LINEAR REPRESENTATIONS OF BRAID GROUPS

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Abstract

This paper looks at four linear representations of braid groups, the Burau, Gassner and Lawrence-Krammer representations, as well as another, which will be termed the Lawrence-Gassner representation. The Burau and Lawrence-Krammer representations are defined for the full braid group and the other two for just the pure braid group. All four representations are described in terms of topological objects in the n-punctured disc known as forks, which represent elements of a homology group of an infinite covering space associated with the representation.

Complex specialisations of the Burau and Gassner representations are briefly covered as well as the possibility of other representations based on forks.

Acknowledgements

I would like to thank my supervisor, Hans Boden, for all his help and for finding me something to study when I showed up at McMaster without much idea of a subject. I am also grateful to those whose papers and books helped me, especially Drs Joan Birman and Stephen Bigelow. Thanks to McMaster University for accepting me and to Prof Dugald Macpherson whose idea it was for me to come here.

Contents

List of Figures

1 Introduction

1.1 Braids

Braids crop up in various ways in geometry and topology, as well as in areas of group theory, algebra and theoretical physics. There is, of course, the geometric definition of a braid as n disjoint strands in $\mathbb{C} \times [0,1]$, transverse to each plane $\mathbb{C} \times \{t\}$ with endpoints fixed at the top and the bottom. With the operation of adjoining the bottom of one braid to the top of another, this becomes a group, known as the *braid group* and denoted B_n. From this comes the view of B_n as the fundamental group of the space of n unordered, distinct points in the plane. A path in this space then permutes the points, giving a surjection of B_n onto the symmetric group, S_n. The kernel of this map is a subgroup of B_n known as the *pure braid group* and denoted P_n. The pure braid group is also the fundamental group of the complement of an arrangement of hyperplanes in $\mathbb{C}^n$, specifically the arrangement that forms the solution set to $\prod_{1\le i<j\le n}(x_i - x_j) = 0$.

Also connected with this is the definition of braids as certain automorphisms of the free group, F_n, which is the fundamental group of the complement of the space of n points in $\mathbb{C}$. Braids also come into knot theory, since they may be *closed* to form links by connecting the bottoms of the strands to the tops. Two braids close to the same link if and only if they are connected by a series of *Markov moves*, that is conjugation by another braid, and *crossed stabilisation*, i.e. the addition or removal of the last strand in a way that does not alter the closure knot. If a braid invariant is unchanged under the Markov moves, then it defines a knot invariant. The trace and characteristic polynomial of a linear representation are already invariant under conjugation, so may be good candidates for producing knot invariants.

The approach used in this paper is to view braids as orientation-preserving homeomorphisms of the n-punctured disc, up to an equivalence. In this way, the braid group is a mapping class group, see [8].

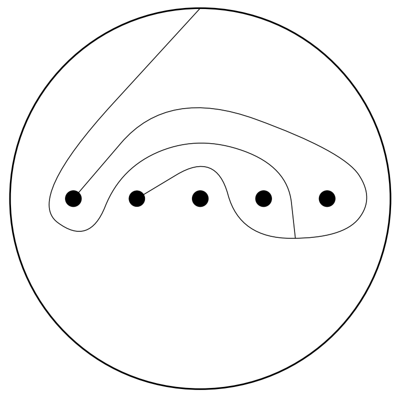


Figure 1: A fork.

1.2 Linear representations

An open problem for a while was the question of whether braid groups are linear, that is isomorphic to a linear group. This comes down to whether or not it is possible to find a faithful linear representation of the braid group.

Several linear representations of B_n have been considered, the first being the Burau representation, which turned out not to be faithful for large enough n [15], [13], [4]. The Gassner representation is a variant on the Burau representation, but which is only defined on pure braids. Its faithfulness is, in general, as yet unknown. The first linear representation to be known to be faithful was the Lawrence-Krammer representation, a deformation of the symmetric square of the Burau [11], [6], [12]. The Gassner and Lawrence-Krammer representations can both be mapped to the Burau, though in different ways, so there is another representation, also faithful, that sits over both, completing a square commutative diagram. This will be termed the Lawrence-Gassner representation. These are the four representations discussed in this paper.

The image of a braid under any of these linear representations may be determined as the braid's effect on a certain infinite branched covering space connected to D_n. Elements of this module can be expressed in terms of a basis of *forks* in D_n, and the action of a braid on these forks naturally defines a matrix of Laurent polynomials (possibly in several variables).

A Laurent polynomial may have one or more of its variables specialised to a complex number, hence the same is true of the representations. This gives a simpler representation; one algebraic proof of the faithfulness of the Burau representation of B_3 uses a specialisation to give generators of a group of known presentation. The kernel of this specialised representation is contained in the centre of B_n. In general, however, the faithfulness of specialised representations is not well understood. This paper looks at some of them.

1.3 Outline

Section 2 covers some of the necessary background for understanding the subject, such as braids, Laurent polynomials and forks.

The next four sections cover the four representations in turn, including details of the modules in terms of forks and what is known of their faithfulness. Complex specialisations are considered for the Burau and Gassner representations.

Finally, section 7 addresses the question of whether there may be other fork-based representations.

2 Background

2.1 Braid groups

Definition 2.1. Let $D = \{x \in \mathbb{C} : \|x\| \leq 1\}$ denote the 2-disc and D_n denote the n-punctured 2-disc. The punctures are normally assumed to be on the real line and labelled $p_1, \ldots, p_n$ in order from smallest to largest. Define $\text{Homeo}^+(D_n)$ to be the space of orientation-preserving homeomorphisms of D_n which preserve ∂D. Consider the equivalence classes of $\text{Homeo}^+(D_n)$, where two homeomorphisms are equivalent if they differ by an isotopy of D_n. These equivalence classes are *braids*. Sometimes the punctures are best thought of as distinguished points in the disc, but they will be called punctures throughout.

With composition, braids form a group, called the *braid group* on n strands and denoted B_n.

There is a natural surjection from B_n, the braid group, to S_n, the symmetric group, defined by sending a braid to the induced permutation on puncture points. The kernel of this map is called the *pure braid group*, denoted P_n.

A braid may be viewed geometrically as n strands embedded disjointly in $D \times [0,1]$. Let $p_1, \dots, p_n$ be disjoint points in D. Then each strand begins at $(p_i, 0)$ and ends at $(p_j, 1)$ for some $i, j \in \{1, \dots, n\}$. Further, each strand intersects each plane $D \times \{t\}$ in just one point. Two geometric braids are equivalent if they are related by an ambient isotopy, that is an isotopy of $D \times [0,1]$ preserving $D \times \{0,1\}$ pointwise. To construct the geometric braid corresponding to a class of homeomorphisms, use the fact that when extended to the entire disc, an orientation-preserving homeomorphism must be isotopic to the identity. The isotopy gives a map $D \times [0,1] \to D$, and the geometric braid is then obtained as the preimage of the puncture points. In a pure geometric braid the strands will begin and end at the same point p_i in D.

In drawing a geometric braid, assume $D \times \{0\}$ to lie above and $D \times \{1\}$ to lie below. This is in keeping with the original convention of how geometric braids act on the free group that is the fundamental group of D_n. However, in keeping with the standard notation for maps, braids act on the left and so compose right-to-left, not left-to-right as they were originally held to do. Because of the symmetry of the relations given below, this does not alter the presentation of the group.

Definition 2.2. 1. For $1 \leq i \leq n-1$, define σ_i to be a half Dehn twist about a curve enclosing points p_i and p_{i+1} (see figure 2). The set $\{\sigma_1, \dots, \sigma_{n-1}\}$ is then a generating set for B_n, known as the *standard generating set*. This becomes a presentation of B_n with the addition of the relations:

$$[\sigma_i, \sigma_j] = 1, \qquad \text{for } i, j = 1, \dots, n \text{ and } |i-j| \geq 2,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \text{for } i = 1, \dots, n-1.$$

2. For $1 \leq i < j \leq n$, define A_{ij} to be a full Dehn twist about a curve enclosing points p_i and p_j and passing above the other puncture points

(see figure 3). The set $\{A_{ij} \mid 1 \leq i < j \leq n\}$ is a generating set for P_n, known as the *standard generating set*. Relations for the pure braid group are as follows:

$$\begin{aligned} [A_{ij}, A_{kl}] &= 1, && \text{for } i < j < k < l \text{ and } i < k < l < j, \\ [A_{ij}, A_{jk}A_{ik}] &= 1, && \text{for } i < j < k, \\ [A_{ik}, A_{ij}A_{jk}] &= 1, && \text{for } i < j < k, \\ [A_{ik}, A_{jk}^{-1}A_{jl}A_{jk}] &= 1, && \text{for } i < j < k < l. \end{aligned}$$

In terms of the σ_is, the A_{ij}s can be written:

$$A_{ij} = \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} \cdots \sigma_{j-1}.$$

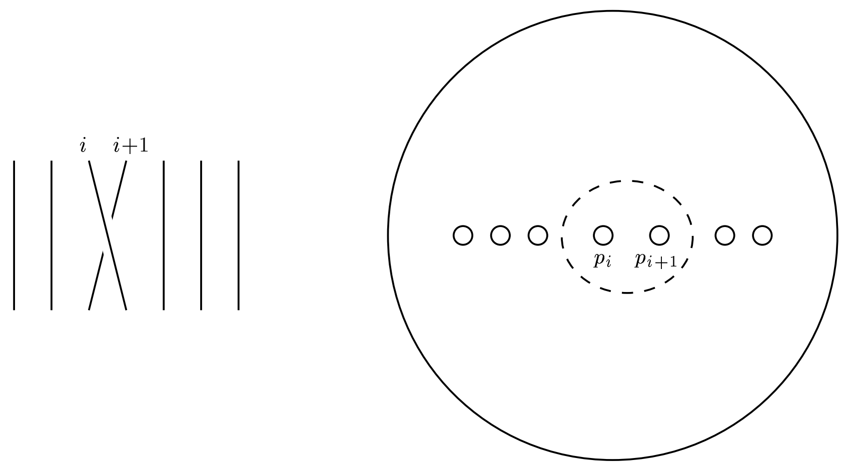


Figure 2: The braid σ_i, represented geometrically on the left. As a homeomorphism, it is a half Dehn twist about the curve on the right.

One notable braid is the half Dehn twist about a curve parallel to the boundary, which will be denoted Δ. Notice that Δ^2 is a pure braid as it is a full Dehn twist about the same curve.

Although an $n-1$ element generating set is standard, for any n, B_n is generated by two elements. Let $\sigma = \sigma_1$ and $a = \sigma_1\sigma_2 \cdots \sigma_n$. Then $\sigma_i = a^{i-1}\sigma a^{-(i-1)}$ and B_n has the presentation

$$\langle \sigma, a \mid a^n = (a\sigma)^{n-1}, [\sigma, a^{-j}\sigma a^j] = 1 \rangle.$$

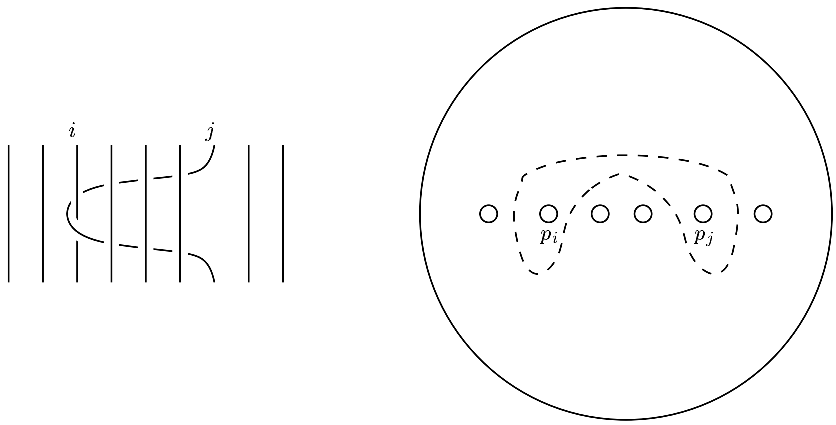


Figure 3: The braid A_{ij}, represented geometrically on the left. As a homeomorphism, it is a Dehn twist about the curve on the right.

The group B_4 has another, neater, 2-generator presentation. Putting $x = \sigma_1\sigma_2\sigma_3$ and $y = \sigma_1\sigma_2\sigma_3\sigma_2$ it has the presentation:

$$\langle x, y \mid x^4 = y^3, [x^2, yxy] = 1\rangle$$

It is natural to ask whether P_n can ever have a smaller generating set than the standard. Unfortunately the answer turns out to be no.

Theorem 2.3. *The standard set of $\binom{n}{2}$ generators for P_n is minimal.*

Proof. The relations in P_n are all of the form $A = BAB^{-1}$, that is an element is equal to some conjugate of itself. These relations would hold in any abelian group, hence the abelianisation of P_n is a free abelian group on $\binom{n}{2}$ generators. $\square$

2.2 Laurent polynomials

Definition 2.4. A *Laurent polynomial* in variables $q_1, \ldots, q_n$ is a polynomial with integer coefficients and terms of the form $q_1^{a_1}q_2^{a_2}\cdots q_n^{a_n}$, where $a_1, \ldots, a_n \in \mathbb{Z}$. The ring of Laurent polynomials in these variables is denoted $\Lambda[q_1, \ldots, q_n]$ and may be thought of as $\mathbb{Z}[q_1, q_1^{-1}, \ldots, q_n, q_n^{-1}]$.

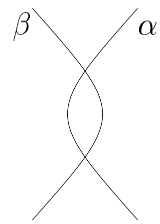


Figure 4: A digon.

2.3 Digons

In some topological arguments, we will need to decide whether or not one curve can be homotoped off another. There is a fairly simple criterion for checking this.

Definition 2.5. A *digon* between two curves α and β on a surface is an embedded disc whose boundary consists of one subarc of α and one subarc of β, see figure 4.

The following lemma and proof are taken from [10].

Lemma 2.6. *Suppose α and β are simple closed curves on a surface which intersect transversely at finitely many points. Then α and β can be freely homotoped to simple closed curves which intersect at fewer points if and only if there exists a digon between the two curves.*

Proof. ($\Leftarrow$) If a digon exists, one of the curves can be homotoped across it in order to reduce the number of intersection points.

($\Rightarrow$) Let $H : [0,1] \times S^1 \to M$ be a homotopy from $\alpha = H(\{0\} \times S^1)$ to a curve, $H(\{1\} \times S^1)$, meeting β transversely at fewer intersection points than α does. Let $h_t : S^1 \to M$ be defined for $t \in I$ by $h_t(x) = H(t,x)$. Assume h_t is in general position with respect to β for all values of t. Then H is transverse to β and hence $H^{-1}(\beta)$ is a 1-manifold, which has four possible types of component, see figure 5.

There must be at least one component of type I, call it Γ_1, with endpoints q_1 and q_2. Let Γ_0 be the arc of α between q_1 and q_2 which is

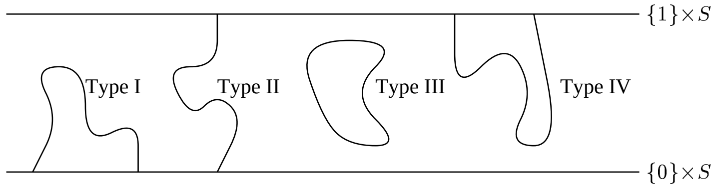


Figure 5: The four possible types of component of $H^{-1}(\beta)$.

homotopic to Γ_1 in $I \times S^1$. Then $H(\Gamma_0\Gamma_1^{-1})$ is a closed curve homotopic to 0 formed from one subarc of α and one subarc of β.

The part of the disc bounded by $H(\Gamma_0\Gamma_1^{-1})$ will be a union of one or more digons, since it lies in the image of H. □

2.4 Forks, whiskers and noodles

There are three geometric objects in D_n that will be used in defining the representations. Take the basepoint of D_n to lie in ∂D_n and denote it y_0.

Definition 2.7. A *fork* in D_n is an embedded tree consisting of an arc, known as the *tine* and denoted $T(F)$, from one puncture point to another, together with a *handle* from y_0 to a point in the interior of the tine. The tine is oriented so that the handle joins it from the left.

Two forks are *equivalent* if they are homotopic via forks (in particular, the endpoints of the tines will be the same).

A *standard* fork is one contained in the upper half of D_n. A standard fork with tine beginning at p_i and ending at p_j with $i < j$ is denoted f_{ij}. Define a total ordering on the standard forks by $f_{ij} \leq f_{kl}$ if $j < l$, or if $j = l$ and $i \leq k$. That is, in order from smallest to largest,

$$f_{12}, f_{13}, f_{23}, f_{14}, f_{24}, f_{34}, \ldots, f_{1n}, f_{2n}, \ldots, f_{n-1,n}.$$

This will be referred to as the *standard ordering* and has the advantage that the forks on n points come first in the listing of forks on $n+1$ points.

A *simple* fork is a specific standard fork, which begins and ends at adjacent punctures. The simple fork from p_i to p_{i+1} is denoted f_i.

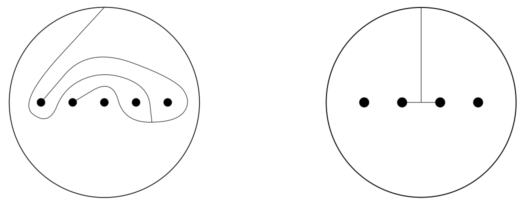


Figure 6: A nonstandard fork in D_5 and the simple fork $f_2 = f_{23}$ in D_4.

Definition 2.8. A *whisker* in D_n is an embedded arc from y_0 to one of the distinguished points. A *standard whisker* is one contained in the upper half of D_n.

Definition 2.9. If D_n is viewed as having two basepoints, $y_0, y_0' \in \partial D$, as will be necessary later, then a *noodle* is defined as an embedded arc from y_0 to y_0'.

In addition, given a fork, F, based at y_0, a fork, F', based at y_0' is a *parallel* to F, if the two tines are disjoint, but homotopic, relative to the endpoints, the handles are disjoint and the area enclosed by the forks and the arc of ∂D_n between y_0 and y_0' contains no punctures. There are two choices of arc of ∂D_n, but if y_0 and y_0' are viewed as being in the top half of D_n then the shorter arc, which is also contained in the top half, is naturally chosen.

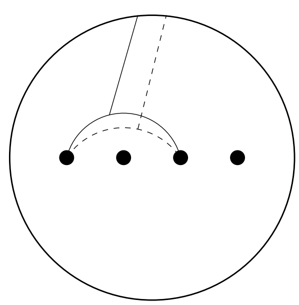


Figure 7: A fork and a parallel fork (dashed) in D_4.

3 The Burau representation

Definition 3.1. Let $x_1, x_2, \ldots, x_n$ be the free generators of $\pi_1(D_n, y_0)$, with x_i represented by a curve passing anti-clockwise around p_i.

Consider the homomorphism $\pi_1(D_n, y_0) \to \mathbb{Z}$ taking a word in the x_is to its exponent sum. Let $\widetilde{D}_n$ be the cover of D_n corresponding to this map and choose $\tilde{y}_0$, a lift of y_0 as the basepoint of $\widetilde{D}_n$.

Claim 3.2. The homology group $H_1(\widetilde{D}_n)$ has rank $n-1$ as a module over $\Lambda[q]$.

Proof. First note that the deformation retract of $\widetilde{D}_n$ is an infinite sequence of vertices with each connected to the next by n edges, see figure 8. The covering translation q acts on this by moving each vertex to the next to the right and each edge to the corresponding edge with the relevant endpoints.

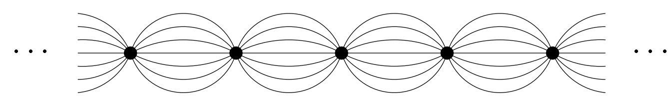


Figure 8: The deformation retract of $\widetilde{D}_7$.

Consider the section of this retract between two adjacent vertices. The first homology group of this section is generated by $n-1$ elements. Any element of $H_1(\widetilde{D}_n)$ can be written in terms of these and Laurent polynomials in q. Hence $H_1(\widetilde{D}_n)$ has rank $n-1$. □

A homeomorphism of D_n, ψ, representing an element of B_n then naturally lifts to $\tilde{\psi}$, a homeomorphism of $\widetilde{D}_n$, which induces $(\tilde{\psi})_*$, a homomorphism of $H_1(\widetilde{D}_n)$. This is independent of the choice of representative of the braid.

Definition 3.3. The *Burau representation* is the map

$$\begin{aligned} \beta_n : B_n &\to GL(n-1, \Lambda[q]), \\ [\psi] &\mapsto (\tilde{\psi})_*. \end{aligned}$$

3.1 The Burau module

Elements of $H_1(\widetilde{D}_n)$ can be represented by forks with relations between them. Given a fork, F, in D_n, lift it to $\widetilde{D}_n$ such that the handle connects with the basepoint of $\widetilde{D}_n$. Then take an embedded circle based at a point on the lift of the tine as representative of an element of $H_1(\widetilde{D}_n)$. Figure 9 shows the projection of this down to D_n, where this circle is immersed as a figure 8 curve. Note that an element of $H_1(\widetilde{D}_n)$ may be represented by two non-isotopic forks, the tines of these forks, however, will be isotopic.

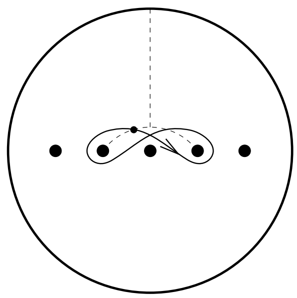
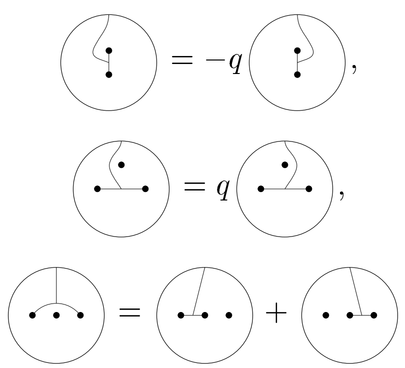


Figure 9: A fork (dashed) with corresponding figure eight curve.

With this in mind, we can define the Burau module in terms of the forks, as Krammer [11] did. As generators take the simple forks. Forks are related by the following fork skein relations:

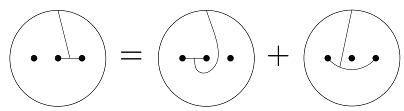


(1)

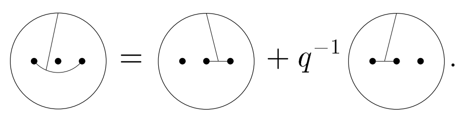
Remark 3.4. There are two things to note about this expression of the relations. Firstly that there may be more punctures than those shown; any remaining punctures lie in the lower part of the disc. Secondly they still hold under an orientation-preserving homeomorphism of D, in fact the first two relations do not apply under the usual assumption that the punctures lie on the real line, unless they are altered by a homeomorphism first.

In particular they still hold under any braid.

Example 3.5. For example, σ_1^{-1} changes the third relation to



or, using the first relation



Proposition 3.6. *Any fork may be written in terms of the simple forks using the given relations.*

In the proof of this we will use the following:

Definition 3.7. A *shaft* in D_n is an arc directly downwards from a puncture point. Thus the shafts are n disjoint arcs from puncture points to the boundary.

The *height* of a fork is the total number of intersections its tine has with the shafts, where the intersections at the ends of the tine count $\frac{1}{2}$ each. A fork with height one is then equivalent to a standard fork via applications of the first two Burau relations.

Proof of 3.6. Assume a fork has height essentially ≥ 2, so the tine cannot be homotoped into the upper half-plane, thus its interior intersects at least one shaft. In fact, it must intersect a shaft not connected to either of its endpoints, see [16] for a proof of this. Taking the topmost intersection on this shaft, apply the third Burau relation (first move the handle out of the way using the first two if necessary) to replace the tine with one that passes the other side of the puncture point and two that end at the point. All three of these have a lower height than the original. By repeating this process, the fork can be written in terms of standard forks. Finally, by the third relation, any standard fork is a sum of simple forks. □

With this presentation of $H_1(\widetilde{D}_n)$, the Burau representation can be considered as matrices showing the effect of a braid on the simple forks. Denote the $k \times k$ identity matrix by I_k.

Example 3.8. 1. If $i = 2, \ldots, n-2$, then

$$\beta_n(\sigma_i) = \left(\begin{array}{c|ccc|c} I_{i-2} & & 0 & & 0 \\ \hline & 1 & 0 & 0 & \\ 0 & q & -q & 1 & 0 \\ & 0 & 0 & 1 & \\ \hline 0 & & 0 & & I_{n-i-2} \end{array}\right).$$

If $i = 1$ or $n-1$, the 3×3 block will lose its first or last row and column respectively, e.g.

$$\beta_4(\sigma_1) = \begin{pmatrix} -q & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. $\beta_n(\Delta^2) = q^n I_{n-1}$.

3. $\beta_2(\sigma_1^i) = (-q)^i$.

3.2 Faithfulness

The group B_2 is infinite cyclic and β_2 can easily be shown to be faithful, since its image is generated by a 1×1 matrix. Magnus and Peluso [14] showed β_3 to be faithful by considering the images of the two generators of B_3. It then seemed likely that a similar method might decide faithfulness of β_4.

Unfortunately the properties of matrix groups are not understood sufficiently well for much more headway to have been made on this and it was by a different, topological argument that β_n was shown to be unfaithful for $n \geq 5$. This was the culmination of work done by several different people.

In 1991, Moody [15] showed β_n to be unfaithful for $n \geq 9$. The method was refined by Long and Paton [13], who improved the bound to $n \geq 6$, and later by Bigelow [4], who showed β_5 to be unfaithful. In [6], Bigelow demonstrated that the topological criterion from [4] can also be used to show faithfulness of β_3.

A version of the criterion used in these results is given below. The criterion will be adapted in this paper to apply to other representations, starting with complex specialisations of the Burau representation in the next section.

Definition 3.9. Given a whisker, W, in D_n, let $\widetilde{W}$ be the lift of W to $\widetilde{D}_n$ that is based at $\tilde{y}_0$. Define the *Burau pairing* between a whisker and a fork in the Burau module by

$$\langle W, F\rangle_\beta = \sum_{a\in\mathbb{Z}} q^a (q^a\widetilde{W}, \widetilde{F}).$$

Note that if F and F' are forks with identical tines, but different handles, then

$$\langle W, F\rangle_\beta = q^a \langle W, F'\rangle_\beta,$$

for some $a \in \mathbb{Z}$.

The following theorem gives the criterion used in [4].

Theorem 3.10. *The following are equivalent:*

1. *The Burau representation of B_n is faithful.*
2. *If W, F are any whisker and fork in D_n such that $\langle W, F\rangle_\beta = 0$, then $T(F)$ is homotopic, rel endpoints, to an arc which is disjoint from W.*

Bigelow then proved that β_5 is unfaithful by presenting an example of a whisker and tine with pairing zero, but with no digons between them. Hence it is not possible to homotope one so as to make the two disjoint.

There is one case remaining — it is still not known whether or not β_4 is faithful. Notably, a large computer search organised by Bigelow (mentioned in [6]) based on a result equivalent to Theorem 3.10 did not find a fork and whisker with pairing zero. The search covered most pairings with up to 2300 intersections before it was called off [7]. It thus seems likely that either β_4 is faithful or that a counterexample will not be found by such brute-force techniques, at least not without quite a bit of refinement.

3.3 Complex specialisations

Specialising $q = z \in \mathbb{C}^*$ (i.e. z is a nonzero complex number) turns the Burau representation into a representation over $GL(n-1, \mathbb{C})$. Clearly this representation is unfaithful for $n \geq 5$, so the only interesting cases are when $n = 3$ or 4. If the number z is an ith root of unity, then Δ^{2i} is in the kernel of the Burau representation specialised at z, since $\beta_n(\Delta^{2i}) = q^{ni}I_{n-1}$. Bigelow [5] noted that a criterion for faithfulness of the specialised Burau representation can be found similarly to that of the unspecialised version.

Theorem 3.11. *If z is not a root of unity, then the following are equivalent:*

1. *The Burau representation of B_n is faithful when specialised to z.*
2. *If W, F are any whisker and fork in D_n such that z and $1/z$ are both roots of $\langle W, F\rangle_\beta$, then $T(F)$ is isotopic to an arc which is disjoint from W.*

Proof. (2.)⇒(1.) Let w_i denote the standard whisker ending at point p_i. For $i \neq j, j+1$, $\langle w_i, f_j\rangle_\beta = 0$ for the simple reason that the whisker is disjoint from the tine of the fork.

Now if ψ is in the kernel of $\beta_n(z)$, then we have $\langle w_i, \psi(f_j)\rangle_\beta|_{q=z} = 0$ and $\langle w_i, \psi(f_j)\rangle_\beta|_{q=1/z} = 0$, so, by hypothesis, $T(\psi(f_j))$ is isotopic to an arc which is disjoint from w_i. It is simple to show that this arc can be chosen to be disjoint from all such w_i ,where $i \neq j, j+1$, by using Lemma 2.6. Thus ψ must fix all the $T(f_j)$s, and so may be considered as the identity except on the annulus formed by removing a regular neighbourhood of these from the original disc. Hence ψ can only be some power of Δ^2. However, $\beta_n(z)(\Delta^2) = z^n I_{n-1}$, so ψ must be trivial unless z is a root of unity.

(1.)⇒(2.) Let τ_N be a Dehn twist about a curve parallel to $\partial D \cup W$. Let τ_F be a half Dehn twist about the boundary of a regular neighbourhood of $T(F)$.

Now consider the effect of τ_F on an arc, α that crosses it once. This is shown in figure 10. Up to homology, this is equivalent to adding the curve F' shown in figure 11 to α.

Similarly, if α crosses W once, then τ_W has the effect shown in figure 10. Up to homology, this is equivalent to adding the curve W' shown in figure 11.

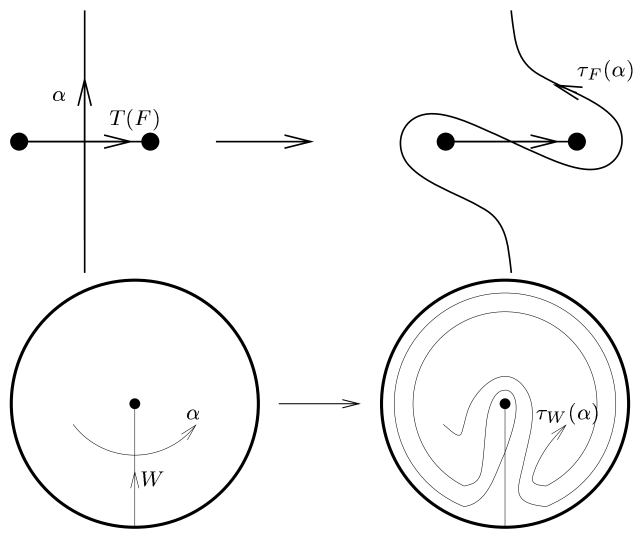


Figure 10: The effects of τ_F and τ_W on a curve.

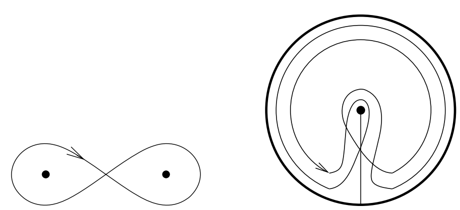


Figure 11: The curves added by τ_F and τ_W.

Now, both F' and W' lift to closed curves in $\widetilde{D}_n$, call some choices of these $\widetilde{F}'$ and $\widetilde{W}'$ respectively. Now the effects of $(\tilde{\tau}_F)_*$ and $(\tilde{\tau}_W)_*$, being the maps on $H_1(\widetilde{D}_n)$ induced by τ_F and τ_W respectively, on a closed curve, $\tilde{\alpha}$ in $\widetilde{D}_n$ are to add copies of $\widetilde{F}'$ or $\widetilde{W}'$ respectively.

Now if $\langle W, F\rangle_\beta|_{q=z} = 0$ and $\langle W, F\rangle_\beta|_{q=1/z} = 0$, then, working under

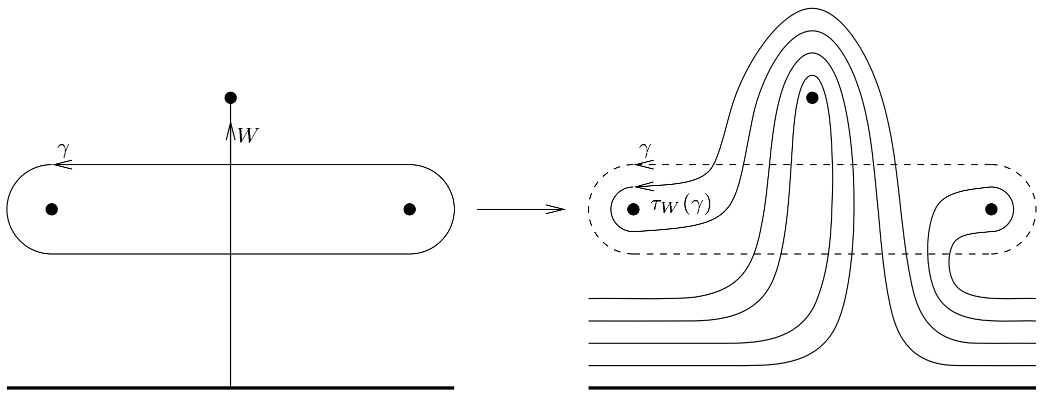


Figure 12: The effect of τ_W on γ.

the specialisation $q = z$, $(\tilde{\tau}_W)_*(\widetilde{F}') = \widetilde{F}'$ and $(\tilde{\tau}_F)_*(\widetilde{W}') = \widetilde{W}'$, so

$$(\tilde{\tau}_W)_*(\tilde{\tau}_F)_*(\tilde{\alpha}) = (\tilde{\tau}_W)_*(\tilde{\alpha} + f\widetilde{F}') = \tilde{\alpha} + g\widetilde{W}' + f\widetilde{F}',$$
$$(\tilde{\tau}_F)_*(\tilde{\tau}_W)_*(\tilde{\alpha}) = (\tilde{\tau}_F)_*(\tilde{\alpha} + g\widetilde{W}') = \tilde{\alpha} + f\widetilde{F}' + g\widetilde{W}'.$$

Therefore $(\tilde{\tau}_F)_*$ and $(\tilde{\tau}_W)_*$ commute, meaning $[\tau_W, \tau_F] \in \ker \gamma_n$. It remains only to show that this element in not trivial, that is that τ_W and τ_F do not commute.

To do this, consider the effect of both on a closed curve, γ, which is defined to be the boundary of a regular neighbourhood, Γ of $T(F)$. Similarly, let δ be a closed curve that is the boundary of a regular neighbourhood of $W \cap \partial D$.

By assumption, W cannot be homotoped off $T(F)$ rel endpoints, hence it cannot be homotoped off γ rel endpoints. It follows that γ cannot be freely homotoped off W and hence δ. Now *claim* that $\tau_W(\gamma)$ cannot be freely homotoped off $T(F)$.

For this we use Lemma 2.6: First assume that no digons exist between γ and δ. Now consider $\tau_W(\gamma)$ after a small homotopy to move the parts unaffected by τ_W to inside Γ. Figure 12 shows the effect of τ_W on γ if $T(F)$ intersects W only once.

Notice that outside Γ, every arc of $\tau_W(\gamma)$ is parallel to an arc of δ, any digons between γ and $\tau_W(\gamma)$ must be inside Γ. Now all arcs of $\tau_\beta(\gamma) \cap \Gamma$

intersect $T(F)$ precisely once, and hence split Γ into two parts, each containing one puncture point, (these two points are the endpoints of $T(F)$). Thus there can be no digons between γ and $\tau_W(\gamma)$ within Γ either. The same will be true if $T(F)$ intersects W more times. Hence $\tau_W(\gamma)$ cannot be freely homotoped off γ and so nor can it be freely homotoped off $T(F)$.

Now claim that $\tau_F\tau_W(\gamma)$ cannot be freely homotoped off $\tau_W(\gamma)$. This is a very similar argument to before: Let $\Gamma' = \tau_W(\Gamma)$. Outside Γ', $\tau_F\tau_W(\gamma)$ is parallel to $\tau_W(\gamma)$ and inside it each arc intersects β once, so $\tau_F\tau_W(\gamma)$ cannot be freely homotoped off $\tau_W(\gamma)$ (note that $\tau_F\tau_W(\gamma)$ cannot be wholly contained within Γ' because otherwise $\tau_W(\gamma)$ could be homotoped off $T(F)$).

The required result follows because $\tau_F(\gamma) = \gamma$, so $\tau_W\tau_F(\gamma) = \tau_W(\gamma)$, hence $\tau_W\tau_F(\gamma)$ cannot be freely homotoped off $\tau_F\tau_W(\gamma)$, so in particular,

$$\tau_F\tau_W(\gamma) \neq \tau_W\tau_F(\gamma),$$

which means $\tau_F, \tau_W \neq \mathrm{Id}$. □

4 The Gassner representation

The Gassner representation of the pure braid group, P_n is formed in a similar way to the Burau representation of B_n, but carries more information. Again take $y_0 \in \partial D_n$ as the basepoint of D_n and $x_1, \ldots, x_n$ as free generators of $\pi_1(D_n, y_0)$.

Definition 4.1. Consider $\mathbb{Z}^n$ to be the free abelian group generated multiplicatively by elements $q_1, \ldots, q_n$. Then consider the abelianising map $\pi_1(D_n, y_0) \to \mathbb{Z}^n$, taking $x_i \mapsto q_i$ for $i = 1, \ldots, n$. Let $\widetilde{D}_n$ be the covering space of D_n corresponding to this map. As a basepoint for $\widetilde{D}_n$, choose some lift of y_0 and denote it $\tilde{y}_0$.

Claim 4.2. $H_1(\widetilde{D}_n)$ is a rank $n-1$ module over $\Lambda[q_1, \ldots, q_n]$.

Proof. The deformation retract of $\widetilde{D}_n$ is rather more complex than for the Burau representation. It is an n-dimensional grid (see figure 13), upon which the covering translation q_i acts by moving each vertex to the next vertex in the ith dimension.

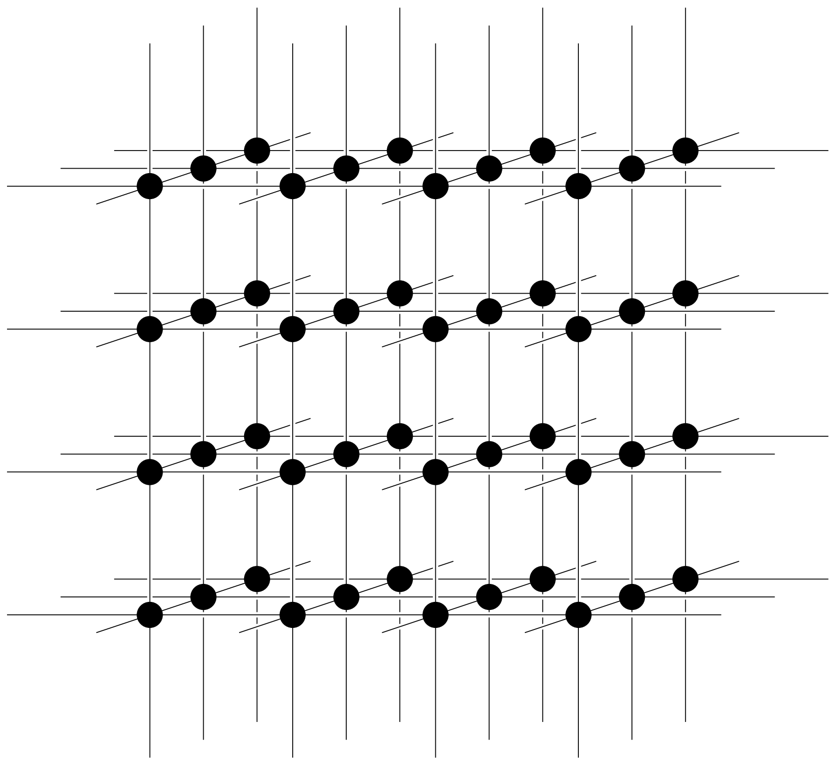


Figure 13: The deformation retract of $\widetilde{D}_3$.

Let η_{ij} denote the loop, based at $\tilde{y}_0$, in this grid that passes anti-clockwise around a square in the ij-plane. These loops are related, up to homology, by the equations:

$$(1-q_i)\eta_{jk} + (1-q_j)\eta_{ki} + (1-q_k)\eta_{ij} = 0,$$

which may be rewritten as

$$(1-q_j)\eta_{ik} = (1-q_i)\eta_{jk} + (1-q_k)\eta_{ij}.$$

Therefore $\eta_{12}, \eta_{23}, \ldots, \eta_{n-1,n}$ is a basis of $H_1(\widetilde{D}_n)$ over $\Lambda[q_1, \ldots, q_n]$. □

A homeomorphism of D_n, ψ, representing an element of P_n then naturally lifts to $\tilde{\psi}$, a homeomorphism of $\widetilde{D}_n$, which induces $(\tilde{\psi})_*$, a homomorphism of $H_1(\widetilde{D}_n)$. This is independent of the choice of representative of the braid.

Definition 4.3. The *Gassner representation* is the map

$$\begin{aligned} \gamma_n : P_n &\to GL(n-1, \Lambda[q_1, \ldots, q_n]), \\ [\psi] &\mapsto (\tilde{\psi})_*. \end{aligned}$$

4.1 The Gassner module

As with the Burau module, elements can be represented as sums of forks, using the simple forks $f_1, \ldots, f_{n-1}$ as a basis, where f_i represents the element of $H_1(\widetilde{D}_n)$ which is $(1-q_1)\cdots(1-q_{i-1})(1-q_{i+2})\cdots(1-q_n)$ times the lift of an embedded circle in $H_1(\widetilde{D}_n)$. The projection of this circle down to D_n is the curve shown in figure 14.

Then the following fork skein relations hold.

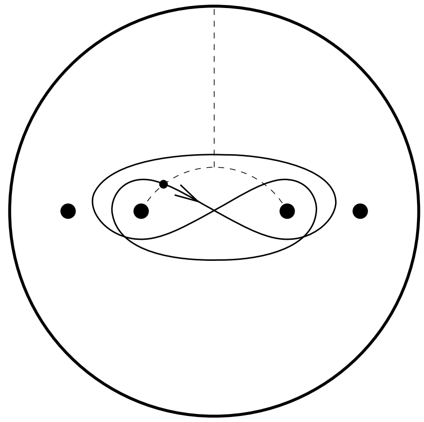
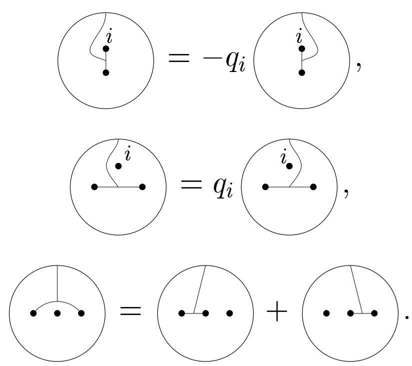


Figure 14: A fork (dashed) with corresponding curve.



(2)

As in the Burau module, any orientation-preserving homeomorphism of D may be applied to these relations and they still hold. In particular note that any braid, not just any pure braid, may be applied.

Example 4.4. Recall the standard generators of P_n, denoted A_{ij}, where $1 \leq i < j \leq n$.

1. If $i + 1 < j$, then

$$\gamma_n(A_{ij}) = \begin{pmatrix} I_{i-2} & 0 & 0 \\ 0 & S_{ij} & 0 \\ 0 & 0 & I_{n-j-1} \end{pmatrix},$$

where S_{ij} is the $(j-i+2)\times(j-i+2)$ matrix

$$S_{ij} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q_i(1-q_j) & q_j & 0 & \cdots & 0 & q_j(q_i-1) & 1-q_i \\ & q_j-1 & 1 & & & & \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ & & & & 1 & q_j(q_i-1) & \\ q_i(1-q_j) & q_j-1 & 0 & \cdots & 0 & q_iq_j-q_j+1 & 1-q_i \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Perhaps a simpler way to write this is as $S_{ij} = I_{j-i+1} - P_{ij}Q_{ij}$, where P_{ij} is the $n\times 1$ matrix with 1s from the ith to $(j-1)$st places and 0s elsewhere and Q_{ij} is the $1\times n$ matrix

$$(0,\ldots,0,\ q_i(q_j-1),\underbrace{1-q_j}_{i\text{th entry}},0,\ldots,0,\ q_j(1-q_i),\underbrace{q_i-1}_{j\text{th entry}},0,\ldots,0).$$

If $i=1$ or $j=n$, then the first or last row and column of S_{ij} will be removed. For example

$$\gamma_4(A_{14}) = \begin{pmatrix} q_4 & 0 & q_4(q_1-1) \\ q_4-1 & 1 & q_4(q_1-1) \\ q_4-1 & 0 & q_1q_4-q_4+1 \end{pmatrix}.$$

2. If $i=j+1$, then

$$\gamma_n(A_{i,i+1}) = \begin{pmatrix} I_{i-2} & 0 & 0 \\ 0 & S_{i,i+1} & 0 \\ 0 & 0 & I_{n-i-2} \end{pmatrix},$$

where $S_{i,i+1}$ is the 3×3 matrix

$$S_{i,i+1} = \begin{pmatrix} 1 & 0 & 0 \\ q_i(q_{i+1}-1) & q_iq_j & 1-q_i \\ 0 & 0 & 1 \end{pmatrix}.$$

3. $\beta_n(\Delta^2) = (q_1q_2\cdots q_n)I_{n-1}$.

The Gassner representation is unitary, as shown by Abdulrahim [1]. The characteristic polynomial of $\gamma_n(A_{ij})$ is $(\mu - q_iq_j)(\mu-1)^{n-2}$.

4.2 Faithfulness

The Gassner representation is faithful for $n \leq 3$ for the simple reason that the Burau representation, which is faithful for these n, factors through it. For larger n, faithfulness is unknown. Bachmuth [3] did claim to have proven the Gassner representation to be faithful for all n, but his article was later refuted by Abramenko and Müller [2].

Definition 4.5. For a whisker, W, define $\widetilde{W}$ to be its lift to $\widetilde{D}_n$ based at y_0.

Define the *Gassner pairing* between a fork and a whisker to be the Laurent polynomial

$$\langle W, F\rangle_\gamma = \sum_{\forall i, a_i \in \mathbb{Z}} q_1^{a_1} \cdots q_n^{a_n} \left(q_1^{a_1} \cdots q_n^{a_n} \widetilde{W}, T(\widetilde{F})\right),$$

where $\left(q_1^{a_1} \cdots q_n^{a_n} \widetilde{W}, T(\widetilde{F})\right)$ represents the algebraic intersection number of the two curves. Consider $q_1^{a_1} \cdots q_n^{a_n} \widetilde{W}$ to be the image of $\widetilde{W}$ under the deck transformation $q_1^{a_1} \cdots q_n^{a_n}$.

The next theorem is the adaption of Theorem 3.10 to the Gassner representation, and a main result of this paper.

Theorem 4.6. *The following are equivalent:*

1. *The reduced Gassner representation of P_n is faithful.*
2. *If W and F are a whisker and a fork in D_n such that $\langle W, F\rangle_\gamma = 0$ then $T(F)$ is isotopic to an arc which is disjoint from W.*

Proof. (2.)⇒(1.) By a similar argument to that given for 3.11, given (2.), a braid, $\psi \in \ker \gamma_n$ preserves the tines of the simple forks, so must be some power of Δ^2. However, $\gamma_n(\Delta^2) = (q_1 \cdots q_n) I_{n-1}$, therefore ψ is trivial.

(1.)⇒(2.) Let τ_W be the same as in the proof of Theorem 3.11. Let τ_F be a Dehn twist about the boundary of a regular neighbourhood of $T(F)$. In the proof of Theorem 3.11 it was a half twist.

The effects of τ_W and τ_F on arcs are shown in figure 15. Up to homology these are equivalent to adding the curves W' and F' shown in figure 16.

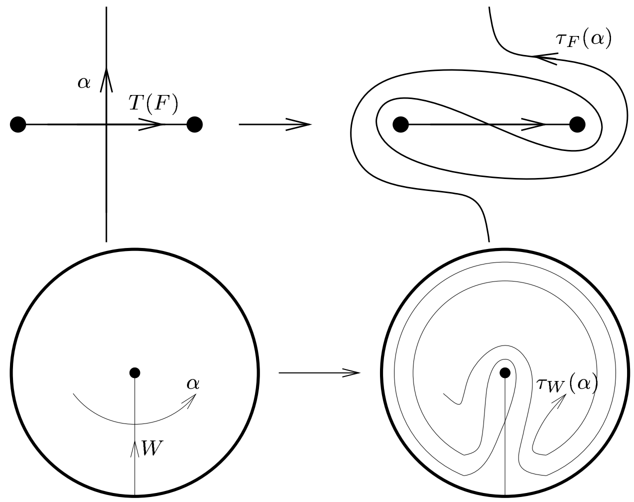


Figure 15: The effects of τ_F and τ_W on curves.

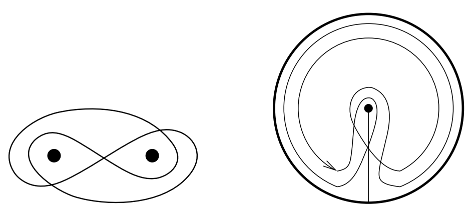


Figure 16: The two curves added by τ_F and τ_W.

Let $\widetilde{F}'$ and $\widetilde{W'}$ be lifts of F' and W' to $\widetilde{D}_n$. These are simple closed curves.

Now if $\langle W, F\rangle_\gamma = 0$, then $(\tilde{\tau}_F)_*(\widetilde{W'}) = \widetilde{W'}$ and $(\tilde{\tau}_W)_*(\widetilde{F}') = \widetilde{F}'$, so

$$(\tilde{\tau}_W)_*(\tilde{\tau}_F)_*(\tilde{\alpha}) = (\tilde{\tau}_W)_*(\tilde{\alpha} + f\widetilde{F}') = \tilde{\alpha} + g\widetilde{W'} + f\widetilde{F}', \tag{3}$$

$$(\tilde{\tau}_F)_*(\tilde{\tau}_W)_*(\tilde{\alpha}) = (\tilde{\tau}_F)_*(\tilde{\alpha} + g\widetilde{W'}) = \tilde{\alpha} + f\widetilde{F}' + g\widetilde{W'}. \tag{4}$$

Therefore $(\tilde{\tau}_F)_*$ and $(\tilde{\tau}_W)_*$ commute, meaning $[\tau_W, \tau_F] \in \ker \gamma_n$. A similar argument to the proof of Theorem 3.11 shows that this element is non-trivial. □

The pairing $\langle W, F\rangle_\gamma$ is calculated by considering the contribution each point in $W \cap T(F)$ makes towards it. To do this, form a closed path, going from the basepoint to the intersection point, x, along the fork and back along the whisker. If a_i is the winding number of this path about the point p_i, then let $m(x) = q_1^{a_1} \cdots q_n^{a_n}$. This is called the *associated monomial* of x. Given two points, $x, y \in W \cap T(F)$, the difference between the two associated monomials can be similarly calculated using the path $\alpha(x, y)$ from x to y along $T(F)$ and back along W. If $\operatorname{sgn}(x)$ denotes the sign of the intersection at x, then

$$\langle W, F\rangle_\gamma = \sum_{x \in W \cap T(F)} \operatorname{sgn}(x) m(x).$$

For the pairing to vanish, the contribution from each intersection point must cancel with another. Therefore, for each point $x \in W \cap T(F)$ there must exist another point $y \in W \cap T(F)$ with the same associated monomial but with the opposite sign.

Bigelow [6] showed that it is impossible for points in D_3 to cancel, even under the weaker pairing associated with the Burau representation. In D_4 cancellation is possible, see figure 17, which shows the tine of a fork and part of a whisker. The two intersection points labelled $*$ have opposite signs, but the same associated monomial, as can be checked using the path from one point to the other along the tine and back along the whisker. The winding number of this path about the point p_i is the power of q_i that appears in the quotient of the two associated monomials. The winding number of a path about a point is equal to its oriented intersection number of the path with the shaft connected to that point. In figure 17 then, the part of the whisker shown has zero intersection number with each fork, which demonstrates that the two points labelled $*$ have the same associated monomials. It is a simple matter to add a fork handle and the remainder of the whisker.

Figure 18 shows an example in D_5, with a simpler whisker. In fact figure 17 was derived from this by changing the tine.

4.3 Relationship to the Burau representation

If the q_is are all specialised to the same variable, q, then the Gassner module becomes the Burau module. So let g_n be the map

$$g_n : \Lambda[q_1, \ldots, q_n] \to \Lambda[q],$$

which acts by sending $q_i \mapsto q$ for all i. Then define

$$G_n : GL(n-1, \Lambda[q_1, \ldots, q_n]) \to GL(n-1, \Lambda[q])$$

to be the map which acts as g_n on each entry of the matrix. Then

$$G_n \circ \gamma_n = \beta_n.$$

4.4 Complex specialisations

Definition 4.7. Define $\gamma_n(z_1, \ldots, z_n)$ to be the reduced Gassner representation of P_n under the specialisation $q_i = z_i$ for all i.

Let $f(z_1, \ldots, z_n)$ be the specialisation of f to $q_1 = z_1, \ldots, q_n = z_n$.

To shorten notation, we use $f|_{q_i=z_i}$ to indicate the polynomial f with q_i specialised to $z_i \in \mathbb{C}$ for all i from 1 to n.

Lemma 4.8. *Δ^2 is in the kernel of $\gamma_n(z_1, \ldots, z_n)$ if and only if $z_1 z_2 \cdots z_n$ is a root of unity.*

Proof. Under the unspecialised, reduced Gassner representation,

$$\gamma_n(\Delta^{2m}) = (q_1 q_2 \cdots q_n)^m I_n,$$

which is equal to I_n if and only if $(q_1 q_2 \cdots q_n)^m = 1$. □

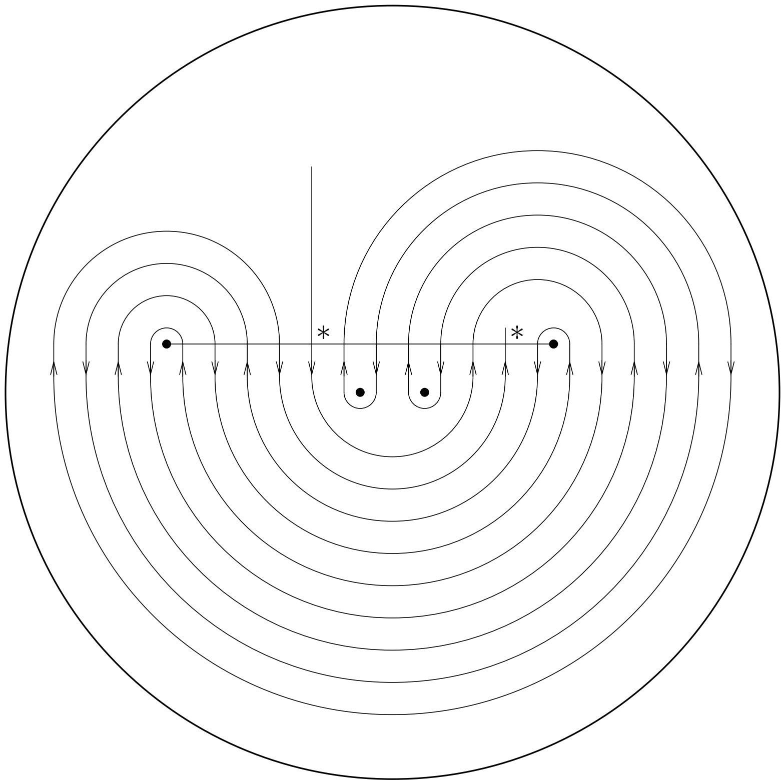


Figure 17: An example of a tine of a fork and part of a whisker in D_4 where two points cancel out in the pairing.

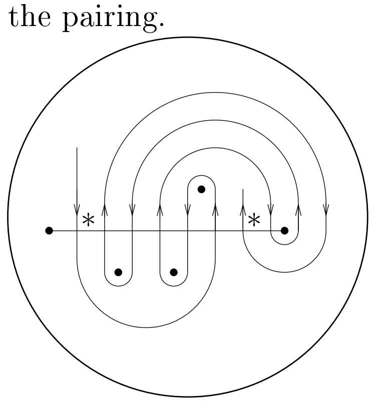


Figure 18: An example in D_5.

Remark 4.9. In P_2, there is just one generator, A_{12}, which equals Δ^2, so $\gamma_2(z_1, z_2)$ is faithful if and only if $z_1 z_2$ is not a root of unity.

Theorem 4.10. *If $z_1 \cdots z_n$ is not a root of unity, then the following are equivalent:*

1. *The reduced Gassner representation of P_n specialised at $q_i = z_i$ for all i is faithful.*
2. *If W and F are a whisker and a fork in D_n such that $\langle W, F\rangle_\gamma|_{q_i=z_i}$ and $\langle W, F\rangle_\gamma|_{q_i=\frac{1}{z_i}}$ are both zero, then $T(F)$ is isotopic to an arc which is disjoint from W.*

Proof. (2.)⇒(1.) This is similar to the proof of Theorem 3.11. For $i \neq j, j+1$, $\langle w_i, f_j\rangle_\gamma = 0$ because the whisker is disjoint from the tine of the fork.

Now if $\psi \in \ker \gamma_n(z_1, \ldots, z_n)$, then we have both $\langle w_i, \psi(f_j)\rangle_\gamma|_{q_i=z_i} = 0$ and $\langle w_i, \psi(f_j)\rangle_\gamma|_{q_i=\frac{1}{z_i}} = 0$, so, by hypothesis, $T(F)$ is isotopic to an arc which is disjoint from f_i. Using Lemma 2.6, it is simple to show that this arc can be chosen to be disjoint from all such w_i (where $i \neq j, j+1$). Thus ψ must fix all the $T(f_j)$s, which, as in the proof of Theorem 3.11, this means ψ is some power of Δ^2, but $\gamma_n(z_1, \ldots, z_n)(\Delta^2) = z_1 \cdots z_n I_{n-1}$, so ψ must be trivial unless $z_1 \cdots z_n$ is a root of unity.

(1.)⇒(2.) Again, this is similar to the proof of 3.11. Let τ_W, τ_F be the same as in the proof of Theorem 4.6.

Let $(\tilde{\tau}_F)_*$ and $(\tilde{\tau}_W)_*$, be the maps on $H_1(\widetilde{D}_n)$ induced by τ_F and τ_W respectively.

Now if $\langle W, F\rangle_\gamma|_{q_i=z_i} = 0$ and $\langle W, F\rangle_\gamma|_{q_i=1/z_i} = 0$, then, working under this specialisation, $(\tilde{\tau}_F)_*(\widetilde{F}') = \widetilde{F}'$ and $(\tilde{\tau}_F)_*(\widetilde{W}') = \widetilde{W}'$, so

$$(\tilde{\tau}_W)_*(\tilde{\tau}_F)_*(\tilde{\alpha}) = (\tilde{\tau}_W)_*(\tilde{\alpha} + f\widetilde{F}') = \tilde{\alpha} + g\widetilde{W}' + f\widetilde{F}',$$
$$(\tilde{\tau}_F)_*(\tilde{\tau}_W)_*(\tilde{\alpha}) = (\tilde{\tau}_F)_*(\tilde{\alpha} + g\widetilde{W}') = \tilde{\alpha} + f\widetilde{F}' + g\widetilde{W}'.$$

Therefore $(\tilde{\tau}_F)_*$ and $(\tilde{\tau}_W)_*$ commute, meaning $[\tau_W, \tau_F] \in \ker \gamma_n$. That this element in not trivial follows in precisely the same way as in Theorem 3.11 □

4.4.1 Faithfulness in some cases

If all the q_i are specialised to the same value, then this reduces to the case for the Burau representation. Alternatively, if they are specialised at algebraically independent z_is, (i.e. there is no polynomial in n variables which is zero when specialised at $z_1, \ldots, z_n$), then the specialised Gassner representation is faithful if and only if the unspecialised one is.

Consider a less extreme case, where one of the z_is, say z_1 is equal to one.

1. $\gamma_2(1, z_2)$ is faithful, if and only if z_2 is not a root of unity, see remark 4.9

2. $\gamma_n(1, z_2, \ldots, z_n)$ for $n \geq 3$ is unfaithful for any choice of the remaining z_is. In the case $n \geq 4$, this is easy to see since figure 19 shows a tine and a whisker in D_4 with unspecialised pairing equal to $1 - q_1$, up to a sign or a deck transformation. The addition of extra punctures will not change the pairing, provided they are added outside the digon in D that lies between $T(F)$ and W. If the fork and whisker are labelled F and W respectively, then figure 20 shows the geometric braid $[\tau_F, \tau_W]$, which is in the kernel of this specialisation of the Gassner representation.

   For $n = 3$, note that the images of the generators under γ'_3, which is defined to be $\gamma_3(1, z_2, z_3)$, are

$$\gamma'_3(A_{12}) = \begin{pmatrix} z_2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma'_3(A_{13}) = \begin{pmatrix} z_3 & 0 \\ 1 - z_3 & 1 \end{pmatrix},$$
$$\gamma'_3(A_{12}) = \begin{pmatrix} 1 & 0 \\ z_2(z_3 - 1) & z_2 z_3 \end{pmatrix},$$

   and so

$$\gamma'_3(A_{23}A_{13}) = \begin{pmatrix} z_3 & 0 \\ 0 & z_2 z_3 \end{pmatrix}.$$

   Now, since two diagonal matrices always commute, $[A_{12}, A_{23}A_{13}]$ is in the kernel of γ'_3. Figure 21 shows this braid geometrically, demonstrating that it is not trivial.

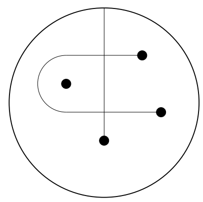


Figure 19: A tine and a whisker showing $\gamma_n(1, z_2, \ldots, z_n)$ to be unfaithful.

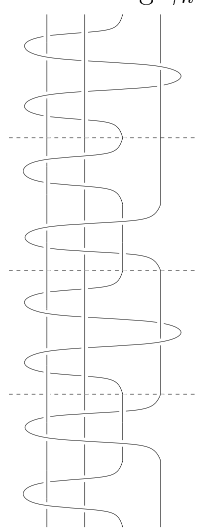


Figure 20: The geometric braid formed from this tine and whisker.

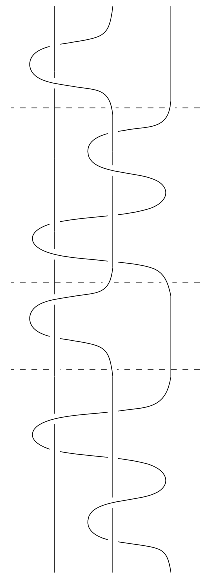


Figure 21: A geometric braid in the kernel of $\gamma_3(1, z_2, z_3)$.

If a product of two or more z_i is a root of unity, then $\gamma_n(z_1, \ldots, z_n)$ is also unfaithful. For $n = 2$, this is already shown. For $n \geq 3$, consider figure 22. This shows a whisker and tine with Gassner pairing $1 + q_1 q_2 + \ldots + q_1^{k-1} q_2^{k-1}$, where k is the number of intersection points between the two. If $z_1 z_2$ is a kth root of unity other than 1 itself, then under the specialisation $q_1 = z_1$, $q_2 = z_2$, this pairing is zero.

5 The Lawrence-Krammer representation

Definition 5.1. In order to define the Lawrence-Krammer representation, consider the space, denoted C, of all unordered pairs of distinct points in D_n. That is

$$C = \frac{(D_n \times D_n) \setminus \Delta(D_n)}{(x, y) \sim (y, x)},$$

where $\Delta(D_n)$ is the diagonal of $D_n \times D_n$. A point in C may be represented by $\{x, y\}$, where $x, y \in D_n$ and $x \neq y$. Choose a basepoint for C to be

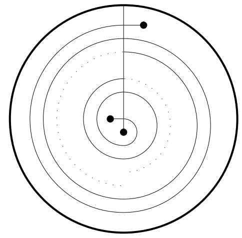


Figure 22: A whisker and tine in D_3.

$Y_0 = \{y_0, y_0'\}$, where y_0, y_0' are distinct points in ∂D_n.

A path, $\alpha : [0,1] \to C$ may be represented by $\alpha = \{\alpha_1, \alpha_2\}$, where $\alpha_1, \alpha_2 : [0,1] \to D_n$, are two paths such that $\alpha_1(t) \neq \alpha_2(t)$, for all $t \in I$. Then $\alpha(t) = \{\alpha_1(t), \alpha_2(t)\}$. Notice that the paths may intersect, as long as they are not in the same place for the same value of t. Now if α_1 and α_2 both have start and endpoints at y_0 or y_0' then $\alpha \in \pi_1(C)$. There are two ways for this to happen,

1. α_1 is a closed path based at y_0 and α_2 is a closed path based at y_0'.

2. $\alpha_1(0) = \alpha_2(1) = y_0$ and $\alpha_1(1) = \alpha_2(0) = y_0'$.

In both cases, the rôles of α_1 and α_2 may be interchanged.

Notice also that a homotopy, $\Gamma : I \times I \to C$ of γ (i.e. $\Gamma(0,t) = \gamma(t)$) is defined by any two homotopies, $\mathrm{A}_1 : I \times I \to C$ of α_1 and $\mathrm{A}_2 : I \times I \to C$ of α_2, such that $\mathrm{A}_1(s,t) \neq \mathrm{A}_2(s,t)$ for all $s, t \in I$, that is each pair of paths corresponding to fixed s defines a valid path in C.

Definition 5.2. If α_1, α_2 are paths in D_n, such that $\alpha_1(t) \neq \alpha_2(t)$ for all $t \in [0,1]$, we define

$$a = \frac{1}{2\pi i} \sum_{j=1}^{n} \left(\int_{\alpha_1} \frac{dz}{z - p_j} + \int_{\alpha_2} \frac{dz}{z - p_j} \right) \tag{5}$$

$$b = \frac{1}{\pi i} \int_{\alpha_1 - \alpha_2} \frac{dz}{z}. \tag{6}$$

For closed paths, the value of a is the sum of the winding numbers of the paths about the puncture points. For the other case, it is the sum of the winding numbers of the closed path $\alpha_1\alpha_2$ about the puncture points.

On the other hand, b is twice the winding number of the two paths about each other, a value which cannot be calculated from the images of α_1 and α_2, though it only needs a little extra information:

First assume the intersection points between α_1 and α_2 are transverse and that there are no triple points. Now index the preimage of the intersection points under α_1 as $t_1, \ldots, t_n \in [0,1]$ and the preimage under α_2 as $t'_1, \ldots, t'_n$, so that $\alpha_1(t_j) = \alpha_2(t'_j)$, for $j = 1, \ldots, n$. Then, for each point, define

$$c_i = \begin{cases} +1, & \text{if the crossing of } \alpha_1 \text{ over } \alpha_2 \text{ is positive,} \\ -1 & \text{if the crossing is negative,} \end{cases}$$

$$d_i = \begin{cases} +1, & \text{if } t'_i < t_i, \\ -1, & \text{if } t_i < t'_i. \end{cases}$$

Lemma 5.3. *If α_1 and α_2 are closed paths, based in ∂D, then the value of b can be calculated as*

$$b = \sum_{i=1}^{k} c_i d_i.$$

Proof. If the paths do not intersect, the sum is 0, which is also clearly the value of b. Then proceed by induction: Consider the effect of homotoping one path to produce more intersection points (the fact that any two paths in a disc may be homotoped off each other together with lemma 2.6 show that this is sufficient as the induction step).

In the left-hand part of figure 23, the two paths are oriented in the same direction near the new crossing points, which will be labelled $k+1$ and $k+2$. Notice, $c_{k+1} = +1$ and $c_{k+2} = -1$. The dashed lines join points $\alpha_1(t)$ and $\alpha_2(t)$, so show the value of $\alpha_1 - \alpha_2$ at the point t. From this it is easily seen that $d_{k+1} = +1$ and $d_{k+2} = -1$ and that this move increases both the value of b by 2, which is as required since $\sum_{i=1}^{k+2} c_i d_i = \sum_{i=1}^{k} c_i d_i + 2$. The cases with other values of d_{k+1} and d_{k+2} may be similarly calculated.

The right-hand part of figure 23 shows the situation where the paths are oriented in opposite directions near the new crossings. Here $c_{k+1} = -1$, $c_{k+2} = +1$, $d_{k+1} = -1$ and $d_{k+2} = +1$, giving an increase of +2 in the sum, which is also the increase in the value of b. □

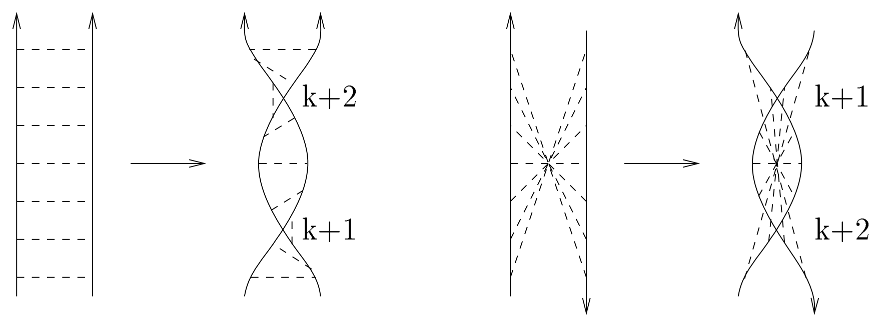


Figure 23: The introduction of two new crossings and its effect on b.

Corollary 5.4. *If $\alpha_1(0) = \alpha_2(1) = y_0$ and $\alpha_1(1) = \alpha_2(0) = y_0'$, then the same formula holds, except that the intersections at y_0 and y_0' contribute only* $\pm\frac{1}{2}$

Proof. To prove this, it is only necessary to consider other basis cases. Figure 24 shows one such, for which the equality holds. □

Definition 5.5. For any path, α, representing an element of $\pi_1(C, Y_0)$, let $\phi(\alpha) = q^a t^b$. Then let $\widetilde{C}$ be the covering space corresponding to ϕ.

A braid, ψ, can be lifted to $\tilde{\psi}$, a homeomorphism of $\widetilde{C}$. It then induces a map on $H_2(\widetilde{C})$, which is denoted $(\tilde{\psi})_*$.

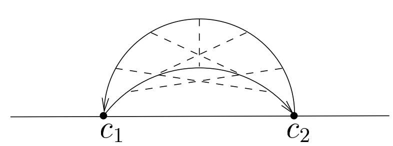


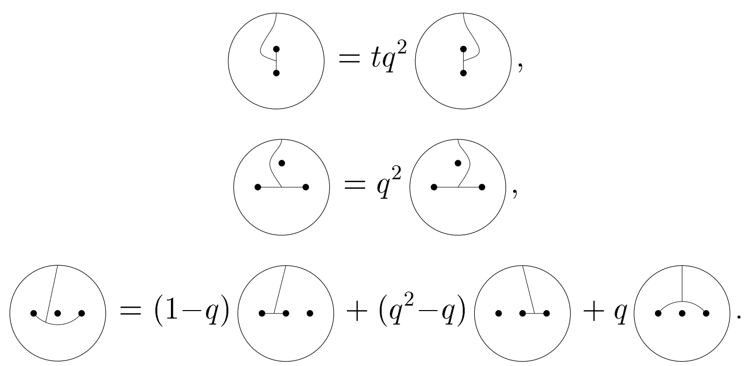
Figure 24: A basis case in the calculation of b.

Definition 5.6. The *Lawrence-Krammer representation* of the braid group is the map

$$\kappa_n : B_n \to GL(\tbinom{n}{2}, \Lambda[t,q]),$$
$$[\psi] \mapsto (\tilde{\psi})_*.$$

5.1 The Lawrence-Krammer module

Bigelow [5] showed that elements of $H_2(\widetilde{C})$, considered as a module over $\Lambda[t,q]$, can be represented in terms of the standard forks (i.e. those contained in the upper half plane). Any fork may be reduced to a linear combination (in q and t) of standard forks by using the fork skein relations below.



We can now write elements of the image of κ_n as matrices over the basis of standard forks. Recall the standard ordering: $f_{12}, f_{13}, f_{23}, f_{14}$, etc.

Example 5.7. 1. $\kappa_4(\sigma_1) = \begin{pmatrix} tq^2 & tq(q-1) & 0 & tq(q-1) & 0 & 0 \\ 0 & 1-q & 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q & 1 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

2. $\kappa_n(\Delta^2) = t^2 q^{2n} I_{\binom{n}{2}}$.

Claim 5.8. The characteristic polynomial of $\kappa_n(\sigma_i)$ is $(\mu - tq^2)(\mu+q)^{n-2}(\mu - 1)^{\frac{1}{2}(n-1)(n-2)}$.

Proof. Since the σ_i are conjugate to each other, it suffices to show the result for σ_1.

First note $\det(\mu I_{\binom{n}{2}} - \kappa_2(\sigma_1)) = \mu - tq^2$. Now

$$\kappa_n(\sigma_1) = \left(\begin{array}{c|c} \kappa_{n-1}(\sigma_1) & * \\ \hline 0 & \kappa'_n(\sigma_1) \end{array}\right),$$

where $\kappa'_n(\sigma_1)$ is the $(n-1) \times (n-1)$ matrix

$$\kappa'_n(A_{12}) = \left(\begin{array}{cc|c} 1-q & 1 & \multirow{2}{*}{0} \\ q & 0 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-3} \end{array}\right).$$

Then for $n \geq 2$, by induction

$$\begin{aligned} \det(\mu I_{\binom{n}{2}} - \kappa_n(\sigma_1)) &= \det(\mu I_{\binom{n-1}{2}} - \kappa_{n-1}(\sigma_1)) \cdot \det(\mu I_{n-1} - \kappa'_n(\sigma_1)) \\ &= (\mu - tq^2)(\mu + q)^{n-3}(\mu - 1)^{\frac{1}{2}(n-2)(n-3)} \cdot (\mu + q)(\mu - 1)^{n-2} \\ &= (\mu - tq^2)(\mu + q)^{n-2}(\mu - 1)^{\frac{1}{2}(n-1)(n-2)}. \end{aligned}$$

□

5.2 Faithfulness

The Lawrence-Krammer representation is faithful for all n, in fact it was the first known example of an always faithful linear representation of B_n. This was shown by Krammer [11] for κ_4, using an algebraic argument, then in general by Bigelow [5] using a topological argument, outlined below. Krammer [12] then adapted his argument to the general case.

Just as for the Burau and Gassner representations, this depends upon a pairing, defined in terms of the intersections between two surfaces, though since this representation uses the second homology group, these surfaces are not just lifts of the fork and noodle.

Definition 5.9. Given a fork, F, and a parallel fork, F', in D_n let $\Sigma(F)$ be the surface in C consisting of all points that can be written $\{x, y\}$ where

$x \in T(F)$ and $y \in T(F')$. This surface has a natural orientation as $F \times F'$. $\Sigma(F)$ is given a handle, defined by the handles of the fork and parallel fork. A path along each of these define a path in C. Finally, let $\widetilde{\Sigma}(F)$ be the lift of $\Sigma(F)$ that has handle based at $\tilde{Y}_0$

If N is a noodle in D_n, define a surface, $\Sigma(N)$ in C to consist of all unordered pairs of distinct points in N. This is isomorphic to the section of $N \times N$ that lies above the diagonal, take this to give the orientation. Then define $\widetilde{\Sigma}(N)$ to be the lift of $\Sigma(N)$ to $\widetilde{C}$ that is based at $\tilde{Y}_0$.

A fork may also stand for a representative of an element of $H_2(\widetilde{C})$, which is identical to a multiple of $\widetilde{\Sigma}(F)$ away from the lifts of the punctures, see [5].

Definition 5.10. The *Lawrence-Krammer pairing* is defined for any fork, F, and noodle, N, in D_n, as

$$\langle N, F \rangle_\kappa = \sum_{a,b \in \mathbb{Z}} q^a t^b \big(q^a t^b \widetilde{\Sigma}(N), \widetilde{\Sigma}(F) \big).$$

This pairing, like the Burau pairing (see theorem 3.10), gives a criterion for faithfulness of the representation.

Theorem 5.11. *The following are equivalent:*

1. *The Lawrence-Krammer representation of B_n is faithful.*
2. *If N, F are any noodle and fork in D_n such that $\langle W, F \rangle_\kappa = 0$, then $T(F)$ is homotopic, rel endpoints, to an arc which is disjoint from N.*

Unlike, the Burau representation, however, Bigelow argued that faithfulness followed from this.

Theorem 5.12 (Bigelow). *If N is a noodle and F is a fork in D_n with $\langle N, F \rangle_\kappa = 0$ then $T(F)$ can be homotoped rel endpoints to be disjoint from N.*

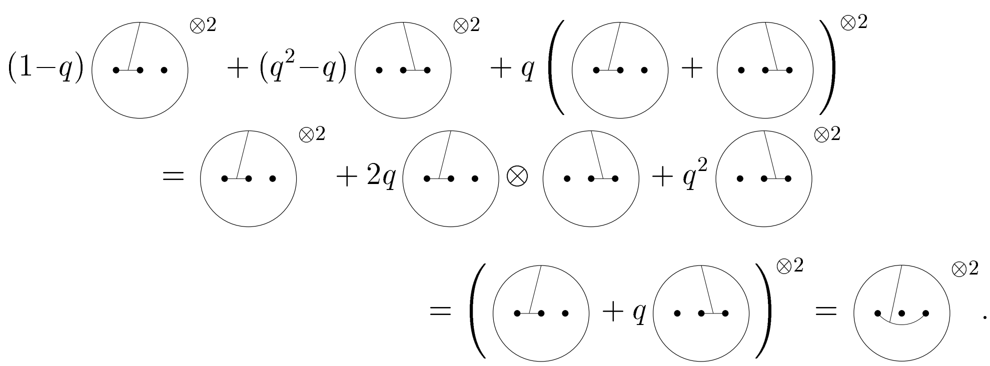
Corollary 5.13 (Bigelow). *The Lawrence-Krammer representation of B_n is faithful for all n.*

5.3 Relationship to the Burau representation

Under the specialisation $t = 1$, the Lawrence-Krammer module becomes the symmetric square of the Burau module. In terms of the modules, a fork in the Lawrence-Krammer module may be viewed as the tensor square of the equivalent fork in the Burau module. In particular, the standard bases are related as so:

$$f_{ij} = (f_i + \cdots + f_{j-1}) \otimes (f_i + \cdots + f_{j-1}).$$

With these in mind, the relations in the Lawrence-Krammer module with t specialised to 1 follow naturally from those in the Burau module. The first two are immediate, since the Lawrence-Krammer module then has a factor of q^2 where the Burau has $\pm q$. The third can be seen as follows:



Since the Lawrence-Krammer representation is faithful, there is a well-defined, surjective map $K_n : \kappa_n(B_n) \longrightarrow \beta_n(B_n)$ taking the image of a braid, ψ under κ_n to its image under β_n. Using the relations, this map can be calculated more explicitly.

The entries in column $(k, k{+}1)$ of $\kappa_n(\psi)$ may be viewed as coefficients of f_{ij}s in a sum, in which case column k of $\beta_n(\psi)$, viewed as listing coefficients of f_is, can be calculated by mapping, term by term,

$$t^a q^b f_{ij} \mapsto (-1)^a q^{\frac{b}{2}} (f_i + \cdots + f_{j-1}).$$

The map K_n can be extended to a map of general linear groups, however the codomain has to include some half powers of q as well. So it maps

$$GL\left(\binom{n}{2}, \Lambda[t,q]\right) \to GL(n-1, \Lambda[q^{\frac{1}{2}}])$$

Note that K_n is injective (indeed, bijective) precisely when β_n is faithful. So if β_4 is faithful, it must be possible to reconstruct the powers of t in each entry of $\kappa_4(\psi)$, for any $\psi \in B_4$, from $\beta_4(\psi)$. It seems likely that the signs in the Burau module will be important here, as they are not reflected in the symmetric product, which is the Lawrence-Krammer module with t specialised to 1.

6 The Lawrence-Gassner representation

The Gassner and Lawrence-Krammer representations sit over the Burau representation in different ways. Another representation may be produced that completes the diagram:

$$\begin{array}{ccc} \lambda_n(P_n) & \longrightarrow & \kappa_n(B_n) \\ \big\downarrow & & \big\downarrow{\scriptstyle K_n} \\ \gamma_n(P_n) & \xrightarrow{G_n} & \beta_n(B_n)\ . \end{array}$$

To define this representation, we again consider C, the space of all unordered pairs of distinct points in D_n, which was introduced for the definition of the Lawrence-Krammer representation. Recall that a point in C is represented by $\{x, y\}$, where $x, y \in D_n$ and $x \neq y$ and that $Y_0 = \{y_0, y_0'\}$ denotes the basepoint of C.

Recall the values a and b from the definition of the Lawrence-Krammer representation.

Definition 6.1. If α_1, α_2 are paths in D_n, such that $\alpha_1(t) \neq \alpha_2(t)$ for all $t \in [0, 1]$, define

$$a_j = \frac{1}{2\pi i}\left(\int_{\alpha_1} \frac{dz}{z - p_j} + \int_{\alpha_2} \frac{dz}{z - p_j}\right),$$

For closed paths, the value of a_j is the sum of the winding numbers of the paths about the puncture point p_j. For the other case, it is the sum of the winding numbers of the closed path $\alpha_1\alpha_2$ about the point. Note that $\sum_{j=1}^{n} a_j = a$.

Definition 6.2. For any path α representing an element of $\pi_1(C, Y_0)$, define $\phi(\alpha) = q_1^{a_1} \cdots q_n^{a_n} t^b$. Then let $\widetilde{C}$ be the covering space corresponding to ϕ.

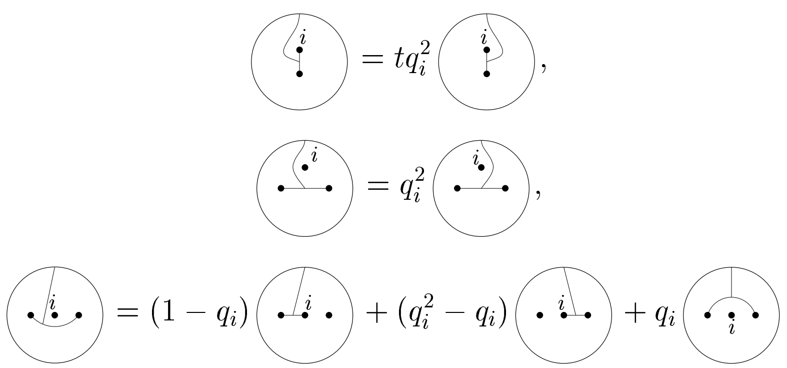
A braid, ψ, can be lifted to $\tilde{\psi}$, a homeomorphism of $\widetilde{C}$. It then induces a map on $H_2(\widetilde{C})$, which is denoted $(\tilde{\psi})_*$.

Definition 6.3. The *Lawrence-Gassner representation* of the braid group is the map

$$\begin{aligned} \lambda_n : B_n &\longrightarrow GL(\tbinom{n}{2}, \Lambda[t, q_1, \ldots, q_n]), \\ [\psi] &\mapsto (\tilde{\psi})_*. \end{aligned}$$

6.1 The Lawrence-Gassner module

Now $H_2(\widetilde{C})$ is a module over $\Lambda[t, q_1, \ldots, q_n]$ and its elements may be represented by forks. The relations in equation 7 hold and can be used to write any fork in terms of the standard forks.



(7)

The effect of one of the standard generators of P_n on the standard forks is then as follows:

If $1 \le i < j \le n,$ then,

$$A_{ij}(f_{ij}) = t^2 q_i^2 q_j^2 f_{ij}.$$

If $1 \leq i < j < k \leq n,$ then,

$$\begin{aligned}
A_{ij}(f_{ik}) &= tq_i(q_j-1)(tq_iq_j-q_i+1)f_{ij} + (q_iq_j-q_i+1)f_{ik},\\
&\quad + q_i(1-q_j)f_{jk},\\
A_{ij}(f_{jk}) &= tq_i(q_i-1)f_{ij} + (1-q_i)f_{ik} + q_if_{jk},\\
A_{ik}(f_{ij}) &= q_kf_{ij} + q_k(q_k-1)f_{ik} + t^{-1}(1-q_k)f_{jk},\\
A_{ik}(f_{jk}) &= tq_k(1-q_i)f_{ij} + tq_k(q_i-1)(tq_iq_k-q_k+1)f_{ik},\\
&\quad + (q_iq_k-q_k+1)f_{jk},\\
A_{jk}(f_{ij}) &= (q_jq_k-q_j+1)f_{ij} + q_j(1-q_k)f_{ik},\\
&\quad + q_j(q_k-1)(tq_jq_k-q_j+1)f_{jk},\\
A_{jk}(f_{ik}) &= (1-q_j)f_{ij} + q_jf_{ik} + q_j(q_j-1)f_{jk}.
\end{aligned}$$

If $1 \leq i < j < k < l \leq n,$ then,

$$\begin{aligned}
A_{ij}(f_{kl}) &= f_{kl}, \qquad A_{jk}(f_{il}) = f_{il},\\
A_{il}(f_{jk}) &= f_{jk}, \qquad A_{kl}(f_{ij}) = f_{ij},\\
A_{ik}(f_{jl}) &= t(1-q_i)(q_k-1)f_{ij} + t(q_i-1)(q_k-1)(tq_iq_k-q_i-q_k+1)f_{ik},\\
&\quad + (q_i-1)(q_k-1)f_{jk} + (q_i-1)q_kf_{il},\\
&\quad + f_{jl} + (q_i-1)(1-q_k)f_{kl},\\
A_{jl}(f_{ik}) &= (q_j-1)(q_l-1)f_{ij},\\
&\quad + f_{ik} + (q_j-1)(1-q_l)f_{jk},\\
&\quad + (q_j-1)(1-q_l)f_{il},\\
&\quad + (q_j-1)(q_l-1)(tq_jq_l-q_j-q_l+1)f_{jl},\\
&\quad + t^{-1}(q_j-1)(q_l-1)f_{kl}.
\end{aligned} \tag{8}$$

Claim 6.4. The characteristic polynomial of $\lambda_n(A_{ij})$ is

$$\left(\mu - t^2q_i^2q_j^2\right)\left(\mu-1\right)^{\binom{n-1}{2}}\left(\mu - q_iq_j\right)^{n-2} \tag{9}$$

Proof. First suppose $n = 2$. Then the characteristic polynomial of $\lambda(A_{12})$ is easily calculated as $\mu - t^2q_1^2q_2^2$.

Now proceed by induction. Suppose that

$$\det\left(\mu I - \lambda_{n-1}(A_{12})\right) = \left(\mu - t^2q_1^2q_2^2\right)\left(\mu-1\right)^{\binom{n-2}{2}}\left(\mu - q_1q_2\right)^{n-3},$$

then observe that for $n \geq 2$

$$\lambda_n(A_{12}) = \left(\begin{array}{c|c} \lambda_{n-1}(A_{12}) & * \\ \hline 0 & \lambda'_n(A_{12}) \end{array}\right), \tag{10}$$

where $\lambda'_n(A_{12})$ is the $(n-1) \times (n-1)$ matrix:

$$\lambda'_n(A_{12}) = \begin{pmatrix} q_1q_2 - q_1 + 1 & 1 - q_1 & 0 & \dots & 0 \\ q_1(1-q_2) & q_1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{11}$$

Now calculate

$$\begin{aligned} &\det\left(\mu I - \lambda_n(A_{12})\right) \\ &\quad = \det\left(\mu I - \lambda_{n-1}(A_{12})\right)\left(\mu - 1\right)^{n-3} \begin{pmatrix} \mu - q_1q_2 - q_1 + 1 & q_1 - 1 \\ q_1(q_2 - 1) & \mu - q_1 \end{pmatrix} \\ &\quad = \left(\mu - t^2q_1^2q_2^2\right)\left(\mu - 1\right)^{\binom{n-1}{2}}\left(\mu - q_1q_2\right)^{n-2}, \end{aligned}$$

as required.

Another A_{ij}, with $i > 2$ is conjugate to A_{12} in a larger subgroup of B_n which allows the first and ith, and second and jth strands to be interchanged. In this group, A_{12} and A_{ij} are conjugate. If q_1 and q_i are specialised to the same variable, and the same for q_2 and q_j, then λ_n becomes a representation of this group. So under this specialisation, $\lambda_n(A_{ij})$ has the same characteristic polynomial as $\lambda_n(A_{12})$, but none of the entries in $\lambda_n(A_{ij})$ contains a q_1 or q_2, and so the characteristic polynomial must be

$$\left(\mu - t^2q_i^2q_j^2\right)\left(\mu - 1\right)^{\binom{n-1}{2}}\left(\mu - q_iq_j\right)^{n-2}.$$

If i is 1 or 2, then there is a similar proof. □

6.2 Faithfulness

The Lawrence-Gassner representation is faithful simply because if all the q_is are specialised to the same variable, q, then it becomes the Lawrence-Krammer representation.

Alternatively, this could also be argued as in the Lawrence-Krammer representation, using a pairing between a fork and a noodle defined below.

Definition 6.5. Given a fork, F, and a parallel fork, F', in D_n let $\Sigma(F)$ be the surface in C consisting of all points that can be written $\{x, y\}$ where $x \in T(F)$ and $y \in T(F')$. This surface is isomorphic to, and given the natural orientation of $F \times F'$. $\Sigma(F)$ is given a handle, defined by the handles of the fork and parallel fork. A path along each of these define a path in C. Finally, let $\widetilde{\Sigma}(F)$ be the lift of $\Sigma(F)$ that has handle based at $\tilde{Y}_0$

If N is a noodle in D_n, define a surface, $\Sigma(N)$ in C to consist of all unordered pairs of distinct points in N. This is isomorphic to the section of $N \times N$ that lies above the diagonal, take this to give the orientation. Then define $\widetilde{\Sigma}(N)$ to be the lift of $\Sigma(N)$ to $\widetilde{C}$ that is based at $\tilde{Y}_0$.

Definition 6.6. Given any fork, F, and noodle, N, in D_n, the *Lawrence-Gassner pairing* is defined as

$$\langle N, F \rangle_\lambda = \sum_q q\big(q\widetilde{\Sigma}(N), \widetilde{\Sigma}(F)\big),$$

where $q = q_1^{a_1} \dots q_n^{a_n} t^b$ with $a_1, \dots, a_n, b \in \mathbb{Z}$.

6.3 Relationship to the other representations

By its very construction, this representation does indeed complete the diagram:

$$\begin{array}{ccc} \lambda_n(P_n) & \xrightarrow{G'_n} & \kappa_n(B_n) \\ \Big\downarrow{\scriptstyle K'_n} & & \Big\downarrow{\scriptstyle K_n} \\ \gamma_n(P_n) & \xrightarrow{G_n} & \beta_n(B_n) \end{array},$$

where G'_n is defined, like G_n by mapping all the q_is in the entries of the matrix to q, and K'_n is similar to K_n. That is, for a braid, ψ, treating the $(i, i+1)$st column of $\kappa_n(\psi)$ as coefficients of a sum of standard forks,

$$K'_n : t^a q_1^{b_1} \cdots q_n^{b_n} f_{ij} \mapsto (-1)^a q_1^{b_1/2} \cdots q_n^{b_n/2} (f_i + \cdots + f_{j-1}),$$

allows column i of β_n to be calculated as coefficients of the f_i.

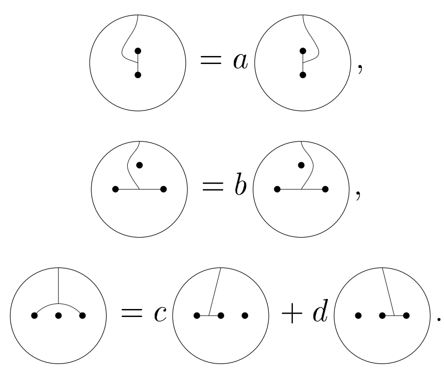
Since both λ_n and κ_n are faithful, G'_n is injective. Similarly, the Gassner representation is faithful iff K'_n is injective. So the question of faithfulness of γ_n amounts to whether or not the exponents of t in $\lambda_n(\psi)$ can be calculated from $\gamma_n(\psi)$.

7 On generalised skein fork modules

7.1 The Burau module

The Burau representation may be unfaithful for large n, but it is natural to ask "Why?" Is the basis chosen deficient, or could another set of relations produce a faithful representation?

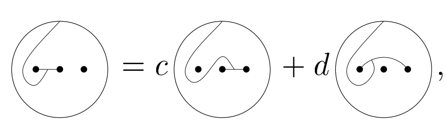
Start with the following *generalised Burau relations*, assuming a, b, c and d are invertible elements of a commutative ring.



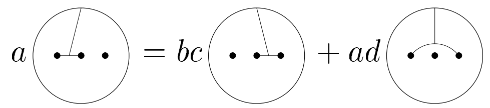
(12)

Claim 7.1. The Burau module is the unique module with relations as given in equation 12.

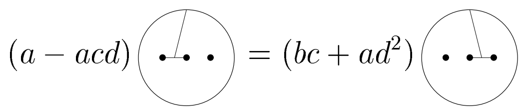
Proof. The relations must still hold when a braid is applied to them. Under the braid $\sigma_1\sigma_2$, the third relation becomes



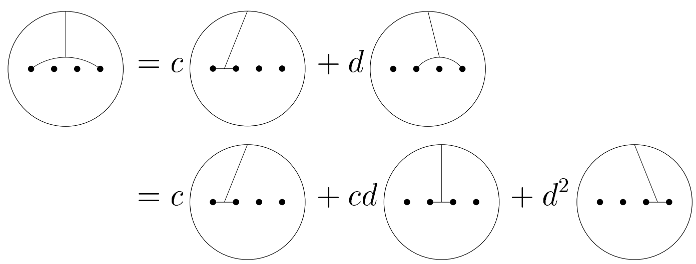
i.e.



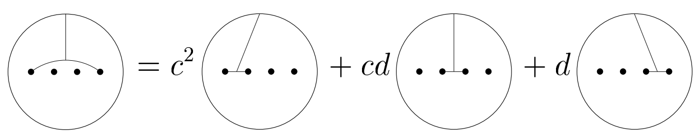
Applying the original third relation gives



But, for $f_1, \ldots, f_{n-1}$ to form a basis for the module, they must be independent, and so $c = d^{-1}$ and $b = -ad^3$. This is not the whole story, however. Consider decomposing the following fork in two different ways:



and, similarly,

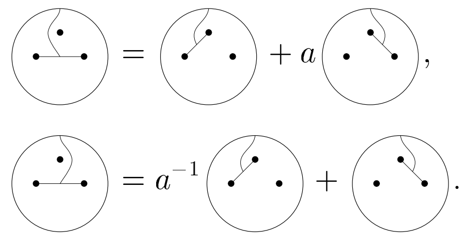


from which, $c = d = 1$. Hence this is simply the Burau module. □

One might also imagine that the second Burau relation could be generalised further — it involves three punctures so could relate the left-hand fork to two others. This is not possible.

Claim 7.2. The second generalised Burau relation is, in fact, a consequence of the other two.

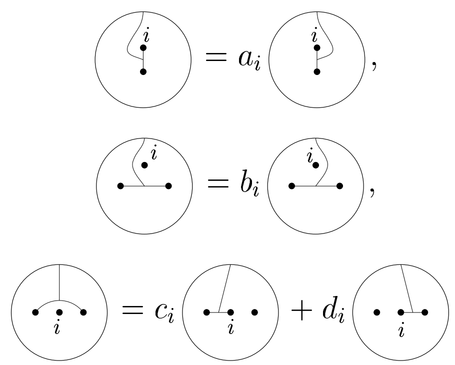
Proof. The third relation gives



∎

7.2 The Gassner module

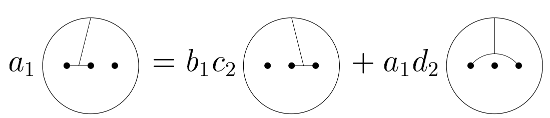
Start with the following *generalised Gassner relations*, assuming that a_i, b_i, c_i, d_i and e_i are invertible elements of a commutative ring for all $i = 1, \dots, n$.



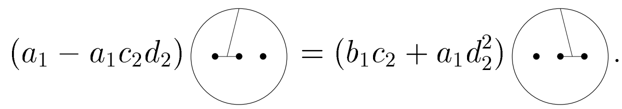
(13)

Claim 7.3. The Gassner module is the unique module with relations as given in equation 13.

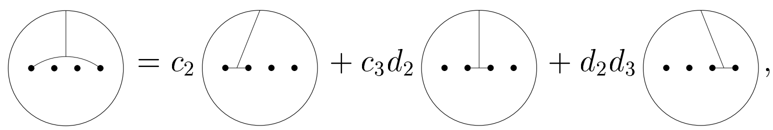
Proof. This is similar to the case of the generalised Burau relations. Under the braid $\sigma_1\sigma_2$, the third relation becomes



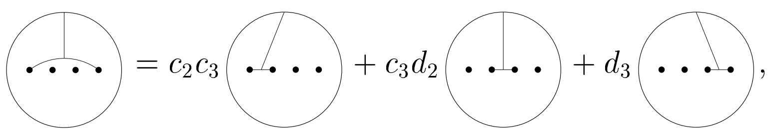
For ease of notation, we assume the punctures to be labelled 1, 2 and 3, but in fact they may be any punctures, with the disc homotoped to place them as shown. Applying the original third relation gives



But, for $f_1, \ldots, f_{n-1}$ to form a basis for the module, they must be independent, and so $c_2 = d_2^{-1}$ and $b_1 = -a_1 d_2^3$. This is not the whole story, however. Consider decomposing the following fork:



and, similarly,

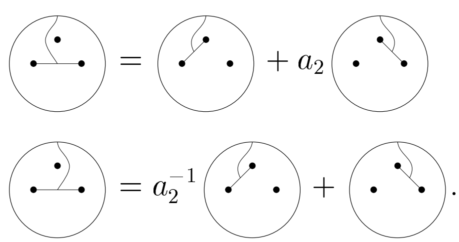


from which, $c_2 = d_2 = c_3 = d_3 = 1$. Hence $c_i = d_i = 1$ and $b_i = -a_i$ for all i and this is simply the Gassner module. □

Just as with the Burau module, the second relation cannot be generalised.

Claim 7.4. The second generalised Gassner relation is a consequence of the other two.

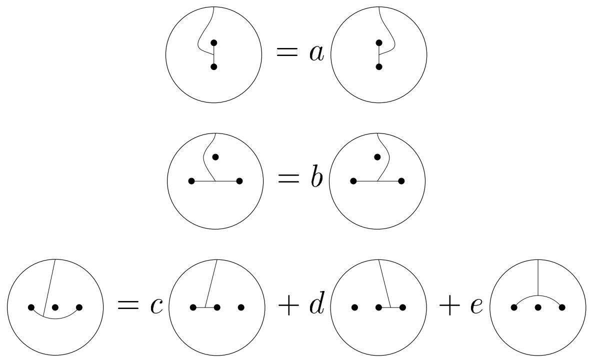
Proof. The third relation gives



□

7.3 The Lawrence-Krammer module

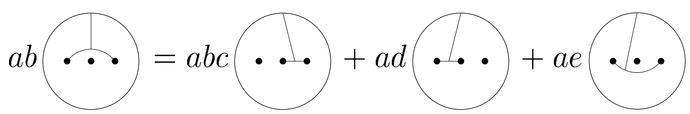
The *generalised Lawrence-Krammer relations*, are as follows, where a, b, c, d and e are invertible elements of a commutative ring.



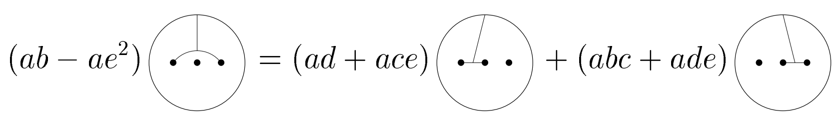
(14)

Claim 7.5. The Lawrence-Krammer module is the unique module with relations as given in 14.

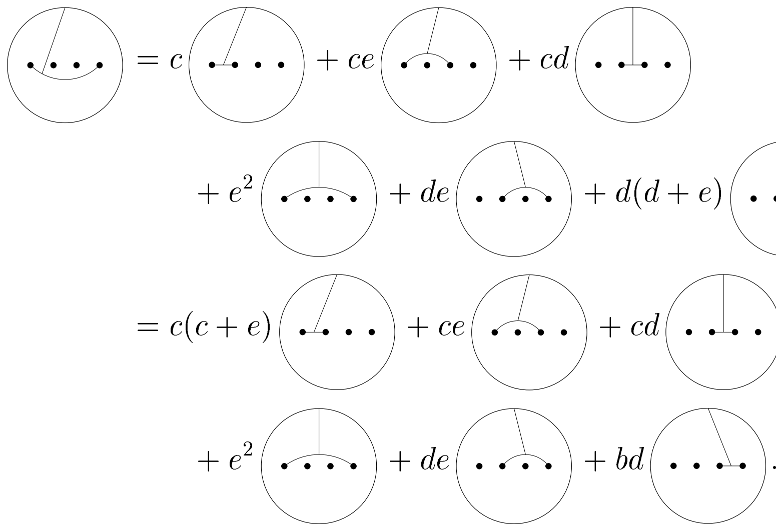
Proof. The relations must still hold when a braid is applied to them. Under the braid Δ, the third relation becomes



Apply the original third relation:

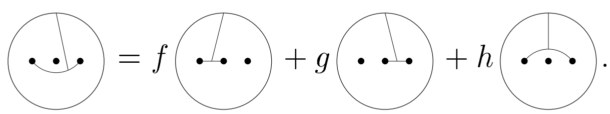


Again, the f_{ij}s must be independent, so $b = e^2$ and $d = -ce$. Now, the following fork may be decomposed in two different ways, by applying the third relation to the second puncture and then the third, and by applying it to the third, then the second.



Comparing coefficients gives $c = c(c+e)$ and $d(d+e) = bd$, so $c = 1 - e$ and $d = e^2 - e$. Putting q in place of e, this is the Lawrence-Krammer module. □

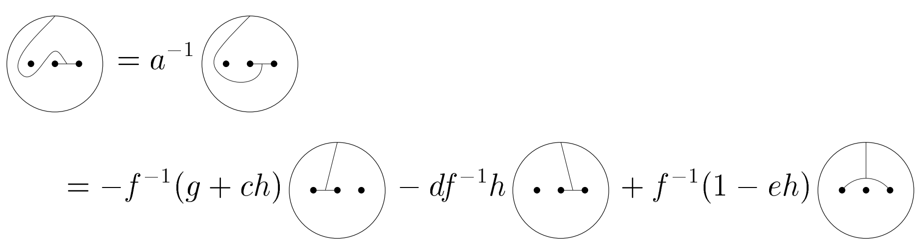
Again, one might imagine that replacing the second relation will yield a larger module, however this is not the case if it is replaced with the following relation:



(15)

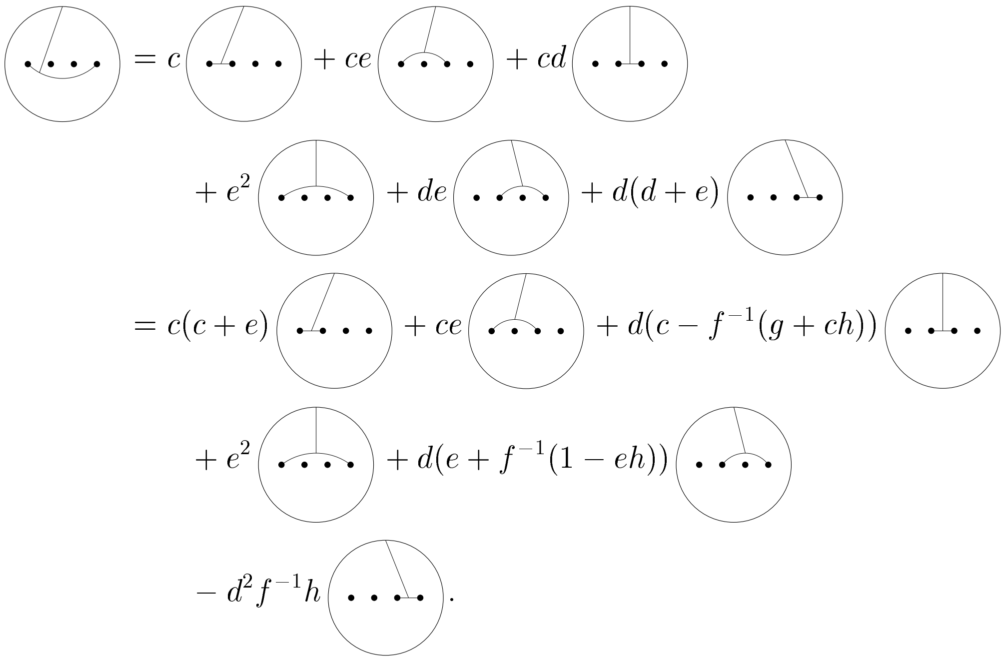
Claim 7.6. The second generalised Lawrence-Krammer relation follows from the other two and equation 15.

Proof. First note this decomposition:

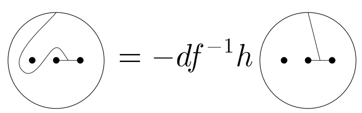


(16)

Now, using the above, decompose the following fork in two ways:



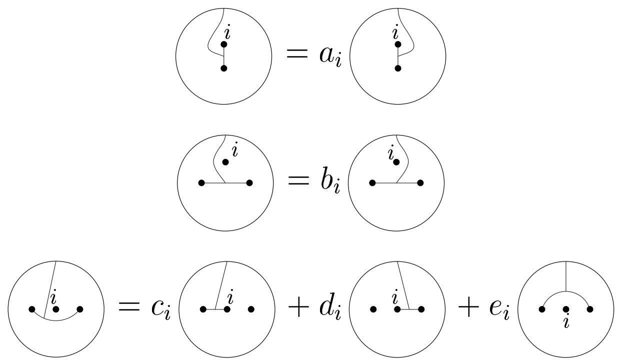
Comparison of coefficients gives $g + ch = 0$ and $1 + eh = 0$, which makes equation 16



indicating that these relations yield the generalised Lawrence-Krammer relations as presented in equation 14. ☐

7.4 The Lawrence-Gassner module

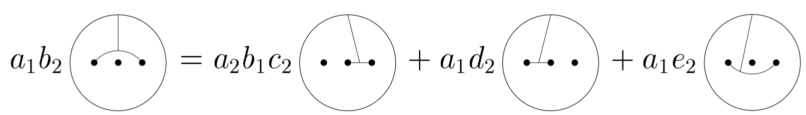
The *generalised Lawrence-Gassner relations*, are as follows, where a_i, b_i, c_i, d_i and e_i are invertible elements of a commutative ring for all i.



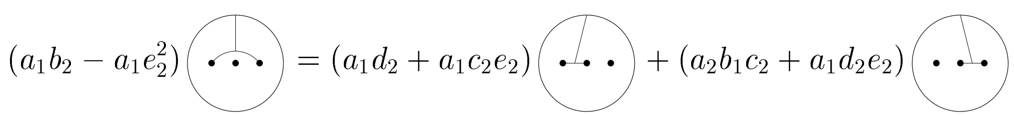
(17)

Claim 7.7. The Lawrence-Gassner module is the unique module with relations as given in 17.

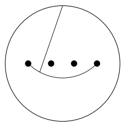
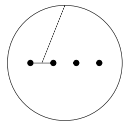
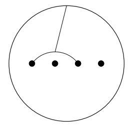
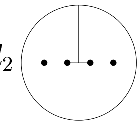
Proof. The relations must still hold when a braid is applied to them. Under the braid Δ, the third relation becomes

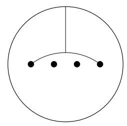
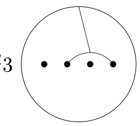
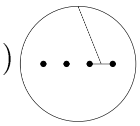


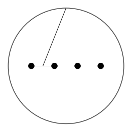
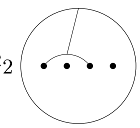
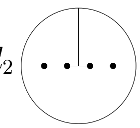
Apply the original third relation:

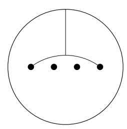
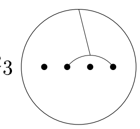
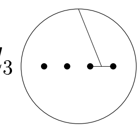


Again, the f_{ij}s must be independent, so $b_i = e_i^2$ and $d_i = -c_i e_i$ (remembering that here the punctures are labelled in the standard way for convenience and that the same calculation could be done with the punctures permuted). Now, the following fork may be decomposed in two different ways, by applying the third relation to the second puncture and then the third, and by applying it to the third, then the second.

 $= c_2$  $+ c_3e_2$  $+ c_3d_2$ 

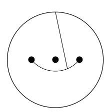
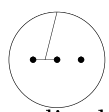
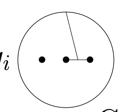
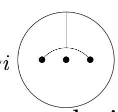
$+ e_2e_3$  $+ d_2e_3$  $+ d_3(d_2 + e_2)$ 

$= c_2(c_3 + e_3)$  $+ c_3e_2$  $+ c_3d_2$ 

$+ e_2e_3$  $+ d_2e_3$  $+ b_2d_3$  .

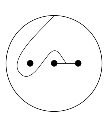
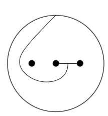
Comparing coefficients gives $c_2 = c_2(c_3 + e_3)$ and $d_3(d_2 + e_2) = b_2d_3$, so $c_i = 1 - e_i$ and $d_i = e_i - e_i^2$. Putting q_i in place of e_i, this is just the original module. □

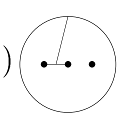
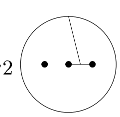
Again, if the second relation is removed and replaced with this,

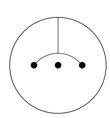
 $= f_i$  $+ g_i$  $+ h_i$  , (18)

Claim 7.8. The second generalised Lawrence-Gassner relation follows from the other two and equation 18.

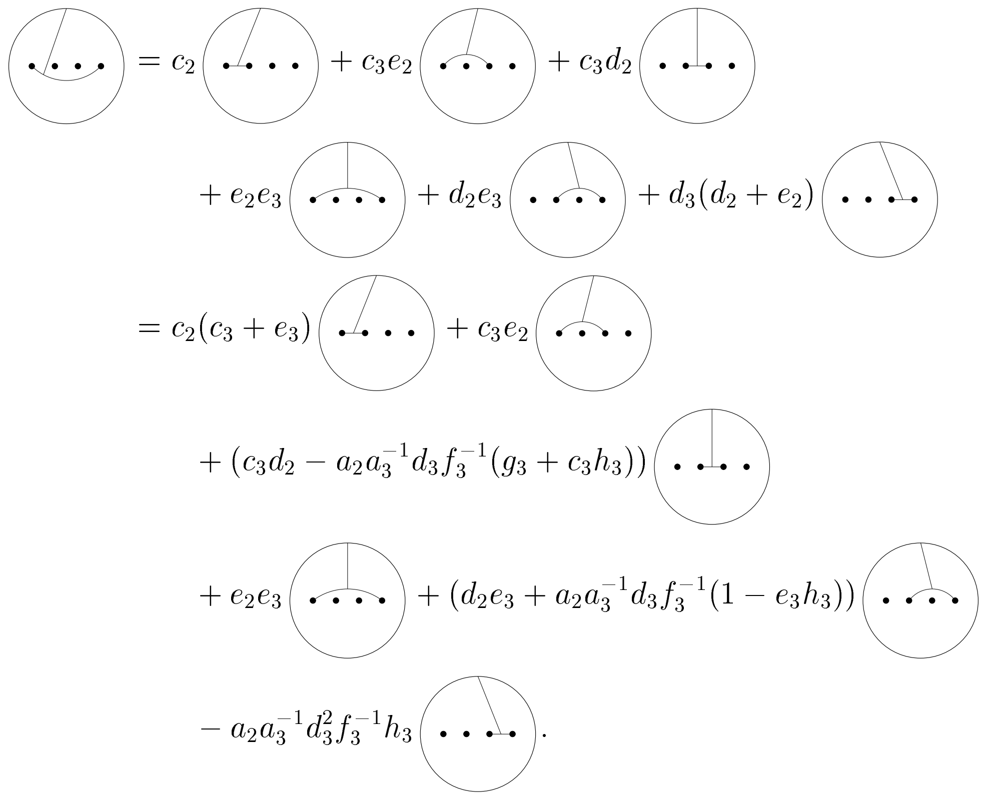
Proof. First note this decomposition:

 $= a_2^{-1}$ 

$= -a_1a_2^{-1}f^{-1}(g_2 + c_2h_2)$  $- a_1a_2^{-1}d_2f_2^{-1}h_2$  (19)

$+ a_1a_2^{-1}f_2^{-1}(1 - e_2h_2)$  .

Now, using the above, decompose the following fork in two ways:



Comparison of coefficients gives $g_i + c_i h_i = 0$ and $1 + e_i h_i = 0$, which makes all the coefficients zero except for the coefficient of f_{23} in equation 19 indicating that these relations yield the generalised Lawrence-Gassner relations. □

7.5 Further possibilities

In each module, the first relation involves two points and the second and third involve three. In the Gassner and Lawrence-Gassner modules the relations could use all these, taking coefficients a_{ij}, b_{ijk}, etc. Also, the third relation could write a fork in terms of more than just the three forks. There could potentially be $n-1$ or $\binom{n}{2}$ terms on the right-hand side, but this seems to get away from forks actually representing elements of a homology group.

8 Concluding remarks

The two major outstanding problems are faithfulness of the Burau representation of B_4 and the Gassner representation of P_n for $n \geq 4$. If the Gassner representation is faithful, then γ_4 is probably the easiest one to start with. If not, then non-faithfulness of γ_n for large n is probably the easiest to prove, just as it was with the Burau representation.

A characterisation of which specialisations of these representations are faithful is another open problem, although if the faithfulness of the unspecialised version is unknown then this cannot be determined in general. The Lawrence-Krammer and Lawrence-Gassner representations may also be specialised and, since both are known to be faithful, which specialisations remain faithful is an interesting question.

The Burau, Gassner and Lawrence-Krammer relations are known to be unitary with respect to some Hermitian form. For the Burau representation, this was shown by Squier [18], for the Gassner by Abdulrahim [1], and for the Lawrence-Krammer recently by Song [17] and Budney [9]. With this is mind, it is natural to ask whether the Lawrence-Gassner representation is also unitary.

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