trivial prime knots. If* $K = K_1 \# K_2 \# ... \# K_n$ *and* $K = J_1 \# J_2 \# ... \# J_m$*, with each* K_i *and* J_i *non-trivial and prime, then* $m = n$ *and, re-ordering if necessary, each* $K_i \sim_C J_i$.

5.2 Extensions to Virtual Knots

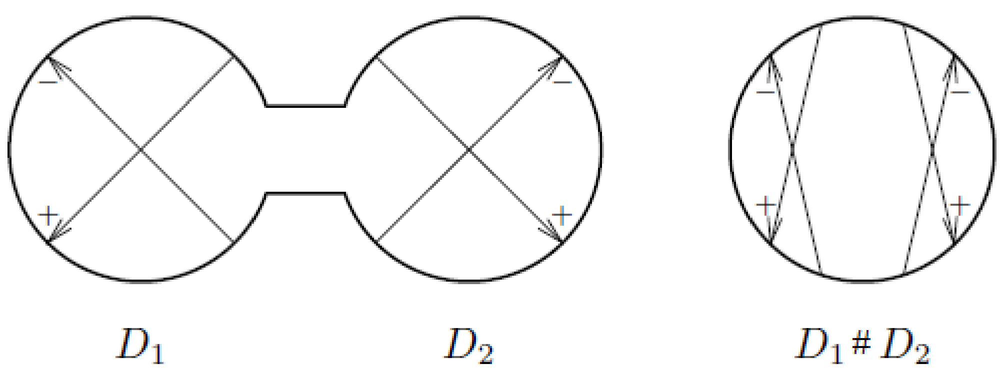


Figure 5.3: The connected sum of Gauss diagrams

We can also operate on the level of virtual knot diagrams in an analogous way, and define the connected sum operation on Gauss diagrams. Choose a basepoint on each Gauss diagram, and amalgamate the two as in Figure 5.3. There is also a notion of a prime virtual knot. We say a virtual knot is prime if its Gauss code (or diagram) cannot be written as the juxtaposition of two Gauss codes (or diagrams) on disjoint collections of markers.

The connected sum fails to be unique for virtual knots, however. The classical proof of the uniqueness of connected sum entails sliding the first basepoint until it coincides with the second choice, performing Reidemeister moves as necessary. Unfortunately, in order for that argument to work for virtual knots we would require the use of the forbidden moves, F_h and F_t, as we would occasionally need to pass over virtual crossings.

It turns out that there are two distinct problems that arise:

1. The knot $K = K_1 \# K_2$ obtained by connected sum depends on the particular knot diagrams, for K_1 and K_2.

2. The $K = K_1 \# K_2$ also depends on the choice of basepoints where connected sum is performed.

We illustrate these points by considering a few examples.

Example 5.2. *Consider the virtual knots in Figure 5.4. These are readily seen to be trivial knots. Depending on the choice of basepoint for the connected sum, we may have a trivial knot (Figure 5.4) or a non-trivial knot (Figure 5.5).*

The non-trivial virtual knot in Figure 5.5 is due to Kishino. It is not that hard to see that Kishino's knot is trivial as a welded knot. However any attempt at showing this makes essential use of the forbidden move F_h, as it is known Kishino's knot is a non-trivial in $\mathscr{V}$. Nontriviality of Kishino's knot was an open problem for quite some time, but now there are several approaches for answering this rather subtle question.

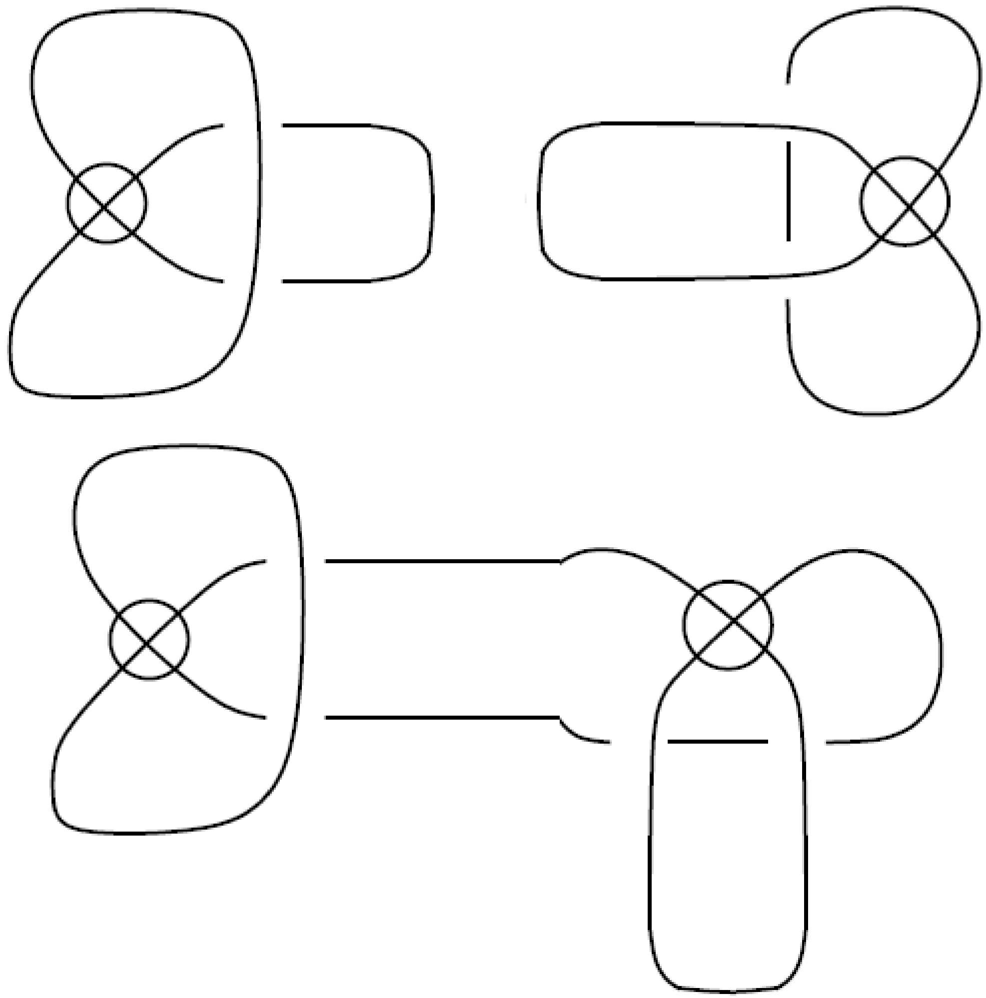


Figure 5.4: A trivial connected sum of trivial virtual knots

This knot is not distinguishable from $\bigcirc$ under most other easily computable knot invariants. Indeed, $V_K(t) = 1$, and $\pi_K = \mathbb{Z}$. In fact, its 2-stranded Jones polynomial is also trivial, $V^2(K) = -t^{-\frac{1}{2}} - t^{\frac{1}{2}} = V^2(\bigcirc)$. But it can be shown to be non-trivial by the 3-stranded Jones Polynomial, and was first done so by Kishino and Satoh (See [22]). A few other knot invariants (of equal or greater difficulty to compute) have been used to successfully detect the Kishino Knot. These include the X_i-Polynomial, the quaternionic biquandle [19], and the Surface-Bracket Polynomial [7].

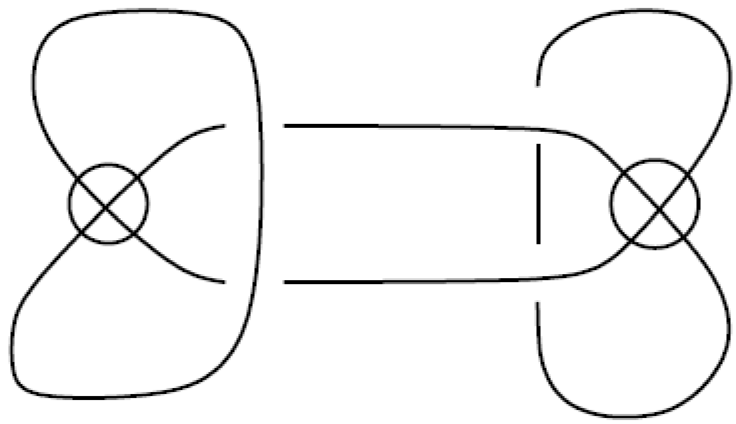


Figure 5.5: Kishino's knot - a non-trivial connected sum of trivial virtual knots

Example 5.3. *Given this example, we can create an infinite collection of nontrivial virtual knots by simply iterating connected sums of Kishino's knot as in Figure 5.6. Each of these decomposes into factors consisting entirely of* $\bigcirc$.

Remark 5.4. *We do not get a Prime Decomposition Theorem analogue for virtual knots. As we have seen, there are non-trivial knots which arise as the iterated connected sum of* $\bigcirc$*, which contradicts the uniqueness condition. Indeed, if* K^* *is Kishino's knot:*

$$\begin{aligned} K &= K \# \bigcirc \\ &= K \# \bigcirc \# \bigcirc \\ &= K \# K^* \end{aligned}$$

While it has been shown that Kishino's knot is non-trivial, other more elementary proofs of its non-triviality are sought, to add insight to the following problem:

Problem 5.5. *Classify those virtual knots which can be written as iterated connected sums of* $\bigcirc$.

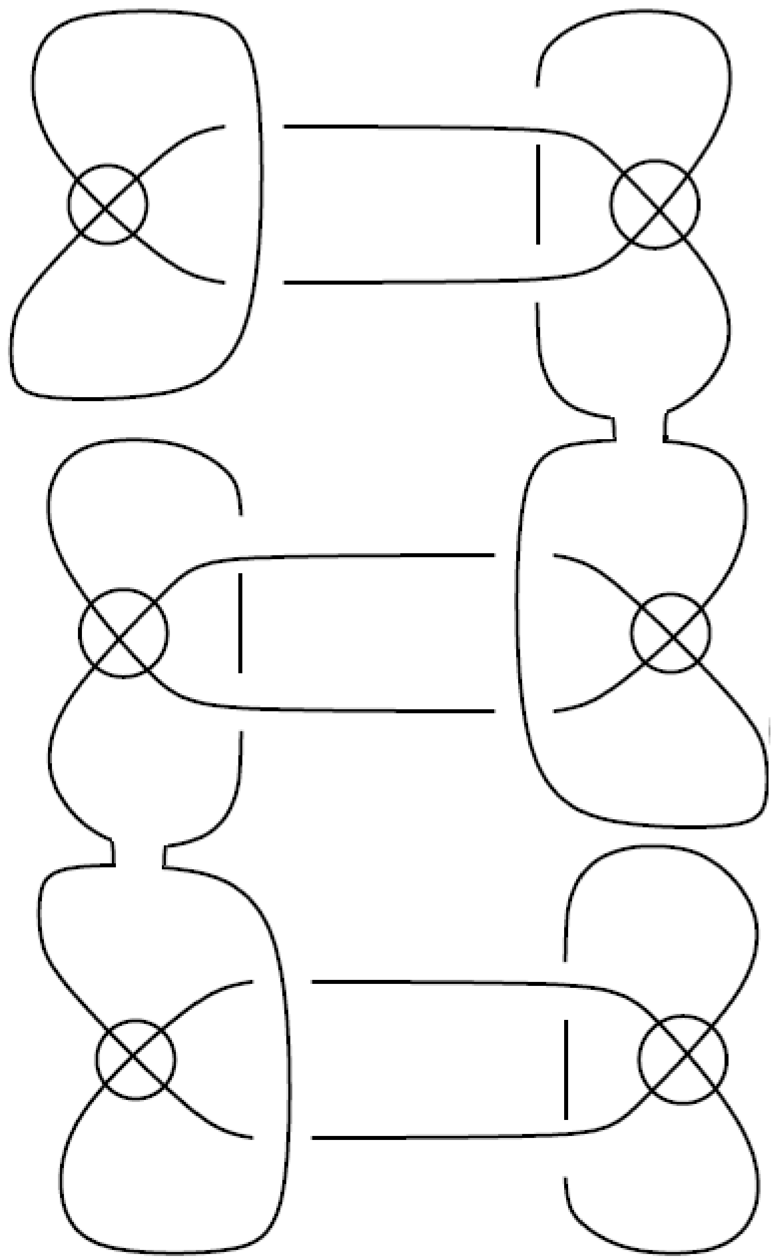


Figure 5.6: A family of nontrivial virtual knots whose factors are trivial

It makes sense to wonder how the connected sum behaves under the class of welded knots, as Kishino's knot is trivial when regarded as a welded knot. We conclude the chapter with an example of a non-trivial welded knot that arises as the connected sum of two trivial knots. This shows that the connected sum is not well-defined as an operation on equivalence classes of welded knots either.

Example 5.6 ([21])**.** *Consider the virtual knot with Gauss code O1+U2-U1+O2-O3-U4+U3-O4+ (Figure 5.7). Its knot group is isomorphic to that of the Trefoil,* 3_1*, and since* F_t *preserves the knot group, it must represent a non-trivial welded knot. However it arises as the connected sum of O1+U2-U1+O2- with O3-U4+U3-O4+, each of which is trivial (we need only apply* $R-II$ *to each to yield* $\bigcirc$*).*

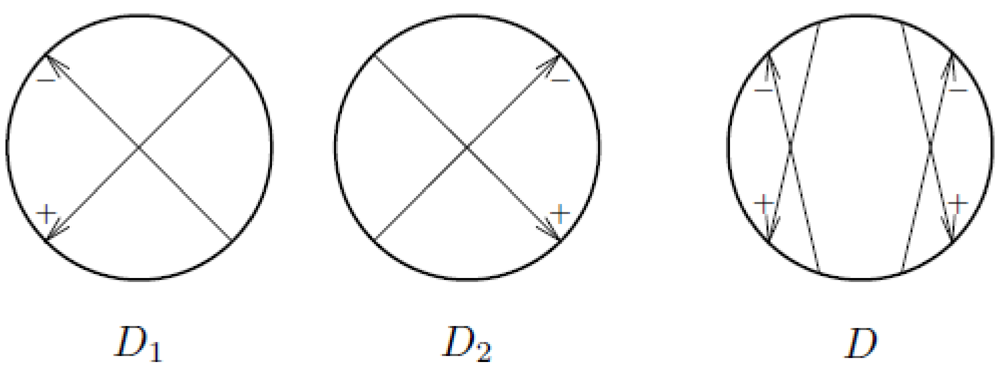


Figure 5.7: A non-trivial welded knot as a connected sum of two trivial knots

Chapter 6

The Forbidden Moves, Reprise

In this chapter we compare and contrast the four knot classes. We examine the forbidden moves in greater detail, and we shall see that F_t is, in some sense, less forbidden than F_h. We shall see in Section 6.2 that using both forbidden moves, one can untie any virtual knot, and we conclude from this that every unwelded knot is trivial. (The analogous statement for unwelded links is not true, and their classification was achieved by Okabayashi [28]).

6.1 F_h, F_t and the Knot Group

F_h and F_t, while both forbidden, differ drastically in the amount they change a virtual knot:

- F_t does not change the Wirtinger presentation at all. (Figure 6.1) On the left hand side, we conjugate x and y by z, and on the right hand side we do precisely the same thing. Hence we have the same generators and relations after an application of F_t and the Wirtinger presentation is unchanged.

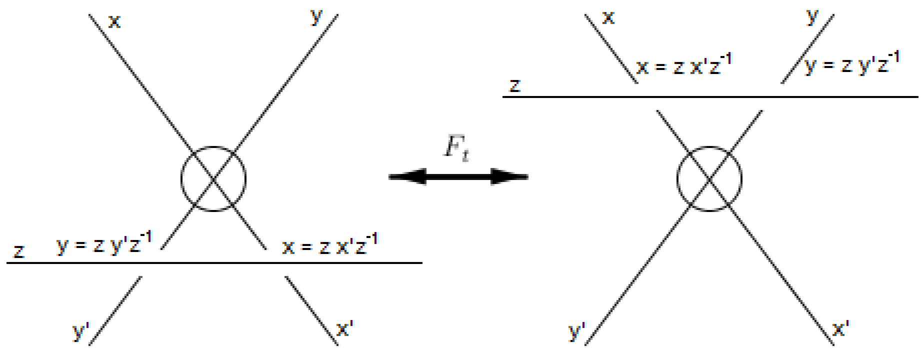


Figure 6.1: F_t does not change the knot group

- F_h changes the knot group. (Figure 6.2) The Gauss diagram on the left hand side admits the Wirtinger presentation:

$$\begin{aligned} G_D &= < a, b, c, d \mid a = ddd^{-1}, d = dcd^-1, c = bbb^{-1}, b = bab^{-1} > \\ &\cong < a, b, c, d \mid a = d, d = c, c = b, b = a > \\ &\cong \mathbb{Z} \end{aligned}$$

After applying F_h twice, we obtain the Gauss diagram on the right. But the resulting knot is the classical trefoil, whose knot group is non-trivial.

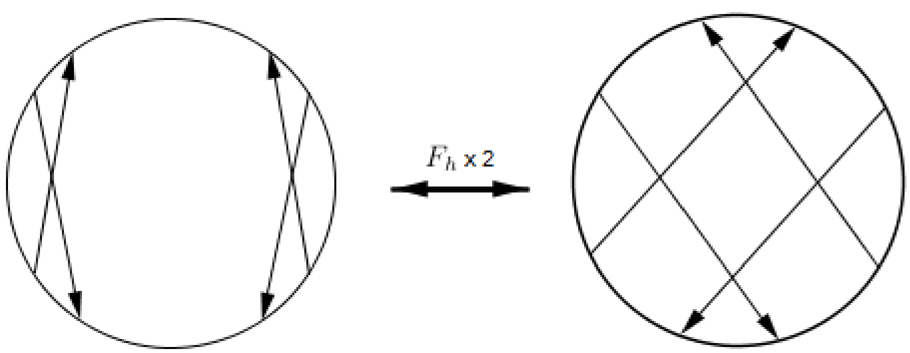


Figure 6.2: F_h does change the knot group

Thus π_D is invariant under F_t, and is hence a welded knot invariant, but fails to be an unwelded knot invariant.

Corollary 6.1. *There are nontrivial welded knots.*

The trefoil 3_1 serves as a sufficient example. Since $\pi_{3_1} \not\cong \mathbb{Z}$, and π_D is a welded knot invariant, it follows that $3_1 \notin [\bigcirc]_{\sim_W}$ and so $[3_1]_{\sim_W}$ is a distinct knot type.

6.2 Triviality of Unwelded Knots

In 2000, Nelson [27] and Kanenobu [15] independently, and with very different methods, showed that the forbidden moves render the study of virtual knots trivial. We have seen that the forbidden moves can untie the trefoil in Section 2.6, (Figure 2.12) and here we will extend this to see that any virtual knot diagram can be untied using the forbidden moves. In other words every unwelded knot is trivial.

Theorem 6.2. *The following are true (See page vi for notation):*
a) $\mathscr{V} \overset{f_t}{\twoheadrightarrow} \mathscr{W} \overset{f_h}{\twoheadrightarrow} \mathscr{U}$
b) $K_1 \sim_C K_2 \iff K_1 \sim_W K_2$, *i.e. two classical knot diagrams* K_1 *and* K_2 *are welded isotopic if and only if they are equivalent as classical knots.*
c) Every unwelded knot is trivial, i.e. $\forall\ K \in \mathscr{D}_{\mathscr{V}}, K \sim_U \bigcirc$

Proof. a) f_t and f_h are maps which take an isotopy class of a Gauss diagram to its corresponding class when the corresponding forbidden move is allowed. These are clearly surjections, as there are fewer isotopy classes modulo more allowable moves.

b) Recall that the group system of a knot is unchanged by F_t. We saw that this move has no impact on the knot group, nor does it change the peripheral structure. With that said, the proof of Theorem 2.3 carries over to this context and shows that every classical knot $K \in \mathscr{C}$ is represented by a unique virtual knot $K \in \mathscr{V}$.

c) We modify the proof in [27].

Take any Gauss diagram corresponding to a virtual knot, with n arrows (crossings). The goal is to show that we can always reduce this to a diagram with at most $n-1$ arrows. We would then proceed by induction and yield $\bigcirc$. The base case can be unknotted via a single application of $R-I$. The forbidden moves allow us to cross and uncross two arrow heads (F_h) and two arrow tails (F_t), regardless of their local orientation ϵ. If we could show that an arrow head and arrow tail can be interchanged, then we would be able to repeatedly interchange the crossings until we had a Gauss diagram resembling that in Figure 6.5 (where the part that is not shown is arbitrary).

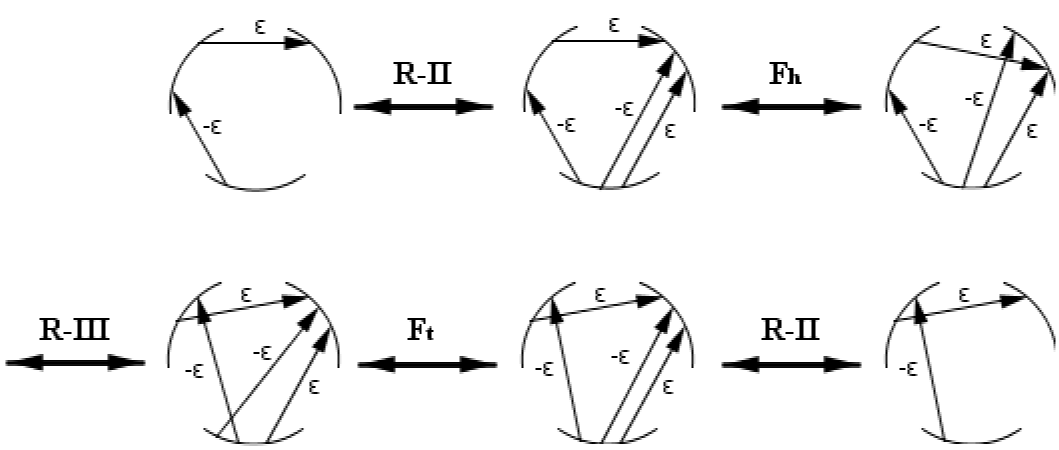


Figure 6.3: The first move sequence - F_o

There are two "move sequences" which Nelson presents that work to accomplish precisely this: one for an arrow head/tail of opposite local orientation (F_o, Figure

6.3), and another for an arrow head/tail of the same orientation (F_s, 6.4).

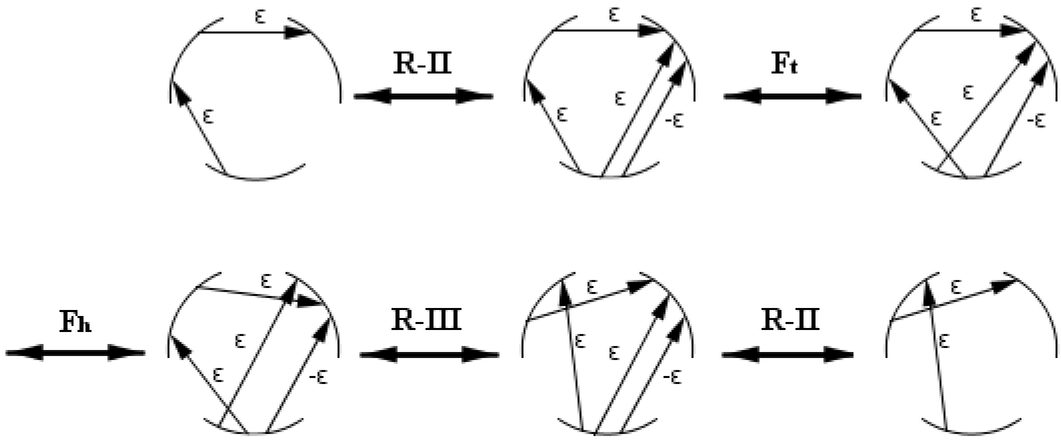


Figure 6.4: The second move sequence - F_s

Now we simply choose any arrow tail. Repeatedly interchange its arrow head with any adjacent heads/tails until the Gauss diagram is as in Figure 6.5. We can then apply $R - I$ to remove this arrow. Hence, by induction, every unwelded knot can be

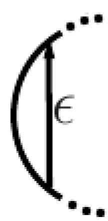


Figure 6.5: Theorem 6.2c - The Inductive Step

transformed into the trivial knot.

□

6.3 Concluding Remarks

After seeing that every unwelded knot is trivial, one may close the book on the forbidden moves entirely. However, the study of unwelded links is non-trivial: there exist links which are not equivalent to the unlink, even when using both forbidden moves (Figures 6.6 and 6.7). Note that the linking number is preserved by the forbidden moves, and is hence an invariant of unwelded links. Okabayashi [28] has shown that unwelded links form a group isomorphic to $\mathbb{Z}^{2n(n-1)}$ under the operation of connected sum.

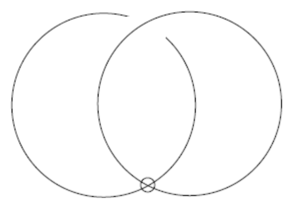


Figure 6.6: A virtual Hopf link

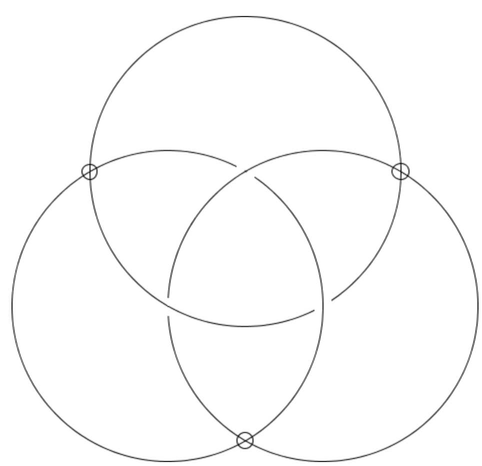


Figure 6.7: Virtual Borrowmean rings

There are a number of interesting topics and open problems that we have not touched upon in this thesis, and here we collect a few of them and provide references for the curious reader.

1. The **Crossing Number** of a classical knot K, denoted $C(K)$, is defined as the least number of crossings that occur in any possible knot diagram of K. Let the **Virtual Crossing Number** of a virtual knot K, denoted $V(K)$ be the least number of virtual crossings that occur in any possible knot diagram of K. It is clear that $V(K) = 0$ if and only if K is a classical knot. Let the **Total Crossing Number**, of a knot K, denoted $T(K)$, be the sum of the virtual and classical crossings in a diagram, minimized over all knot diagrams for K.

   Problem 6.3. *Does every virtual knot have a diagram that simultaneously minimizes both $C(K)$ and $V(K)$?*

   Fenn, Kauffman and Manturov ask whether it is true that $T(K) = V(K) + C(K)$. If so, then it implies a positive answer to Problem 6.3.

   A similar problem is whether or not classical knot diagrams are minimal with respect to the crossing number.

   Problem 6.4. *Suppose K is a classical knot and $C(K) = n$, that is, there is some minimial diagram representing K that has n crossings. Can we construct K' with $K \sim_V K'$ with some virtual crossings, but fewer than n real crossings?*

   A recent invariant of Dye and Kauffman [8], the *arrow polynomial*, gives a lower-bound on $V(K)$. This invariant is computed in a similar manner to the bracket polynomial, but differs in that its skein relations involve oriented crossings. They show that $V(K)$ is greater than or equal to the maximum k-degree of the arrow polynomial $< K >_{\rightarrow}$. Interestingly, this invariant also detects Kishino's knot.

2. The **Unknotting Number** of a classical knot K, denoted $u(K)$, is defined as the least number of crossing changes that are required for the knot to become untied. This ranges over any possible knot diagram of K. There are many estimates for $u(K)$ such as the inequality $u(K) \leq \frac{c(K)}{2}$.

   Problem 6.5. *Can we sensibly define a virtual unknotting number? Is there an analogue of these estimates for virtual crossings?*

   Fenn, Kauffman and Manturov define the **Virtual Unknotting Number** to be the number of crossings one needs to convert from classical to virtual (by direct flattening) in order to untie the knot, minimized over all virtual diagrams for the knot. However, very little is known about the virtual unknotting number.

   In a recent paper [14], Henrich, McNaughton, Narayan, Pechenik and Townsend have defined *pseudodiagrams* to be virtual knot diagrams where some (not necessarily all) crossings, called *pre-crossings* were drawn as double points. These pre-crossings may be resolved as either type of classical crossing, or as a virtual crossing. The theory was motivated by the study of DNA in which the knot structure is not always clearly visible. The study of pseudo-diagrams allows inferences to be made with the prescribed information that is visible. Henrich et. al. have defined the **Trivializing Number** as the least number of pre-crossings which must be resolved to necessarily untie the knot. Similarly, the *knotting number* is the least number of resolutions which yield a knot that is necessarily non-trivial. Also, the *virtualizing number* and *classicalizing number* are the least number of resolutions which must be made to yield a necessarily virtual or classical knot respectively. If no such resolution is possible, we say the corresponding numbers are infinite.

3. Many of the classical invariants can be extended to virtual knots, as we have seen with the Jones polynomial and knot group. It makes sense to ask how other

invariants extend to virtual knots. Does the Alexander-Conway Polynomial extend to virtual knots? Jaeger, Kauffman and Saleur introduced an invariant of links in thickened surfaces which satisfies a Conway-type skein relation. In the same way as in the classical case, the one-variable Alexander polynomial of a virtual link can be derived from the virtual link group (see for instance, [1]). But the skein-relation for (a normalized version of) the classical Alexander polynomial does not extend in an obvious way to the class of virtual links. This is because there exist virtual knot diagrams which cannot be transformed into the unknot by changing classical crossings. Therefore, the Alexander polynomial is different from the Conway polynomial for virtual knots, contrary to the classical case. For more details, see [31].

4. The following series of problems of Fenn, Kauffman and Manturov suggest that virtual knots may be of interest to even the most classical knot theorists. The aim is to determine whether the Jones polynomial detects the unknot.

Problem 6.6. *Suppose K is a virtual knot diagram with $V_K(t) = 1$. Can we recognize whether $K \sim_V K'$, where K' is a classical knot diagram?*

Fenn et al. showed that *crossing virtualization*, the operation of flanking a classical crossing between two virtual crossings (Figure 6.8), is an operation that does not change the Jones polynomial.

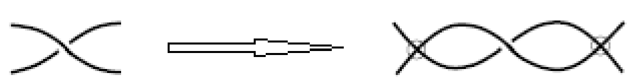


Figure 6.8: Crossing virtualization

Problem 6.7. *Suppose K is a virtual knot diagram with $V_K(t) = 1$. Can we transform K into a classical knot using virtual equivalence and virtualization?*

Problem 6.8. *Does crossing virtualization change the classical knot type? In other words, suppose* K *and* K'*, are both classical, and are related by crossing virtualization and classical Reidemeister moves. (Note that this implies that* $V_K(t) = V_{K'}(t)$*) Is it true that* $K \sim_C K'$*?*

If Problem 6.7 and Problem 6.8 have affirmative answers, then we will have answered Problem 6.6 and shown that the Jones polynomial detects the unknot.

5. The HOMFLY polynomial is a quantum invariant often called the generalized Jones polynomial. Does it extend to virtual knots? We know the Jones polynomial fails to detect the virtual unknot, and so it makes sense to ask if the extended HOMFLY polynomial detects the virtual unknot. But as with the Conway polynomial, difficulties arise in making a skein theoretic definition.

   Problem 6.9. *Can the HOMFLY be suitably extended to virtual knots? Does it detect the (virtual) unknot?*

   To the best of our knowledge, a suitable definition of the HOMFLY polynomial for virtual links has not yet been made.

6. Khovanov Homology is a recent development which has many implications in the study of classical knots. Regarded as the categorification of the Jones polynomial, it is an invariant of knots that arises as the homology of a chain complex (see for instance, [3]). Kronheimer and Mrowka showed quite recently that Khovanov Homology does indeed detect the classical unknot [23].

   Problem 6.10. *Can Khovanov Homology be extended to virtual knots, and does it detect the virtual unknot?*

The aforementioned problems illustrate that virtual knot theory, the younger sibling of its classical counterpart, is a field with many prospects for further development.

While the concept of a virtual crossing may seem unsettling at first, the study of this broader class of knots is sure to yield promising results, and have interesting applications to classical knot theory.

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