

Alexander Invariants of Periodic Virtual Knots

Alexander Invariants of Periodic Virtual Knots

By
LINDSAY WHITE, M.SC.

A Thesis
Submitted to the school of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
of Doctor of Philosophy

McMaster University

© Copyright by Lindsay White, December 2016

DOCTOR OF PHILOSOPHY (2016)
(Mathematics)

McMaster University
Hamilton, Ontario

TITLE: Alexander Invariants of Periodic Virtual Knots
AUTHOR: Lindsay White, B.Sc. (McMaster University),
M.Sc. (McMaster University)
SUPERVISORS: Dr. Hans U. Boden, Dr. Andrew J. Nicas
NUMBER OF PAGES: 1, 86

Abstract

In this thesis, we show that every periodic virtual knot can be realized as the closure of a periodic virtual braid. If K is a q -periodic virtual knot with quotient K_* , then the knot group G_{K_*} is a quotient of G_K and we derive an explicit q -symmetric Wirtinger presentation for G_K , whose quotient is a Wirtinger presentation for G_{K_*} . When K is an almost classical knot and $q = p^r$, a prime power, we show that K_* is also almost classical, and we establish a Murasugi-like congruence relating their Alexander polynomials modulo p .

This result is applied to the problem of determining the possible periods of a virtual knot K . For example, if K is an almost classical knot with nontrivial Alexander polynomial, our result shows that K can be p -periodic for only finitely many primes p . Using parity and Manturov projection, we are able to apply the result and derive conditions that a general q -periodic virtual knot must satisfy. The thesis includes a table of almost classical knots up to 6 crossings, their Alexander polynomials, and all known and excluded periods.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my advisors Dr. Hans U. Boden and Dr. Andrew J. Nicas for their invaluable guidance, motivation, and vast expertise. Their enthusiasm and encouragement have been instrumental throughout the research and writing of this thesis.

Besides my advisors, I would like to thank the rest of my thesis committee: Dr. Adam Van Tuyl and Dr. Manfred Kolster, for serving on my supervisory committee and for their insightful comments. I would also like to extend my thanks to my undergraduate professors and instructors at McMaster University, as well as my high school math teachers at Ancaster High School, for providing interesting courses that encouraged me to continue and pursue a graduate degree in mathematics.

Finally, I would like to thank my mom and dad for encouraging me to pursue mathematics, and for their patience and help with various math questions throughout the years.

Contents

1	Introduction	1
2	Preliminaries	7
2.1	Classical knots and Gauss diagrams	7
2.2	Virtual knots	9
2.3	The knot group and Alexander invariants	11
2.4	Almost classical knots	14
2.5	Virtual braids	21
2.6	Alexander invariants (reprise)	26
3	Periodic Virtual Knots	31
3.1	Basic Definitions	31
3.2	Periodicity and almost classical knots	36
3.3	Periodic virtual braids	39
4	Circulant Matrices	46
5	Murasugi's Theorem for Almost Classical Knots	54
6	Applications	63
7	Conclusion	73

List of Figures

1.1	The Alexander numbering conditions.	3
2.1	The Reidemeister moves $(r1)-(r3)$	7
2.2	A negative and a positive crossing.	8
2.3	The trefoil knot 3_1	8
2.4	Polyak's minimal generating set of Reidemeister moves.	9
2.5	The virtual trefoil and its Gauss diagram.	10
2.6	The detour move.	10
2.7	The relations in G_K from the i -th crossing of K	12
2.8	The relation r_i coming from the chord c_i is $x_k^{-\varepsilon_i} x_i x_k^{\varepsilon_i} x_{i+1}^{-1}$	12
2.9	The forbidden overpass for Gauss diagrams.	14
2.10	The Alexander numbering conditions.	15
2.11	Arc labels at the i -th crossing in terms of Alexander numbers.	19
2.12	Two consecutive Alexander numbers.	20
2.13	Generators of VB_k	21
2.14	The braiding algorithm for $C_K = 01-U2+03-U1-02+U3-$	23
2.15	The classical braid $\beta = (\sigma_1\sigma_2)^2$	27
2.16	The Alexander numbers at the i -th crossing for $\varepsilon_i = 1$ on the left and $\varepsilon_i = -1$ on the right.	29
3.1	A periodic virtual knot and its quotient knot.	32
3.2	A 3-periodic knot diagram for the pretzel knot 9_{35}	32
3.3	On left, $c_{i,j}$ separates $a_{i,j}$ and $a_{i+1,j}$ and has arrowfoot on $a_{k,\ell}$. On right, $a_{i,j}$ is the dotted arc between the arrowheads of $c_{i-1,j}$ and $c_{i,j}$	34
3.4	A Gauss diagram with two odd chords and projection $P_f(D)$ the trefoil.	39
3.5	Converting a periodic tangle into a periodic virtual braid for 9_{35}	43
5.1	The braid $\beta = \sigma_1\tau_2\sigma_3\tau_4\tau_5$ is almost classical.	60

7.1 Gauss diagrams of almost classical knots with up to six crossings. 82

Chapter 1

Introduction

An oriented knot diagram K (or link) in S^3 is called *periodic* of *period* $q > 1$ if there is an orientation preserving diffeomorphism $\varphi: (S^3, K) \rightarrow (S^3, K)$ of finite order q whose fixed point set is a nonempty subset C homeomorphic to S^1 and disjoint from K . The solution of the Smith Conjecture implies that C is unknotted and that φ is equivalent to a rotation about an axis in S^3 . The quotient knot (or link), denoted K_* , is the image of K under the orbit map $S^3 \rightarrow S^3/\varphi \cong S^3$.

An easy way to construct examples of periodic knots is to realize them as the closure of a proper power of a braid. Indeed, if the quotient knot K_* can be written as the closure of a braid β of index k , and if q is a positive integer relatively prime to k , then we take K to be the closure of β^q . It follows that K is a periodic knot with period q and k is equal to the linking number of K with the rotation axis. One can relax the assumption that k and q are relatively prime, and in that case the closure of β^q will be a link L with $n = \gcd(k, q)$ components [Liv93, §8.2].

We say that the periodic knot or link L admits a periodic braid representative if L is isotopic to the closure of β^q for some braid β . In [LP97], Lee and Park establish conditions on a link L for it to admit a periodic braid representative, and in particular they show that not all periodic knots and links admit periodic braid representatives.

In this thesis we study periodic virtual knots (virtual knots are discussed in Section 2.2), defined as follows.

Definition 1.1. A virtual knot K is called **periodic** with **period** q if it admits a virtual knot diagram which misses the origin and is invariant under a rotation in the plane by an angle of $2\pi/q$ about the origin.

One useful result that we establish is the following.

Theorem 1.2. *Any periodic virtual knot or link L admits a periodic virtual braid representative.*

Although this result is not true in the classical case, one can nevertheless apply Theorem 1.2 to a periodic classical knot or link L to show that it admits a periodic *virtual* braid representative. We believe this may be useful in establishing alternative proofs of conditions that classical knot and link invariants must satisfy in the periodic case. From this point of view, virtual knot theory may provide a useful short-cut in establishing classical results.

Indeed, for a classical knot K , there are many known conditions on the invariants of K for it to be periodic. These include Murasugi's conditions on the Alexander polynomial $\Delta_K(t)$ [Mur71], conditions on the Jones polynomial $V_K(t)$ proved by Murasugi [Mur88] and Yokota [Yok91], as well as Traczyk's conditions on the Kauffman bracket [Tra90].

In [Fla85], Flapan proved that a nontrivial classical knot admits only finitely many periods, and in [Hil84] Hillman extended her result to links. In [Edm84], Edmonds used minimal surface theory to establish a strong upper bound on the period of a given knot K in terms of its Seifert genus. In his study of periodic virtual knots [Lee12], S.Y. Lee raised the following important questions.

Question 1.3. *Does a non-trivial virtual knot admit only finitely many periods?*

Question 1.4. *Given a classical knot K , can it admit a q -periodic virtual knot diagram without admitting any q -periodic classical knot diagrams?*

Despite significant effort by many knot theorists, these basic questions remain unsolved. In fact, there are now a whole host of constraints on the invariants of a virtual knot or link for it to be periodic, including conditions on the arrow and index polynomials [IL12], the Miyazawa polynomial [KLS09], the VA -polynomial [KLS13], the writhe and odd writhe polynomials [BL15], and the virtual Alexander polynomial [KLS14].

For various reasons, Murasugi's theorem [Mur71] has not been extended to the virtual category. One of the obstacles is that the Alexander polynomial does not generalize in an entirely straightforward manner to virtual knots and links. For instance, see [Saw99] for a discussion of the difficulties involved.

In order to explain our next result, it is useful to recall the statement of Murasugi's theorem. We begin with some notation. Given two Laurent polynomials $f(t), g(t) \in \mathbb{Z}[t^{\pm 1}]$, we write $f(t) \doteq g(t)$ if $f(t) = \pm t^k g(t)$ for some $k \in \mathbb{Z}$.

Theorem 1.5 (Murasugi). *Suppose p is prime and $q = p^r$ is a prime power. If K is a q -periodic knot diagram with quotient knot K_* and linking number k , then we have:*

1. $\Delta_{K_*}(t)$ divides $\Delta_K(t)$ in $\mathbb{Z}[t^{\pm 1}]$, and
2. $\Delta_K(t) \doteq (\Delta_{K_*}(t))^q (1 + t + t^2 + \dots + t^{k-1})^{q-1} \pmod{p}$.

Various extensions and alternative proofs of Murasugi's Theorem for classical knots and links have been considered in [Hil81, Hil83, DL91a, DL91b, HLN06].

Although the Alexander polynomial does not give a well-behaved invariant for all virtual knots and links, it does extend nicely to the subcategory of "almost classical" knots, defined below.

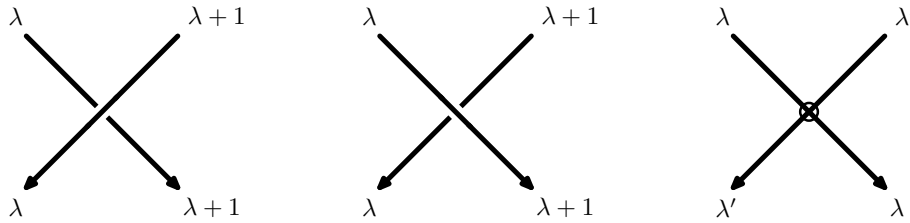


Figure 1.1: The Alexander numbering conditions.

Definition 1.6 (Almost Classical Knots and Links).

- (i) A virtual knot diagram is *Alexander numberable* if there exists an integer-valued function λ on the set of short arcs satisfying the relations in Figure 1.1.
- (ii) Given a virtual knot or link K , we say K is *almost classical* if it admits a virtual knot diagram that is Alexander numberable.

Although almost classical knots are defined in terms of Alexander numberings, it is helpful to keep in mind that a virtual knot is almost classical if and only if it admits a representative knot in a thickened surface which is homologically trivial; see [BGH⁺16] for more details. Another useful observation is that Alexander numberability and periodicity are compatible in the sense that if a knot K is almost classical and periodic, then it admits a periodic virtual knot diagram which is also Alexander numberable (see Theorem 3.8).

The main aim of this thesis is to establish a generalization of Murasugi's theorem for almost classical knots, which we state next. We present some of

the ingredients of the proof and explain how to extend these conditions from almost classical knots to the full category of all virtual knots using Gaussian parity and Manturov projection.

Recall that if K is a q -periodic virtual knot, then by Theorem 1.2, we can write $K = \widehat{\beta}^q$ for some k -strand braid β .

Theorem 1.7. *Let $K = \widehat{\beta}^q$ be a q -periodic almost classical knot with $q = p^r$ a prime power. Then $K_* = \widehat{\beta}$, and we have:*

1. $\Delta_{K_*}(t)$ divides $\Delta_K(t)$ in $\mathbb{Z}[t^{\pm 1}]$, and
2. $\Delta_K(t) \doteq (\Delta_{K_*}(t))^q (f(t))^{q-1} \pmod{p}$, where $f(t) = \sum_{i=1}^k t^{\lambda_i}$ and λ_i is the Alexander number on the i -th strand of β .

The main difference between this result and Theorem 1.5 is that the polynomial term $1 + t + t^2 + \dots + t^{k-1}$ in the original statement of Murasugi's theorem has been replaced with the general polynomial $f(t) = \sum_{i=1}^k t^{\lambda_i}$. This factor $f(t)$ can be read off from the Alexander numbering on the braid strands once one has realized the periodic virtual knot K as the closure of a periodic virtual braid (Theorem 1.2). In fact, assuming β is a braid on k -strands, in the formula $f(t) = \sum_{i=1}^k t^{\lambda_i}$, the exponent λ_i denotes the Alexander number on the i -th strand of β . (This quantity is actually independent of which horizontal line is used to compute it.) In the classical case, assuming that K is the closure of a classical braid, it is immediate that these exponents $\lambda_1, \dots, \lambda_k$ take on the values $0, 1, \dots, k-1$, and this explains why $f(t) = 1 + t + t^2 + \dots + t^{k-1}$ in that case. (The argument is actually a little more involved, because it is not true that every periodic classical knot is the closure of a periodic classical braid.)

The proof of Theorem 1.7 also involves studying circulant block matrices in characteristic $p > 0$. These arise because for a periodic virtual knot K , we can write the Jacobian matrix as a block circulant matrix, with the sum of the blocks equalling the Jacobian matrix of the quotient. The next result summarizes the situation.

Theorem 1.8. *Let K be a virtual knot diagram with period q , and let K_* be its quotient knot. If A and B are the Jacobian matrices of the Wirtinger presentations of the knot groups G_{K_*} and G_K , respectively, then*

$$B = \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ A_{q-1} & A_0 & \cdots & A_{q-2} \\ \vdots & \ddots & \ddots & \vdots \\ A_1 & \cdots & A_{q-1} & A_0 \end{bmatrix},$$

where A_0, A_1, \dots, A_{q-1} are square matrices satisfying $A_0 + A_1 + \dots + A_{q-1} = A$.

In case $q = p^r$ is a prime power and K is almost classical, we perform operations to the block circulant matrix above to show that $\Delta_K(t) \doteq (\Delta_{K_*}(t))^q (f(t))^{q-1} \pmod{p}$, for some polynomial $f(t)$, and we then determine $f(t)$ by using a periodic braid diagram for the knot as previously explained.

This theorem allows us to eliminate certain periods for almost classical knots by testing the Alexander polynomial to see if it can be factored in the desired form after reduction modulo p . For instance, in Theorem 6.4 we show that any almost classical knot K with non-trivial $\Delta_K(t)$ is p -periodic for at most finitely many primes p . In case $\Delta_K(t) \pmod{p}$ is non-trivial for all primes p , Corollary 6.3 provides the stronger conclusion that K can be q -periodic for at most finitely many q . For example, it applies to classical fibered knots K , giving a positive answer to Question 1.3 for such knots.

It also allows us to show that many classical knots do not exhibit additional periods in their virtual knot diagrams.

We extend the techniques described above to eliminate composite periods in many cases. In Example 6.9, we show how to eliminate $6 = 2 \cdot 3$ as a possible (virtual) period for the trefoil. More generally, we give criteria that can be used to eliminate periods of the form $2p$ where p is an odd prime, see Propositions 6.10, 6.11.

Although Theorem 1.7 applies only to almost classical knots, we now explain how to apply parity projection and use it to obtain constraints on any virtual knot. In the following, let f denote the total Gaussian parity (see Equation (2.5) on page 16 for its definition).

In general, the parity $f(c_i)$ of a chord in a virtual knot diagram is either even or odd, and it must satisfy the so-called parity axioms. The projection $P_f(D)$ of a diagram is obtained by eliminating all the odd chords, and the parity axioms are defined to ensure that if two diagrams D_1 and D_2 are equivalent through Reidemeister moves, then so are the diagrams $P_f(D_1)$ and $P_f(D_2)$ obtained by projection. It follows that the knot type of $P_f(K)$ is well-defined and independent of the representative diagram for K .

Since any virtual knot diagram can have only finitely many chords, for any D , there is a positive integer n such that $P_f^{n+1}(D) = P_f^n(D)$ (and therefore, for any $m > n$, the diagram will remain the same, so we have $P_f^m(D) = P_f^n(D)$). Stable Manturov projection is defined as $P_f^\infty(K) = \lim_{n \rightarrow \infty} P_f^n$ (in other words, if n' is the smallest n such that $P_f^{n+1}(D) = P_f^n(D)$, then $P_f^\infty(K) = P_f^{n'}(D)$). It is a general fact that the image $P_f^\infty(K)$ is an almost classical diagram for any

virtual knot diagram K , where f is the total Gaussian parity.

Theorem 1.9. *If K is a q -periodic virtual knot diagram, then $\bar{K} = P_f^\infty(K)$ is a q -periodic almost classical diagram.*

In particular, we can apply Theorem 1.7 to \bar{K} to give conditions that must be satisfied in order for K to be periodic. For instance, any virtual knot K whose projection $\bar{K} = P_f^\infty(K)$ has nontrivial Alexander polynomial is p -periodic for only finitely many primes p .

Table 7.2 lists the known periods and excluded periods for almost classical knots up to 6 crossings. This is based on applying Theorem 1.7 to their Alexander polynomials, which were computed in [BGH⁺16] and are listed here in Table 7.1. Gauss diagrams of almost classical knots with up to six crossings, including the nine classical knots $3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3, 3_1\#3_1$, and $3_1\#3_1^*$, are exhibited in Figure 7.1.

Chapter 2

Preliminaries

In this chapter, we recall the basic notions from knot theory and virtual knot theory.

2.1 Classical knots and Gauss diagrams

In this section, we present the basic definitions for classical knots, knot diagrams, Gauss codes, and Gauss diagrams.

Definition 2.1. A **knot** K is a smooth embedding $S^1 \rightarrow \mathbb{S}^3$ of the circle into 3-space. Two knots are considered **equivalent** if there is an ambient isotopy of \mathbb{S}^3 taking one to the other.

More generally, a **link** is a smooth embedding $S^1 \cup \dots \cup S^1 \rightarrow \mathbb{S}^3$ of a disjoint union of circles, up to ambient isotopy of \mathbb{S}^3 taking one to another.

We will usually work with oriented knots and links, and when necessary we indicate the choice of orientation using an arrow.

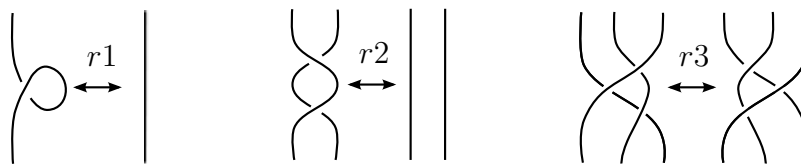


Figure 2.1: The Reidemeister moves (r1)–(r3).

Definition 2.2. A **knot diagram** is the regular projection of a knot K in \mathbb{R}^3 to the plane. Double-points occur whenever one strand crosses over another, and we leave a gap in the diagram to indicate which strand crosses over the

other. Two knot diagrams are **equivalent** if one can be transformed into the other through a series of Reidemeister moves ($(r1)$ – $(r3)$ from Figure 2.1) and planar isotopies.



Figure 2.2: A negative and a positive crossing.

Associated to a knot diagram is its **Gauss code**, which is a word that records the crossings and their signs. Given a knot diagram with n crossings, we number the crossings $1, 2, \dots, n$ arbitrarily. Picking a basepoint on the knot and direction in which to traverse it, we record each crossing as it is encountered. Each crossing will be recorded twice, once as an over-crossing (written $0i$) and then as an under-crossing (written U_i). At the same time, we record the sign of the crossing, which is positive if the crossing is right-handed and negative if it is left-handed. See Figure 2.2. For example, the standard diagram of the trefoil has Gauss code $01+U2+03+U1+02+U3+$; see Figure 2.3.

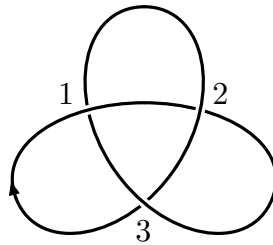


Figure 2.3: The trefoil knot 3_1 .

The Gauss code is determined by the oriented knot up to relabeling of the crossings and altering the choice of basepoint. A relabeling of the crossings amounts to permuting the numbers $1, 2, \dots, n$ within the Gauss code, and altering the choice of basepoint amounts to a cyclic permutation of the Gauss code.

A **Gauss diagram** is a convenient tool for packaging the same information as the Gauss code, and it consists of a base circle, which represents the underlying knot, along with directed chords c_1, \dots, c_n , one for each crossing. The i -th chord c_i points from the over-crossing arc to the under-crossing arc, and its writhe, $\varepsilon_i = \pm 1$, is given by the sign of the i -th crossing.

The Reidemeister moves can be translated into moves between Gauss diagrams, and in [Pol10], Polyak shows that the complete set of Reidemeister moves on oriented diagrams is generated from the four moves $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, and $\Omega 3a$ illustrated in Figure 2.4. In this way one can regard a classical knot as an equivalence class of Gauss diagrams. Every classical knot diagram is uniquely determined by its associated Gauss diagram, but not all Gauss diagrams correspond to classical knots.

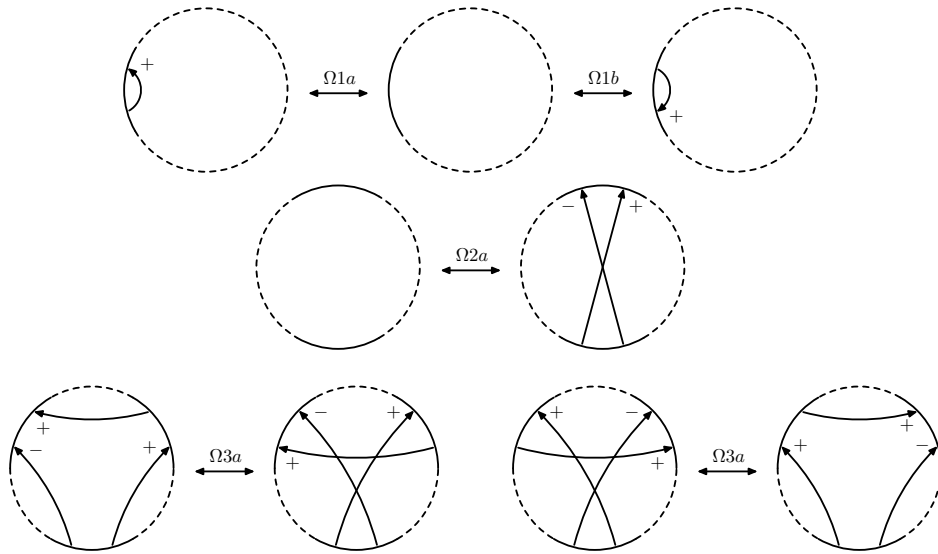


Figure 2.4: Polyak's minimal generating set of Reidemeister moves.

2.2 Virtual knots

Virtual knot theory was invented by Kauffman [Kau99], and virtual knots represent the complete set of all Gauss diagrams modulo Reidemeister moves. As with classical knots, virtual knots can be represented in terms of virtual knot diagrams, which are described next.

Definition 2.3. A **virtual knot diagram** is an immersion of a circle in the plane with only double points, such that each double point is either classical (indicated by over- and under-crossings) or virtual (indicated by a circle). **Virtual link diagrams** are defined similarly. Such a diagram is **oriented** if every component has an orientation.

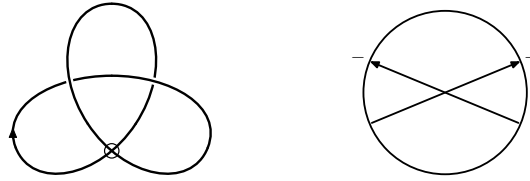


Figure 2.5: The virtual trefoil and its Gauss diagram.

Definition 2.4. Two oriented virtual link diagrams are **virtually isotopic** (or **equivalent**) if they can be related by planar isotopies and a series of *Reidemeister moves* ($r1$)–($r3$) in Figure 2.1 and the *detour move* in Figure 2.6.

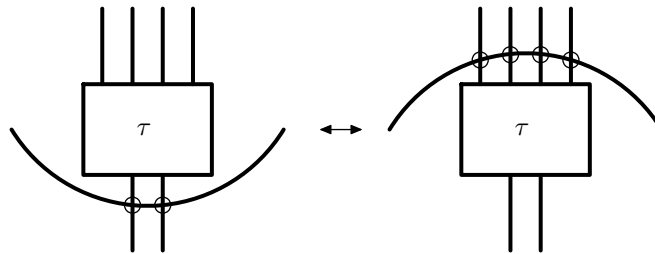


Figure 2.6: The detour move.

Virtual isotopy defines an equivalence relation on virtual link diagrams, and a **virtual knot or link** is defined to be an equivalence class of virtual knot or link diagrams under virtual isotopy.

Throughout the thesis, we will refer to various knots (up to virtual equivalence) by labelling them with a decimal number (for example, 6.90099), which comes from the enumeration by [Gre04]. The number before the decimal refers to the real crossing number of the knot (that is, the minimum number of real crossings in an equivalent diagram of the knot).

Just as with classical knot diagrams, every virtual knot diagram determines a Gauss code and Gauss diagram, either of which uniquely determines the virtual knot diagram. Indeed, an alternative but equivalent way to define virtual knots is as equivalence classes of Gauss diagrams by Reidemeister moves, as proved by Goussarov, Polyak, and Viro in [GPV00].

Virtual knots and links can also be viewed as stable equivalence classes of knots on surfaces, and we take a moment to explain this point of view. Let Σ be an oriented surface and $K \subset \Sigma$ a knot diagram. Thus, K is given by an embedded 4-valent graph with over- and under-crossing information at each vertex. Alternatively, one can think of K as a knot embedded in $\Sigma \times [0, 1]$. In

any case, an orientation of K determines a Gauss code C_K in the usual way; one first numbers the crossings of K and then uses the over- and under-crossing information and signs of the crossings to write out C_K . One can reverse this process; given a virtual knot diagram one can construct a surface Σ and knot diagram on it with the same Gauss code.

Under this correspondence, for a knot K in $\Sigma \times [0, 1]$, the associated virtual knot diagram is obtained by projecting to the plane. It will have real crossings for each crossing of K , and virtual crossings whenever strands of the knot from two different parts of the surface are projected to the same point in the plane.

In [CKS02], the authors established a one-to-one correspondence between virtual knots and stable equivalence classes of knots on surfaces (see [CKS02] for the more details about stable equivalence). In [Kup03], Kuperberg shows that every stable equivalence class of knots in a surface has a unique irreducible representative. For instance, a virtual knot is classical if and only if its unique irreducible representative lives on a surface of genus zero. All of these results hold in the more general setting of virtual links.

2.3 The knot group and Alexander invariants

Definition 2.5. Suppose K is an oriented virtual knot with n classical crossings, and choose a basepoint on K . Starting at the basepoint, we label the arcs x_1, x_2, \dots, x_n so that at each under-crossing, x_i is the incoming arc and x_{i+1} is the outgoing arc. (If $i = n$, then we set $i + 1 := 1$; that is, we take i modulo n .) We use a consistent labeling of the crossings so that the i -th crossing is as shown in Figure 2.7. For $i = 1, \dots, n$ let $\varepsilon_i = \pm 1$ be according to the sign of the i -th crossing. Then the **knot group** of K is the finitely presented group given by

$$G_K = \langle x_1, \dots, x_n \mid x_{i+1} = x_k^{-\varepsilon_i} x_i x_k^{\varepsilon_i}, i = 1, \dots, n \rangle. \quad (2.1)$$

Note that virtual crossings are ignored in this construction.

As explained by S.G. Kim [Kim00], the Wirtinger presentation of the knot group G_K can also be easily read from a Gauss diagram D for K as follows. Pick a basepoint on D and number the chords c_1, \dots, c_n sequentially in the order in which one encounters their arrowheads when going around D counterclockwise. The long arcs of D are the subarcs of D from one arrowhead to the next, and we label them x_1, \dots, x_n sequentially so that x_i and x_{i+1} are separated by the

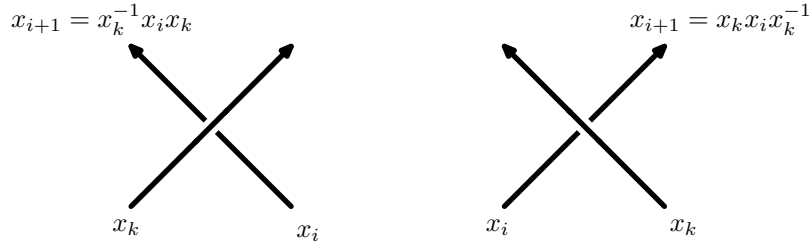


Figure 2.7: The relations in G_K from the i -th crossing of K .

arrowhead of c_i (here i is taken modulo n). Then the knot group admits the presentation

$$G_K = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle, \tag{2.2}$$

where the relation r_i arises from the chord c_i as follows. If the arrowtail of c_i lies on the arc labeled by x_k , then r_i is the relation $x_k^{-\varepsilon_i} x_i x_k^{\varepsilon_i} x_{i+1}^{-1}$, where ε_i is the writhe of c_i and i is taken modulo n (cf. Figure 2.8).

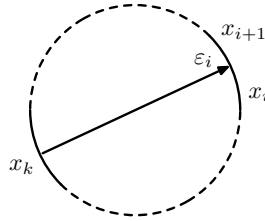


Figure 2.8: The relation r_i coming from the chord c_i is $x_k^{-\varepsilon_i} x_i x_k^{\varepsilon_i} x_{i+1}^{-1}$.

The knot group G_K is an invariant of virtual isotopy, and in fact, as we will explain in a moment, it is unchanged by a forbidden overpass (see Figure 2.9) and thus defines an invariant of the underlying welded knot type of K (see Definition 2.7). In case K is classical, we have $G_K \cong \pi_1(S^3 - N(K))$, the fundamental group of the complement of K .

We recall the construction of the Alexander module associated to the knot group G_K of a virtual knot K . Let $G'_K = [G_K, G_K]$ and $G''_K = [G'_K, G'_K]$ be the first and second commutator subgroups. The Alexander module is then the quotient G'_K/G''_K . It is a finitely generated module over $\mathbb{Z}[t^{\pm 1}]$, the ring of Laurent polynomials, and it is determined by the $n \times n$ Jacobian matrix obtained by Fox differentiating the relations r_i (appearing in the presentation (2.2) of G_K) with respect to the generators x_j and applying the abelianization map $x_\ell \mapsto t$ for $\ell = 1, \dots, n$. While the matrix A will depend on the choice of

presentation for G_K , the associated sequence of elementary ideals

$$(0) = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathbb{Z}[t^{\pm 1}] \quad (2.3)$$

does not. Here, the k -th elementary ideal \mathcal{E}_k is defined as the ideal of $\mathbb{Z}[t^{\pm 1}]$ generated by all $(n-k) \times (n-k)$ minors of A . The Alexander invariants of K (which are all the invariants you can derive from the Alexander module) are then defined in terms of the ideals (2.3).

We now describe the standard method for deriving a presentation matrix for the Alexander module from the Wirtinger presentation (2.1) of the virtual knot. As before, we assume that K is a virtual knot with n real crossings c_1, \dots, c_n and long arcs x_1, \dots, x_n such that x_i starts at the under-crossing of c_{i-1} and ends at the under-crossing of c_i .

The Fox derivatives of the relations r_i are given by

$$\frac{\partial r_i}{\partial x_j} = \begin{cases} x_k^{-\varepsilon_i} & \text{if } j = i, \\ -x_k^{-\varepsilon_i} x_i x_k^{\varepsilon_i} x_{i+1}^{-1} & \text{if } j = i + 1, \\ 1 - x_k x_i x_k^{-1} & \text{if } j = k \text{ and } \varepsilon_i = -1, \\ -x_k^{-1} + x_k^{-1} x_i & \text{if } j = k \text{ and } \varepsilon_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6. The **Jacobian matrix** $A = A(D)$ is the $n \times n$ matrix with

$$A_{i,j} = \left. \frac{\partial r_i}{\partial x_j} \right|_{x_1, \dots, x_n = t}$$

given by Fox differentiation and applying the abelianization map $G_K \rightarrow \mathbb{Z}$ sending $x_\ell \mapsto t$ for $\ell = 1, \dots, n$. More concretely, if the i -th crossing is as in Figure 2.8, then

$$A_{i,j} = \begin{cases} t^{-\varepsilon_i} & \text{if } j = i, \\ -1 & \text{if } j = i + 1, \\ 1 - t^{-\varepsilon_i} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

For both classical and virtual knots, the zeroth elementary ideal \mathcal{E}_0 is always trivial, and the reason is that the fundamental identity of Fox derivatives implies that one column of A can be written as a linear combination of the other columns. Recall that \mathcal{E}_0 is the ideal generated by all $(n-0) \times (n-0)$ minors of A . Since A is $n \times n$, then $\mathcal{E}_0 = (\det(A)) = 0$ because we can get a column of

zeros by simply summing up the columns (which thus makes the determinant equal to 0). Note that row i has one of each of the entries $t^{-\varepsilon_i}, -1, 1 - t^{-\varepsilon_i}$, while the rest are 0, so that the sum of all entries in each row is 0.

For classical knots, the first elementary ideal \mathcal{E}_1 is in general principal, and the **Alexander Polynomial** Δ_K is defined as the Laurent polynomial which generates it. Thus $\mathcal{E}_1 = (\Delta_K(t))$, and the Alexander polynomial is well-defined up to multiplication by $\pm t^k$ for $k \in \mathbb{Z}$. It is obtained by taking the determinant of the Alexander matrix, which is the $(n-1) \times (n-1)$ matrix obtained by removing a row and column from A .

For virtual knots, the first elementary ideal \mathcal{E}_1 is not necessarily principal. One can nevertheless define the Alexander polynomial $\Delta_K(t)$ to be the generator of the smallest principal ideal containing \mathcal{E}_1 . Since $\mathbb{Z}[t^{\pm 1}]$ is a gcd domain, it is given by taking the gcd of all the $(n-1) \times (n-1)$ minors of A .

The knot group and the associated Alexander ideal theory are not only invariants of virtual isotopy, but they are actually invariant under *welded equivalence*.

Definition 2.7. Two virtual knots or links are said to be **welded equivalent** if one can be obtained from the other by generalized Reidemeister moves plus the forbidden overpass, which is the move that exchanges two adjacent arrowtails on a Gauss diagram as in Figure 2.9.

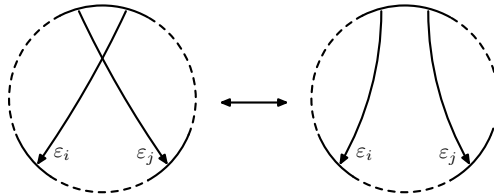


Figure 2.9: The forbidden overpass for Gauss diagrams.

2.4 Almost classical knots

Definition 2.8 (Almost Classical Knots and Links).

- (i) A virtual knot diagram is **almost classical** if there exists an integer-valued function λ on the set of short arcs satisfying the relations in Figure 2.10 at each crossing. (This definition extends naturally to virtual links.) Such diagrams are also called *Alexander numberable*.

- (ii) Given a virtual knot or link K , we say K is **almost classical** if it is represented by an almost classical diagram.

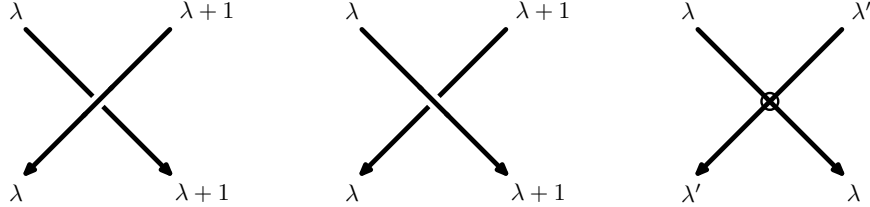


Figure 2.10: The Alexander numbering conditions.

Definition 2.9. Given a Gauss diagram D of a virtual knot with n chords c_1, \dots, c_n , we define the index of the chord c_i by counting the chords c_j that intersect c_i with sign and keeping track of direction (as described next). Orient the diagram so that c_i is vertical with its arrowhead pointing up. Then chords c_j that intersect c_i will cross over c_i either from right to left or from left to right, and they will have sign $\varepsilon_j = \pm 1$. The **index** of the chord c_i is defined to be

$$I(c_i) = \varepsilon_i(r_+ - r_- + \ell_- - \ell_+),$$

where

$$\begin{aligned} r_{\pm} &= \#\{c_j \mid c_j \text{ intersects } c_i \text{ with } \varepsilon_j = \pm 1 \text{ and arrowhead to the right}\}, \\ \ell_{\pm} &= \#\{c_j \mid c_j \text{ intersects } c_i \text{ with } \varepsilon_j = \pm 1 \text{ and arrowhead to the left}\}. \end{aligned}$$

For example, the Gauss diagram in Figure 2.5 has one chord with index 1 and another with index -1 .

Notice that, in order to compute the index of a chord, we only need to look at the nodes in the Gauss diagram between the arrowhead and arrowtail of c_k , or those between the arrowtail and arrowhead (moving counterclockwise around the diagram, as usual). If we count from the arrowtail to arrowhead, we count $+1$ for every positive arrowhead or negative arrowtail we encounter, and a -1 for every positive arrowtail or negative arrowhead. If we count from the arrowhead to arrowtail, we count $+1$ for every negative arrowhead or positive arrowtail we encounter, and a -1 for every negative arrowtail or positive arrowhead.

One can verify that a Gauss diagram D represents an almost classical virtual knot diagram if and only if every chord c_i of D has index $I(c_i) = 0$.

Almost classicality in virtual knot theory is closely related to Gaussian parity, and here we give a brief account. Parity is an important topic in virtual knot theory, and here we will only scratch the surface. For more information, we refer the reader to Manturov’s original article [Man13], his book [MI13], and the monograph [IMN11].

Given a virtual knot diagram, a parity is a function that assigns to each classical crossing a value in $\{0, 1\}$ (or “even” and “odd”) such that the following axioms hold:

1. In a Reidemeister one move, the parity of the crossing is even.
2. In a Reidemeister two move, the parities of the two crossings are either both even or both odd.
3. In a Reidemeister three move, the parities of the three crossings are unchanged. Further, the three crossings can be all even, all odd, or one even and two odd. (In other words, we exclude the case that one crossing is odd and two are even.)

Note that this is the definition of “parity in the weak sense,” cf. Manturov [Man13] and Nikonov [Nik16].

For example, taking

$$f(c_i) = \begin{cases} 0 & \text{if } I(c_i) = 0 \\ 1 & \text{if } I(c_i) \neq 0 \end{cases} \quad (2.5)$$

gives a parity that we call the *total Gaussian parity*.

We give an informal discussion explaining why the total Gaussian parity satisfies the parity axioms. Firstly, it is clear that $I(c_i) = 0$ for any chord c_i involved in a Reidemeister one move, since no other chords intersect the chord c_i . Thus $f(c_i) = 0$, and so axiom 1 holds.

Next we claim that $I(c_i) = -I(c_j)$ for any two chords involved in a Reidemeister two move (see Figure 2.4). To see this, notice that any other chord that intersects c_i will intersect c_j too. Let $m = r_+ - r_- + \ell_- - \ell_+$, the sum of all other chords that cross over both c_i and c_j , excluding contributions by c_i or c_j . Suppose c_i is positive and c_j is negative. Since c_i and c_j intersect one another, we have that $I(c_i) = \varepsilon_i(m+1) = m+1$ and $I(c_j) = \varepsilon_j(m+1) = -(m+1)$. Thus $I(c_i) = -I(c_j)$. It follows that $f(c_i) = f(c_j)$, thus axiom 2 is also satisfied.

Lastly, suppose c_i, c_j, c_k are three chords involved in a Reidemeister three move (see Figure 2.4). Then we will show that $I(c_i), I(c_j)$, and $I(c_k)$ are unchanged throughout the move, and moreover that they satisfy $I(c_i) + I(c_j) =$

$I(c_k)$. To that end, suppose c_k is the negative chord and c_i and c_j are the two positive chords. We will use c_i, c_j, c_k to denote the chords before the move $\Omega 3a$ is performed, and c'_i, c'_j, c'_k for the chords after the move $\Omega 3a$ is performed. Notice that one can compute the index of any chord by counting the nodes from its arrowhead to its arrowtail, counting $+1$ for every negative arrowhead or positive arrowtail encountered, and -1 for every negative arrowtail or positive arrowhead. (Alternatively, one can compute the index by counting the nodes from its arrowtail to its arrowhead, counting $+1$ for every negative arrowtail or positive arrowhead encountered, and -1 for every negative arrowhead or positive arrowtail.)

In any case, we explain why, for any of the four diagrams in $\Omega 3a$ in Figure 2.4, the positive chords c_i and c_j do not contribute to the index $I(c_k)$ of the negative chord. In the first diagram on the left, none of the arrowheads/arrowtails of c_i or c_j occur. In the second and third diagrams, the contributions of c_i and c_j cancel because one arrowhead and one arrowtail of the same sign occur. In the fourth diagram, the arrowheads and arrowtails of both c_i and c_j are encountered, so each cancels itself. It follows that $I(c_k) = I(c'_k)$ in both cases, and a similar argument shows that $I(c_i) = I(c'_i)$ and $I(c_j) = I(c'_j)$. This explains why the indices $I(c_i), I(c_j)$, and $I(c_k)$ are unchanged in undergoing the move $\Omega 3a$.

The same idea shows that $I(c_i) + I(c_j) = I(c_k)$, as we now explain. For instance, applying the second method above for computing indices to the leftmost diagram in Figure 2.4 we note that the arc along which we compute the index of c_k is a disjoint union of the two arcs used to compute the indices of c_i and c_j , along with the cancelling contributions of c_i and c_j . We then see that $I(c_i) + I(c_j) = I(c_k)$ for the leftmost diagram. Likewise, applying the first method for computing the indices in the rightmost diagram in Figure 2.4 and noting that the arc along which we compute the index of c_k is again a union of the arcs used to compute the indices of c_i and c_j , it follows that $I(c_i) + I(c_j) = I(c_k)$ for the rightmost diagram. The same identity holds for the other two diagrams since the indices are unchanged in performing a $\Omega 3a$ move. Thus axiom 3 holds, and it follows that f satisfies all three parity axioms.

Notice that a diagram D has only even chords if and only if $I(c_i) = 0$ for all chords (by definition of f). This is equivalent to the condition that D admit an Alexander numbering.

There is a map

$$P_f: \{\text{Gauss diagrams}\} \longrightarrow \{\text{Gauss diagrams}\}$$

called *Manturov projection* which is defined by removing the odd chords of D . Thus, if all chords of D are even, namely if D admits an Alexander numbering, then $P_f(D) = D$. Otherwise, if $P_f(D) \neq D$ then D contains one or more odd chords and does not admit an Alexander numbering. Its projection $P_f(D)$ will then be a diagram with fewer chords, but because removal of chords may change the parity of the chords that are not removed, it is not generally true that $P_f(D)$ has no odd chords.

However, repeated application of parity projection will eventually give a diagram without odd chords. In fact, since at the start D has only finitely many chords, and since the number of chords in $P_f^k(D)$ is monotone non-increasing in k , eventually we have $P_f^{n+1}(D) = P_f^n(D)$ for some $n \geq 0$. The resulting diagram $\bar{D} = P_f^n(D)$ is called the image of D under stable projection $P_f^\infty = \lim_{n \rightarrow \infty} P_f^n$, and it has no odd chords.

In summary, we have shown that for any virtual knot diagram D , its image $P_f^\infty(D)$ under stable Manturov projection admits an Alexander numbering and therefore is an almost classical virtual knot.

Although Manturov projection P_f is defined at the level of diagrams, the next proposition implies that it is well-defined on virtual knots. The proof is an immediate consequence of the parity axioms, and for details we refer to either [Man13] or [Nik16].

Proposition 2.10. *If two virtual knot diagrams K and K' are virtually isotopic, then so are their images $P_f(K)$ and $P_f(K')$ under Manturov projection.*

Next, we recall that if K is an almost classical knot, then its first elementary ideal \mathcal{E}_1 is principal. This result was proved by Nakamura, Nakanishi, Satoh, and Tomiyama in [NNST12, Theorem 1.2] by using an Alexander numbering to determine a linear combination of the rows of the Jacobian matrix that sum to zero. Because it is central to our later results, we will go through their argument carefully.

Proposition 2.11. *If K is an almost classical knot or link, then its first elementary ideal \mathcal{E}_1 is principal.*

Proof. Let $A = \left(\frac{\partial r_i}{\partial x_j} \Big|_{x_1, \dots, x_n = t} \right)$ be the Jacobian matrix as given in Definition 2.6. Recall that \mathcal{E}_1 is the ideal generated by $(n-1) \times (n-1)$ submatrices

of A . Then equation (2.4) implies that

$$A_{i,j} = \frac{\partial r_i}{\partial x_j} \Big|_{x_1, \dots, x_n=t} = \begin{cases} t^{-\varepsilon_i} & \text{if } j = i, \\ -1 & \text{if } j = i + 1, \\ 1 - t^{-\varepsilon_i} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.6)$$

Let $A_{i,*}$ denote the i -th row of A , and set

$$\vartheta_i = \begin{cases} t^{\lambda_i} & \text{if } \varepsilon_i = -1, \\ t^{\lambda_i+1} & \text{if } \varepsilon_i = +1. \end{cases}$$

where λ_i and λ_{i+1} are the two Alexander numbers showing up at the crossing c_i (see Figure 2.11). Notice that ϑ_i is a unit in $\mathbb{Z}[t^{\pm 1}]$ for $i = 1, \dots, n$.

Claim 2.12. *We have $\sum_{i=1}^n \vartheta_i A_{i,*} = 0$.*

To prove the claim, we compute

$$\vartheta_i A_{i,*} = \begin{cases} (0, \dots, 0, t^{\lambda_i}, -t^{\lambda_i+1}, 0, \dots, 0, t^{\lambda_i+1} - t^{\lambda_i}, 0, \dots, 0) & \text{if } \varepsilon_i = +1, \\ (0, \dots, 0, t^{\lambda_i+1}, -t^{\lambda_i}, 0, \dots, 0, t^{\lambda_i} - t^{\lambda_i+1}, 0, \dots, 0) & \text{if } \varepsilon_i = -1. \end{cases}$$

Recall that a row of A_i corresponds to a crossing c_i of K , and that at a crossing c_i we label the incoming underarc by x_i , outgoing underarc by x_{i+1} , and overarc by x_k . In the case when $\varepsilon_i = +1$, the t^{λ_i} term in row i corresponds to the incoming underarc x_i ; $-t^{\lambda_i+1}$ corresponds to the outgoing underarc x_{i+1} , and $t^{\lambda_i+1} - t^{\lambda_i}$ corresponds to the overarc x_k . Similar considerations apply in the case $\varepsilon_i = -1$ case.

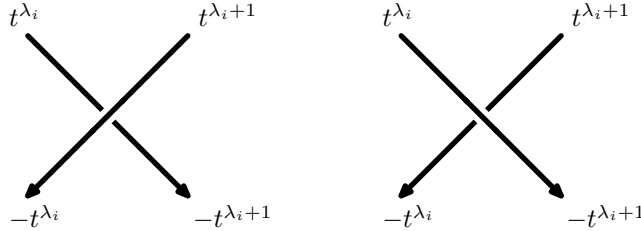


Figure 2.11: Arc labels at the i -th crossing in terms of Alexander numbers.

Notice that the incoming arcs have labels with positive signs, and the outgoing arcs have labels with negative signs. In the linear combination $\sum_{i=1}^n \vartheta_i A_{i,*}$

of rows of A , the j -th entry is given as the sum of all terms in the j -th column (each multiplied by a ϑ_i), namely all the terms as above contributed by the arc x_j . This includes terms for which x_j is the outgoing underarc, incoming underarc, or overarc, and those terms are given by multiplying one of $-1, t^{-\varepsilon_i}, 1 - t^{-\varepsilon_i}$ as in (2.4) with the coefficient ϑ_i as above. This is the same as summing up all the labels as in Figure 2.11 as you move across the arc x_j . Of course, the λ_i term corresponds to the Alexander numbering of the arcs.

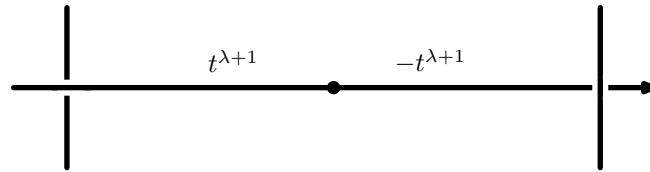


Figure 2.12: Two consecutive Alexander numbers.

Now, on a given arc x_j , it turns out that consecutive labels will cancel and this will show why the sum $\sum_{i=1}^n \vartheta_i A_{i,*} = 0$ is zero. Recall that x_j goes from the $(j - 1)$ -st under-crossing to the j -th under-crossing, so it is an incoming underarc and outgoing underarc exactly once. However, it can be an overarc for multiple crossings. Since the ϑ_i 's correspond to the Alexander numbers, if we have a short arc contributing two terms to the sum, each term must have the same power of t (see Figure 2.12).

The reason is that Alexander numberings are assigned to the short arcs but the labels (as in Figure 2.11) are assigned to half of a short arc. They are still related to the Alexander numbering on the arc, and any two terms on the same short arc must have the same Alexander number. On the other hand, the two terms are of opposite sign (since one will be ingoing and one outgoing), and that is why the sum is zero.

We have now proven the claim that $\sum_{i=1}^n \vartheta_i A_{i,*} = 0$. We now explain why this means that \mathcal{E}_1 is principal. Performing this linear combination on A gives us a row of zeros. We also know that adding up the entries in each row of A gives us a column of zeros. Without loss of generality, let us perform these two operations and assume the last row and last column of A are now zero. Then any $(n - 1) \times (n - 1)$ submatrix of A that does not eliminate the last row and last column will have determinant zero (since we can expand along the row or column of zeros that will be left in). Thus, only the $(n - 1) \times (n - 1)$ submatrix without the row and column of zeros has non-zero determinant, and therefore

\mathcal{E}_1 is generated by a single element, and thus principal. Note however that we could put the row or column of zeros in any row, so any $(n-1) \times (n-1)$ minor will produce a generator of \mathcal{E}_1 . \square

As a consequence of the proof, it follows that for an almost classical knot K with Jacobian matrix A constructed as in Definition 2.6, its Alexander polynomial $\Delta_K(t)$ is given by taking the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing any row and any column from A . The reason is that each of the coefficients ϑ_i (which is t^{λ_i} or t^{λ_i+1}) is a unit in $\mathbb{Z}[t^{\pm 1}]$. Thus, writing out our linear combination from above, we can solve it to write any row as a linear combination of the others, since we have inverses for all the coefficients (the ϑ_i s).

2.5 Virtual braids

In this section, we introduce virtual braids and recall the virtual analogue of Alexander's theorem (stated later in this section), which shows that every virtual knot or link can be realized as the closure of a virtual braid.

The **virtual braid group on k strands**, denoted VB_k , is the group generated by symbols $\sigma_1, \dots, \sigma_{k-1}, \tau_1, \dots, \tau_{k-1}$ subject to the relations given below in (2.7), (2.8), (2.9). Here, σ_i represents a classical crossing and τ_i represents a virtual crossing involving the i -th and $(i+1)$ -st strands as in Figure 2.13. Virtual braids are drawn from top to bottom, the group operation is given by stacking the diagrams, and the closure of a virtual braid represents a virtual link.

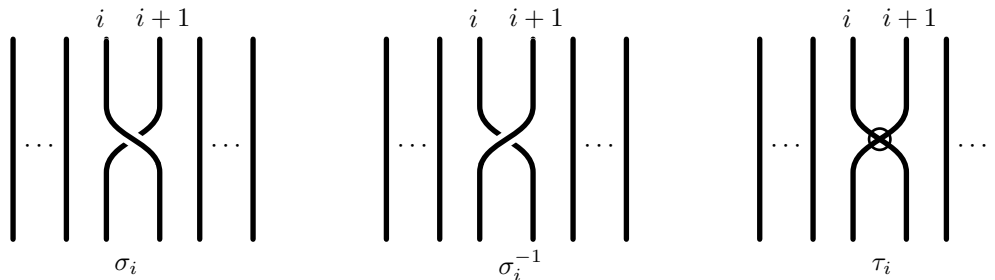


Figure 2.13: Generators of VB_k .

The relations in VB_k come in three families: the first involve only classical crossings σ_i and are the same as in classical braid group B_k ; the second involve only virtual crossings τ_i and are the same as in the permutation group S_k ;

and the third are the mixed relations which involve both classical and virtual crossings:

$$\begin{aligned}\sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1},\end{aligned}\tag{2.7}$$

$$\begin{aligned}\tau_i\tau_j &= \tau_j\tau_i && \text{if } |i-j| > 1, \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1}, \\ \tau_i^2 &= 1,\end{aligned}\tag{2.8}$$

$$\begin{aligned}\sigma_i\tau_j &= \tau_j\sigma_i && \text{if } |i-j| > 1, \\ \tau_i\sigma_{i+1}\tau_i &= \tau_{i+1}\sigma_i\tau_{i+1}.\end{aligned}\tag{2.9}$$

Note that the virtual generators $\tau_1, \dots, \tau_{k-1}$ generate a finite subgroup of VB_k isomorphic to the symmetric group S_k on k letters. Recall that elements in S_k are permutations, and that any permutation can be written as a product of adjacent transpositions, so that S_k can be generated by the set of adjacent transpositions. Since τ_i swaps the i -th and $(i+1)$ -st strands, τ_i corresponds to the permutation $(i \ i+1)$, which is an adjacent transposition.

The next result is Alexander's theorem for virtual knots and links, and it was first proved by Kamada in [Kam07] via a braiding process. Interestingly, the statement of Alexander's theorem is stronger in the virtual setting because, as we shall see, the virtual braid faithfully reproduces the Gauss code of the virtual knot diagram, cf. [Bir74, Theorem 2.1]. Although Alexander's theorem is well-known, we include a proof because similar ideas will come up later in the proof of Theorem 3.11, where we establish an equivariant version of Alexander's theorem for periodic virtual knots.

Theorem 2.13. *Every virtual knot diagram can be realized as the closure of a virtual braid.*

Proof. Let K be a virtual knot diagram with n real crossings and Gauss code C_K . We will show how to construct a virtual braid β on $2n$ strands whose closure is a virtual knot with Gauss code identical with C_K , and it follows that $\widehat{\beta}$ and K are equivalent (as virtual knot diagrams) up to a sequence of detour moves and planar isotopies.

First, we draw the n real crossings side by side pointing downwards according to their sign (see Figure 2.2). For example, considering the Gauss code $C_K = 01-U2+03-U1-02+U3-$, we draw three crossings as in Figure 2.14, with the first and third negative and the second one positive.

We label the $2n$ arcs across the top with $01, U1, U2, 02, \dots$ appropriately (that is, $0i, Ui$ for a negative crossing, and $Ui, 0i$ for a positive crossing), and we draw $2n$ points directly underneath, which we label $01, U1, U2, 02,$

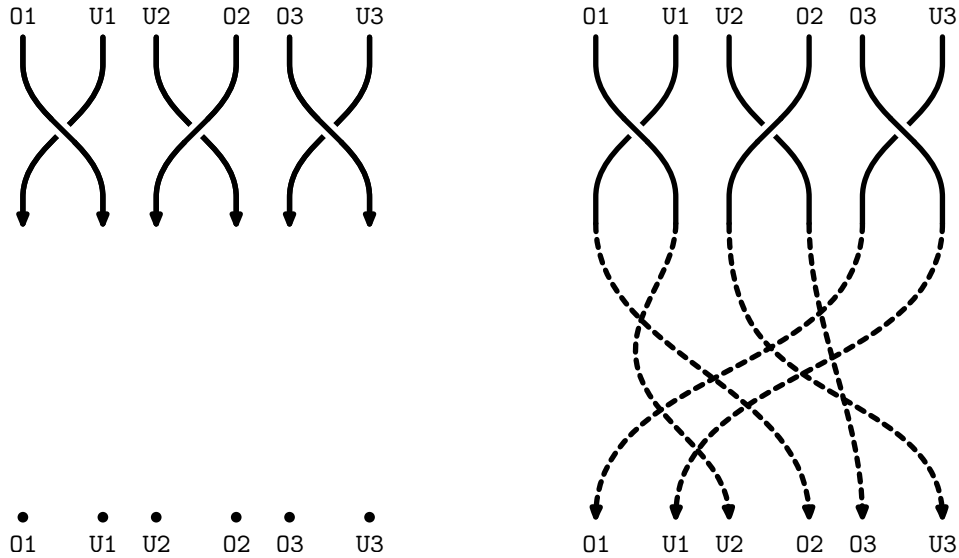


Figure 2.14: The braiding algorithm for $C_K = 01-U2+03-U1-02+U3-$.

... in exactly the same order as along the strands on top. (This is illustrated on the left of Figure 2.14.)

The Gauss word C_K tells us how to connect the outgoing arcs from each of the n crossings to the corresponding points at the bottom. For instance, the first part of the Gauss word $01-U2+03-U1-02+U3-$ tells us to connect the outgoing overarc of the first crossing (01) to the point labelled $U2$ below, and next it tells us to connect the outgoing underarc of the second crossing ($U2$) to the point labelled 03 . Continuing in this way, we connect all the arcs to points, with the last crossing in the Gauss word connected back to the first entry. In the example, it tells us to connect the outgoing underarc of the third crossing ($U3$) to the point labelled 01 . The outcome is a virtual braid as depicted on the right of Figure 2.14.

The connecting arcs are drawn monotonically decreasing, and every new crossing that is created is drawn as a virtual crossing. Basically, the connecting lines, which appear as dashed lines in Figure 2.14, determine an element in S_{2n} , the symmetric group. Because the virtual generators of VB_{2n} generate a subgroup isomorphic to S_{2n} , we can always write this element of S_{2n} as a word in the $\tau_1, \dots, \tau_{2n-1}$ (since we only added virtual crossings). In the example above, we get the word $\tau_1\tau_4\tau_3\tau_5\tau_4\tau_5\tau_2\tau_1\tau_3\tau_2$.

The resulting diagram will be a virtual braid β (in this example, $\beta = \sigma_1\sigma_3^{-1}\sigma_5\tau_1\tau_4\tau_3\tau_5\tau_4\tau_5\tau_2\tau_1\tau_3\tau_2$ on $2n$ strands whose closure is equivalent to the

given knot K . In fact, as one can easily verify, the Gauss code of $\widehat{\beta}$ is equal to C_K . \square

There is also an easy (purely algebraic) way to get a word in the τ_i s (discussed in the proof above), and the entire braidword for β as well, directly from the Gauss code. This method may produce a different braid word than the one above (but they are equivalent). First we map the components of the Gauss code to the integers $1, \dots, 2n$ as follows:

$$\begin{aligned} U_i+ &\mapsto 2i - 1 \\ O_i+ &\mapsto 2i \\ O_i- &\mapsto 2i - 1 \\ U_i- &\mapsto 2i \end{aligned}$$

for $i = 1, 2, \dots, n$ (this represents numbering the strands $1, 2, \dots, 2n$ across the top). We then can rewrite our Gauss code as a permutation in S_{2n} . For example, $O1-U2+O3-U1-O2+U3-$ converts to (135246) . We then write the permutation as a product of adjacent transpositions. In this case, we have

$$\begin{aligned} (135246) &= (13)(35)(52)(24)(46) = (13)(35)(25)(24)(46) \\ &= (12)(23)(34)(45)(23)(34)(45)(23)(34)(45)(56), \end{aligned}$$

where we substituted $(13) = (12)(23)$ for instance. Since τ_i corresponds to $(i \ i + 1)$, we then get our word of τ_i s as $\tau_5\tau_4\tau_3\tau_2\tau_4\tau_3\tau_2\tau_4\tau_3\tau_2\tau_1$. Notice that we read these in reverse, since a permutation is read from right to left, but our braid word is read from left to right. However, we need to add $\tau_1\tau_3\tau_5 \cdots \tau_{2n-1}$ to the beginning of this, as we will explain. Our $1, 2, \dots, 2n$ assignments were labeling the strands before we crossed over the classical crossings. Once we cross over a classical crossing, O_i and U_i have now swapped labels ($2i$ and $2i - 1$), so we need a τ_{2i-1} to return them back to the labels at the top. Then, using the rest of the permutation will give the correct braid (permutation-wise, the τ_{2i-1} s cancel out the strand swapping from the σ_i s). So we have the braid word $\tau_1\tau_3\tau_5\tau_4\tau_3\tau_2\tau_4\tau_3\tau_2\tau_4\tau_3\tau_2\tau_1$ in this example (and we have β equal to $\sigma_1\sigma_3^{-1}\sigma_5\tau_1\tau_3\tau_5\tau_4\tau_3\tau_2\tau_4\tau_3\tau_2\tau_4\tau_3\tau_2\tau_1$). To see that the braid word produced in the proof above is equivalent to this one, we will show that it corresponds to the correct permutation, once we add the beginning $(1 \ 2)(3 \ 4)(5 \ 6)$ swaps

to represent going over the classical crossings (in other words, the $\sigma_1, \sigma_3^{-1}, \sigma_5$). In the proof, we had $\tau_1\tau_4\tau_3\tau_5\tau_4\tau_5\tau_2\tau_1\tau_3\tau_2$, which, once we add in the three permutations mentioned above and write everything in reverse, corresponds to $(2\ 3)(3\ 4)(1\ 2)(2\ 3)(5\ 6)(4\ 5)(5\ 6)(3\ 4)(4\ 5)(1\ 2)(5\ 6)(3\ 4)(1\ 2) = (1\ 3\ 5\ 2\ 4\ 6)$, which is what we want.

We conclude this section by defining almost classical braids and introducing an invariant for them.

Definition 2.14. A braid is called **almost classical** if it admits an Alexander numbering (that is, if one can number the arcs of β such that, at each crossing the conditions of Figure 2.10 are satisfied) and so that the numbers along the bottom of β coincide with the numbers at the top.

Note that a braid is almost classical if and only if its closure $\widehat{\beta}$ is an almost classical diagram. For example, if $\beta \in B_k$ is a classical braid, then taking $\lambda_i = i$ on the i -th strand at the top extends to an Alexander numbering of β such that the i -th strand on the bottom also has the number i , thus any classical braid β is almost classical. Note that taking $\lambda_i = i$ gives a valid Alexander numbering since, for a classical braid, there are no virtual crossings, and each strand's Alexander number must increase by one as we move from left to right across the strands, because of the conditions of Figure 2.10. So, if we start on the left with $\lambda_1 = 1$, then the second strand will need to have $\lambda_2 = 2$, and so on. Since we are dealing with a knot, there must be (classical) crossings between each pair of strands (otherwise we would have a link if there was a pair of strands with no crossings), so it will always be increasing by one as we go along.

If $\beta \in VB_k$ is an almost classical braid, then consider the polynomial $f(t) = \sum_{i=1}^k t^{\lambda_i}$, where λ_i refers to the Alexander number on the i -th strand at a horizontal cross-section of β . Notice that this polynomial is independent of where along the braid the cross-section is taken. When passing a classical crossing, the Alexander numbers on the two strands swap positions, but $f(t)$ remains unchanged. When passing a virtual crossing, the Alexander numbers do not change. Taken up to multiples of t^ℓ , $f(t)$ gives a well-defined invariant of almost classical braids, which is also independent of the choice of Alexander numbering provided β is not a split braid.

2.6 Alexander invariants (reprise)

In this section, we assume that K is a virtual knot diagram, which has been realized as the closure $\widehat{\beta}$ for a virtual braid $\beta \in VB_k$, which we use to give an alternative presentation matrix for the Alexander invariants. The main difference from Definition 2.6 is that we have generators x_1, \dots, x_k for the strands on the top of β and generators z_1, \dots, z_k for the strands on the bottom. This approach is especially convenient in deriving formulas for the Alexander invariants of periodic virtual knots K which are represented as the closures of periodic virtual braids.

Definition 2.15. Suppose K is a virtual knot diagram with n crossings, and apply Theorem 2.13 to write $K = \widehat{\beta}$, where β is a virtual braid on k strands. We label the arcs on top of β by x_1, x_2, \dots, x_k and the arcs on the bottom of β by z_1, z_2, \dots, z_k , and we use y_1, y_2, \dots, y_r to label the internal arcs β , which are the arcs that do not start or end at the top or bottom of β . (Note that we will typically have $r = n - k$, unless $n < k$ or some strands of β pass over all the other strands.)

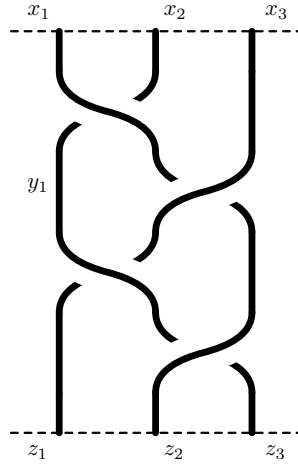
This gives a presentation for the knot group

$$G_K = \langle x_1, \dots, x_k, y_1, \dots, y_r, z_1, \dots, z_k \mid R_1, \dots, R_n, S_1, \dots, S_k \rangle,$$

where the R_i are the usual Wirtinger relations coming from the crossings of β and the S_i are the relations which correspond to setting $x_i = z_i$ for the closure $\widehat{\beta}$.

The **Jacobian matrix** B associated to this presentation of G_K is the $(n + k) \times (n + k)$ matrix with rows ordered by the relations $R_1, \dots, R_n, S_1, \dots, S_k$ and columns ordered by the generators $x_1, \dots, x_k, y_1, \dots, y_r, z_1, \dots, z_k$ and (i, j) entry given by the Fox differentiating the i -th relation with respect to the j -th generator and applying the abelianization map $G_K \rightarrow \mathbb{Z}$ sending each of the generators to t .

For example, consider the classical braid $\beta = (\sigma_1 \sigma_2)^2$ in Figure 2.15. The knot group G_K of $K = \widehat{\beta}$ has a presentation with generators $x_1, x_2, x_3, y_1, z_1, z_2, z_3$ and relations

Figure 2.15: The classical braid $\beta = (\sigma_1\sigma_2)^2$.

$$R_1 = x_1x_2x_1^{-1}y_1^{-1}, \quad R_2 = x_3^{-1}x_1x_3z_2^{-1}, \quad R_3 = y_1x_3y_1^{-1}z_1^{-1}, \quad R_4 = z_2^{-1}y_1z_2z_3^{-1},$$

$$S_1 = z_1x_1^{-1}, \quad S_2 = z_2x_2^{-1}, \quad S_3 = z_3x_3^{-1}.$$

We write the Jacobian matrix of G_K with rows ordered by the relations $R_1, \dots, R_4, S_1, \dots, S_3$ and columns ordered by the generators $x_1, x_2, x_3, y_1, z_1, z_2, z_3$; it is the square matrix

$$B = \begin{bmatrix} 1-t & t & 0 & -1 & 0 & 0 & 0 \\ t^{-1} & 0 & 1-t^{-1} & 0 & 0 & -1 & 0 \\ 0 & 0 & t & 1-t & -1 & 0 & 0 \\ 0 & 0 & 0 & t^{-1} & 0 & 1-t^{-1} & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that the Wirtinger presentation for the virtual knot is

$$G_K = \langle x_1, x_2, x_3, y_1 \mid \tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_4 \rangle,$$

where \tilde{R}_i is obtained from R_i above by substituting x_j for z_j . Applying Defini-

tion 2.6 to this presentation of G_K gives the Jacobian

$$A = \begin{bmatrix} 1-t & t & 0 & -1 \\ t^{-1} & -1 & 1-t^{-1} & 0 \\ -1 & 0 & t & 1-t \\ 0 & 1-t^{-1} & -1 & t^{-1} \end{bmatrix}.$$

Notice that this matrix can also be obtained from the upper left 4×4 block of B by combining (that is, adding) the x_j and z_j columns.

In general, the matrix derived from Definition 2.15 will be given by

$$B = \left[\begin{array}{c|c|c} \partial R_i / \partial x_j & \partial R_i / \partial y_j & \partial R_i / \partial z_j \\ \hline \partial S_i / \partial x_j & \partial S_i / \partial y_j & \partial S_i / \partial z_j \end{array} \right] = \left[\begin{array}{c|c|c} x_* & y_* & z_* \\ \hline -I_k & 0 & I_k \end{array} \right],$$

and it is related to the matrix derived from Definition 2.6, which is

$$A = \left[\begin{array}{c|c} x_* + z_* & y_* \\ \hline 0 & 0 \end{array} \right] \cong [x_* + z_* \mid y_*].$$

We now state and prove a result analogous to Proposition 2.11 for the Jacobian matrix B from Definition 2.15 for an almost classical knot K . We will assume that K has been realized as the closure of an almost classical braid $\beta \in VB_k$.

Lemma 2.16. *Suppose K is an almost classical knot diagram with $\beta \in VB_k$ for an almost classical braid with n crossings. Let B be the Jacobian matrix (constructed as in Definition 2.15) with relations R_1, \dots, R_n from the crossings and S_1, \dots, S_k from the identities $z_i x_i^{-1}$ as above. Note that B is an $(n+k) \times (n+k)$ matrix. For $i = 1, \dots, n$, let λ_i^R be the Alexander number of the i -th crossing of β , as in Figure 2.16, and for $i = 1, \dots, k$, let λ_i^S be the Alexander number of the i -th strand at the top of β .*

Then $\sum_{i=1}^{n+k} \omega_i B_{i,*} = 0$, where

$$\omega_i = \begin{cases} t^{\lambda_i^R} & \text{for } 1 \leq i \leq n \text{ and } \varepsilon_i = -1, \\ t^{\lambda_i^R + 1} & \text{for } 1 \leq i \leq n \text{ and } \varepsilon_i = +1, \\ t^{\lambda_{i-n}^S} & \text{for } n+1 \leq i \leq n+k. \end{cases}$$

Proof. From Claim 2.12, if we have a matrix A constructed from Definition 2.6,

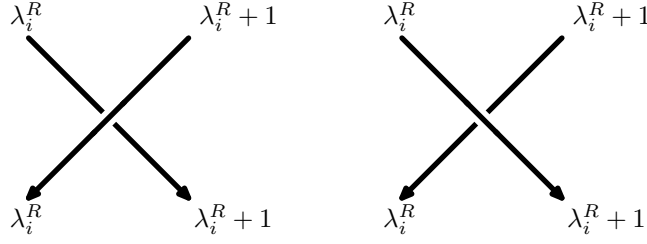


Figure 2.16: The Alexander numbers at the i -th crossing for $\varepsilon_i = 1$ on the left and $\varepsilon_i = -1$ on the right.

then $\sum_{i=1}^n \vartheta_i A_{i,*} = 0$, where

$$\vartheta_i = \begin{cases} t^{\lambda_i^R} & \text{if } \varepsilon_i = -1 \text{ for } 1 \leq i \leq n, \\ t^{\lambda_i^R + 1} & \text{if } \varepsilon_i = +1 \text{ for } 1 \leq i \leq n. \end{cases}$$

Recall from the proof of Claim 2.12, that $\sum_{i=1}^n \omega_i A_{i,j}$ corresponds to the sum of all the $\pm t^k$ labels on the arc corresponding to the j -th column (so if column j was corresponding to an x_k label at the top, then it would be the sum of the labels on the arc that is labeled x_k), where the labels were assigned as in Figure 2.11. When we label the arcs as in Definition 2.15 for a virtual braid, the y_i arcs are not affected, and the only difference is that the arcs that were previously labelled x_i are now cut in half (on the unknotted part of the braid), and we have both an x_i and z_i arc. Previously, all the labels on x_i cancelled in pairs, so we had a sum of 0. But now that we have split x_i into two pieces, there is a pair that gets separated. The label at the top of the braid will be a $t^{\lambda_i^S}$ (positive since it is an ingoing arc), which will sit on the x_i arc. At the bottom of the braid, there will be a $-t^{\lambda_i^S}$ for the outgoing arc, which will sit on the z_i arc. Thus, before considering the S_i relations, we will have a sum of $t^{\lambda_i^S}$ on the x_i arc (all other labels within the braid on x_i will cancel as before), and a sum of $-t^{\lambda_i^S}$ on the z_i arc (all other labels within the braid on z_i will cancel as before). It remains to look at the S_i relations that contribute to the linear combination. Recall that these correspond to $x_i = z_i$, which will give a -1 in the x_i column and a $+1$ in the z_i column, and zeros everywhere else. Hence, if we multiply this row by $t^{\lambda_i^S}$, we will get a sum of zero for the x_i and z_i columns. As stated above, the y_i columns remain unchanged from the previous method, and since the S_i relations do not involve y_i 's, the sum in their columns remains zero as well. Now the row corresponding to S_i is actually the $(n + i)$ -th row of

B , so for $n + 1 \leq i \leq n + k$, we take $\omega_i = t^{\lambda_{i-n}^S}$.

□

Chapter 3

Periodic Virtual Knots

3.1 Basic Definitions

In this section, we recall the definition of periodicity for virtual knot diagrams, Gauss codes and Gauss diagrams. We write out Wirtinger presentations for the knot groups G_K and G_{K_*} of a periodic virtual knot and its quotient, and we show how the Jacobian for K is related to that for K_* in terms of circulant block matrices.

Definition 3.1. A virtual knot diagram K is called **periodic** with **period** q if it misses the origin and is invariant under a rotation in the plane by an angle of $2\pi/q$ about the origin.

Given a periodic virtual knot diagram K , its **quotient knot** K_* is the knot obtained by closing up one fundamental domain of K . To be specific, take one fundamental domain of K , which is a pie-shaped region centered at the origin with angle $2\pi/q$, and connect the arcs along the upper and lower edges with concentric circular arcs.

Figure 3.1 illustrates a periodic virtual knot on the left and its quotient on the right. In that picture, τ is a virtual tangle diagram.

Definition 3.2. The **linking number** k of a periodic virtual knot diagram K is the absolute value of the intersection number of a ray R emanating from the origin with the virtual knot diagram. As usual, we sum up intersection points, and they count positively if they come from an arc of K that winds counter-clockwise around the origin, otherwise they count negatively.

For example, the periodic virtual knot in Figure 3.1 has linking number 3. A well known example is the $(3, 3, 3)$ -pretzel knot 9_{35} , and Figure 3.2 depicts

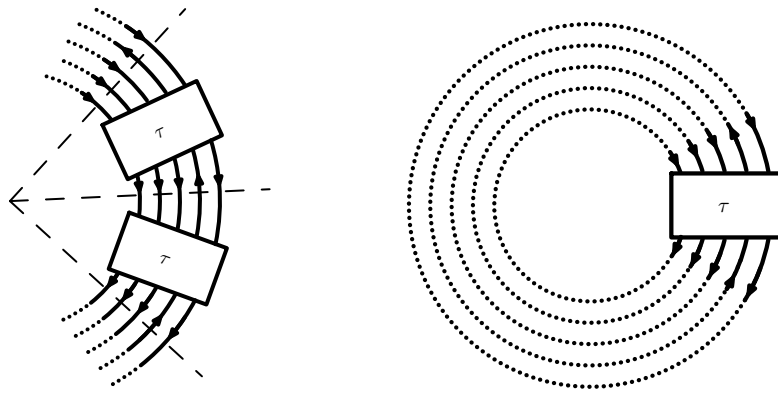


Figure 3.1: A periodic virtual knot and its quotient knot.

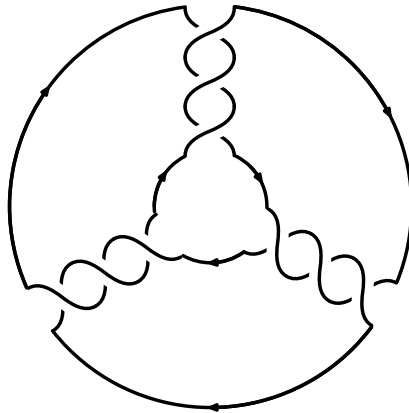


Figure 3.2: A 3-periodic knot diagram for the pretzel knot 9_{35} .

a 3-periodic diagram for 9_{35} with linking number $k = 2$ and quotient K_* a diagram of the unknot.

Periodicity of a virtual knot diagram is reflected in its Gauss code. For instance, the diagram for the pretzel knot 9_{35} has underlying Gauss code

$$C = U1-02-U3-06-U5-04-U7-08-U9-03-U2-01-U4-05-U6-09-U8-07-.$$

In general, we say a Gauss code is q -periodic if the 0/U and +/- patterns repeat with period q , and whenever 0_i goes to 0_j in the next period, then U_i goes to U_j . (In other words, the periodic transformation is a well-defined map on the crossings of K .)

For instance, the Gauss code for 9_{35} is a signed word of length 18, and we

will write it in the following way to emphasize its 3-periodicity.

$$C = \begin{array}{l} \text{U1-02-U3-06-U5-04-} \\ \text{U7-08-U9-03-U2-01-} \\ \text{U4-05-U6-09-U8-07-} \end{array}$$

Thus, the periodic transformation of C is the map (of ordered sets)

$$\{1, 2, 3\} \mapsto \{7, 8, 9\} \mapsto \{4, 5, 6\} \mapsto \{1, 2, 3\}.$$

(Note, a more succinct description of this map is $i \mapsto i + 6 \pmod{9}$.) Applying this transformation to C , the new Gauss code is easily seen to be equivalent to the original one under a cyclic permutation of C .

The Gauss diagram encodes the same information as the Gauss word, and a Gauss diagram is said to be q -periodic if it is invariant under a rotation of an angle of $2\pi/q$. This is equivalent to the condition of q -periodicity for the associated Gauss code, but it is easier to visualize.

Clearly, if K is a q -periodic virtual knot diagram, then its Gauss code and Gauss diagram are also both q -periodic. In that case, the Wirtinger presentation associated to the periodic diagram as in Equation (2.2) is symmetric, and we take a moment to explain what we mean by this.

Suppose K has qn crossings. Pick a basepoint and label the chords

$$c_{1,0}, \dots, c_{n,0}, c_{1,1}, \dots, c_{n,1}, \dots, c_{1,q-1}, \dots, c_{n,q-1}$$

of the Gauss diagram D_K in the order in which their arrowheads are encountered as one travels around the knot. Because K is q -periodic, we can assemble them

$$\begin{array}{ccc} c_{1,0} & \cdots & c_{n,0} \\ \vdots & & \vdots \\ c_{1,q-1} & \cdots & c_{n,q-1} \end{array}$$

so that the periodic action is the vertical shift sending $c_{i,j}$ to $c_{i,j+1}$ for $j = 0, \dots, q-1$, with $j+1$ taken \pmod{q} , which is to say that if $j = q-1$, then $j+1$ equals 0.

We can label the arcs

$$\begin{array}{ccc} a_{1,0} & \cdots & a_{k,0} \\ \vdots & & \vdots \\ a_{1,q-1} & \cdots & a_{n,q-1} \end{array}$$

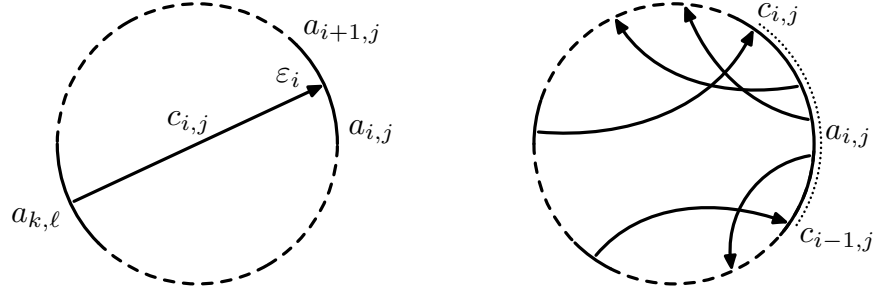


Figure 3.3: On left, $c_{i,j}$ separates $a_{i,j}$ and $a_{i+1,j}$ and has arrowfoot on $a_{k,\ell}$. On right, $a_{i,j}$ is the dotted arc between the arrowheads of $c_{i-1,j}$ and $c_{i,j}$.

accordingly, so that, for $i = 2, \dots, n$, the arc $a_{i,j}$ starts at the arrowhead of $c_{i-1,j}$ and ends at $c_{i,j}$; see Figure 3.3. When $i = 1$, the arc $a_{1,j}$ starts at $c_{n,j-1}$ and ends at $c_{1,j}$.

If $\varepsilon_{i,j} = \pm 1$ is the sign of the chord $c_{i,j}$, then periodicity implies that $\varepsilon_{i,j} = \varepsilon_{i,j+1}$, so we will simply write ε_i . If $a_{k,\ell}$ denotes the arc on which the arrowfoot of $c_{i,j}$ lies, then by periodicity $a_{k,\ell+1}$ is the arc on which the arrowfoot of $c_{i,j+1}$ lies. (Here, $\ell + 1$ and $k + 1$ are taken \pmod{q} .)

With these assumptions, the Wirtinger relation of the crossing $c_{i,j}$ is given by

$$r_{i,j} = a_{k,\ell}^{-\varepsilon_i} a_{i,j} a_{k,\ell}^{\varepsilon_i} a_{i+1,j}^{-1}$$

for $i = 1, \dots, n-1$ and $j = 0, \dots, q-1$. When $i = n$, we get the relation $r_{n,j} = a_{k,\ell}^{-\varepsilon_n} a_{n,j} a_{k,\ell}^{\varepsilon_n} a_{1,j}^{-1}$.

The resulting Wirtinger presentation of the q -periodic virtual knot K is then

$$G_K = \langle a_{i,j} \mid r_{i,j} \rangle, \quad (3.1)$$

where $1 \leq i \leq n$, $0 \leq j \leq q-1$ in (3.1). This presentation admits a \mathbb{Z}/q symmetry, and the Wirtinger presentation for K_* by obtained as the quotient by adding the relations $a_{i,0} = a_{i,1} = \dots = a_{i,q-1}$ for $1 \leq i \leq n$, which gives the presentation

$$G_{K_*} = \langle a_1, \dots, a_n \mid r_1, \dots, r_n \rangle, \quad (3.2)$$

where a_i refers to the equivalence class $\{a_{i,0}, \dots, a_{i,q-1}\}$ of generators and r_i is the relation $a_{i+1}^{-1} a_k^{-\varepsilon_i} a_i a_k^{\varepsilon_i}$ with $i+1$ taken \pmod{n} .

Theorem 3.3. *Let K be a virtual knot diagram with period q , and let K_* be its quotient knot. If A and B are the Jacobian matrices of the Wirtinger presentations (3.2) and (3.1) of G_{K_*} and G_K , respectively, then*

$$B = \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ A_{q-1} & A_0 & \cdots & A_{q-2} \\ \vdots & \ddots & \ddots & \vdots \\ A_1 & \cdots & A_{q-1} & A_0 \end{bmatrix} \quad (3.3)$$

is a block circulant matrix, where A_0, A_1, \dots, A_{q-1} are square matrices satisfying $A_0 + A_1 + \cdots + A_{q-1} = A$.

Proof. Suppose K is q -periodic and label the chords $c_{i,j}$ and arcs $a_{i,j}$ as above. In constructing the Jacobian matrix B associated to (3.1), we order the rows to correspond with the chords

$$c_{1,0}, \dots, c_{n,0}, c_{1,1}, \dots, c_{n,1}, \dots, c_{1,q-1}, \dots, c_{n,q-1}$$

and the columns to correspond with the arcs

$$a_{1,0}, \dots, a_{n,0}, a_{1,1}, \dots, a_{n,1}, \dots, a_{1,q-1}, \dots, a_{n,q-1}.$$

Then the entry of B in the (i, j) -th row and (i', j') -th column, which is the entry in the row corresponding to $c_{i,j}$ and column corresponding to $a_{i',j'}$, is given by

$$B((i, j), (i', j')) = \begin{cases} t^{-\varepsilon_i} & \text{if } (i', j') = (i, j), \\ -1 & \text{if } (i', j') = (i+1, j), \text{ or if } i' = 1, i = n, \text{ and } j' = j+1, \\ 1 - t^{-\varepsilon_i} & \text{if } (i', j') = (k, \ell), \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that $a_{k,\ell}$ is the arc on which the arrowfoot of $c_{i,j}$ lies.)

Notice that for $c_{i,j}$, the $t^{-\varepsilon_i}$ entry will always be in the j -th column block of the i -th block row; the -1 term will always be in the j -th column block of the i -th block row, unless $i = n$, and then the -1 will sit in the $(j+1)$ -st column block of the i -th block row. The $1 - t^{-\varepsilon_i}$ entry, on the other hand, can be in any column block of the i -th block row. Also notice that if the foot of $c_{i,j}$ lies on $a_{k,\ell}$, then periodicity implies that the foot of $c_{i,j+1}$ lies on $a_{k,\ell+1}$ (with $j+1$,

$\ell + 1$ taken mod q). For example, in the i -th block row of B , we will have

$$\begin{array}{c} c_{i,1} \\ c_{i,2} \\ \vdots \end{array} \left[\begin{array}{ccc|cc|ccc|c} a_{i,0} & a_{i+1,0} & a_{k,0} & a_{i,1} & a_{i+1,1} & a_{k,1} & a_{i,2} & a_{i+1,2} & a_{k,2} & \cdots \\ t^{-\varepsilon_i} & -1 & & & & 1 - t^{-\varepsilon_i} & & & & \\ \hline & & & t^{-\varepsilon_i} & -1 & & & & 1 - t^{-\varepsilon_i} & \\ \hline & \cdots & & & \cdots & & \cdots & \cdots & & \end{array} \right]$$

Notice that the matrix B satisfies $B_{i,j} = B_{i+n,j+n}$, and therefore it is block circulant with $n \times n$ blocks of the desired form:

$$B = \begin{bmatrix} A_0 & A_1 & \cdots & A_{q-1} \\ A_{q-1} & A_0 & \cdots & A_{q-2} \\ \vdots & \ddots & \ddots & \vdots \\ A_1 & \cdots & A_{q-1} & A_0 \end{bmatrix}.$$

Next we will show that $\sum_{k=0}^{q-1} A_k = A$, where A is the Jacobian matrix for the quotient knot K_* . Notice that in the Wirtinger presentation (3.2) for G_{K_*} , the relations r_i are obtained from the relations $r_{i,j}$ of (3.1) under setting $a_{i,0} = a_{i,1} = \cdots = a_{i,q-1}$ for $1 \leq i \leq n$. The Jacobian matrix for G_{K_*} has i, j entry

$$A(i, j) = \begin{cases} t^{-\varepsilon_i} & \text{if } j = i, \\ -1 & \text{if } j = i + 1 \pmod{n}, \\ 1 - t^{-\varepsilon_i} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the i -th row of A is equal to the sum of the i -th rows of block matrices appearing in B , which is equivalent to the statement that $\sum_{k=0}^{q-1} A_k = A$. This completes the proof. \square

3.2 Periodicity and almost classical knots

Both periodicity and almost classicality are defined for virtual knots in terms of their representative diagrams, and it remains to show that we can find virtual

knot diagrams that exhibit both properties at the same time. In this section, we use Manturov projection to show that if K is a q -periodic virtual knot diagram representing an almost classical knot, then one can find a q -periodic almost classical diagram equivalent to K .

We begin with a few useful lemmas.

Lemma 3.4. *Suppose K is a virtual knot diagram representing an almost classical knot. Then its image $P_f(K)$ under Manturov projection is virtually isotopic to K .*

Proof. Since K represents an almost classical knot (but may not be Alexander numberable itself), by the definition of almost classical, we have a virtual knot diagram K' which is Alexander numberable and virtually isotopic to K . Equivalently, the Gauss diagram D' corresponding to K' has all of its chords of index 0. Applying Proposition 2.10, it follows that $P_f(K)$ is virtually isotopic to $P_f(K') = K'$, which is virtually isotopic to K . \square

Note that in the above lemma, $P_f(K)$ need not be an almost classical diagram. In fact, even if K represents an almost classical knot, K and $P_f(K)$ may fail to be almost classical diagrams. On the other hand, for any virtual knot diagram K , its image $P_f^\infty(K)$ under stable projection is an almost classical diagram.

Lemma 3.5. *Let K be a q -periodic virtual knot diagram with quotient K_* . For $j = 0, \dots, q - 1$, let $c_{1,j}, \dots, c_{n,j}$ be the chords in the j -th period of the Gauss diagram D_K for K , and let c_1, \dots, c_n be the corresponding chords in the Gauss diagram D_{K_*} for K_* . Then the index satisfies $I(c_{i,j}) = I(c_i)$ for $i = 1, \dots, n$ and $j = 0, \dots, q - 1$. In particular, the index $I(c_{i,j})$ of a chord is independent of its period $j = 0, \dots, q - 1$.*

Proof. Let $\pi : D_K \rightarrow D_{K_*}$ be the mapping of Gauss diagrams. It is a covering map of oriented trivalent graphs preserving the signs.

According to Definition 2.9, the index of c_i is given by counting the arrowheads and arrowtails with sign along the arc α_i of D_{K_*} from the arrowtail of c_i to its arrowhead. One can perform this computation upstairs in D_K after lifting α_i under π . If $\tilde{\alpha}_i$ denotes the lift starting at the arrowtail of $c_{i,j}$, then it will end at the arrowhead of $c_{i,k}$ for some $k = 0, \dots, q - 1$. The index along $\tilde{\alpha}_i$ differs from the index of $c_{i,j}$ by a similar count along an arc β of D_K from the arrowhead of $c_{i,k}$ to the arrowhead of $c_{i,j}$. Taking its image $\pi(\beta)$ under π , we obtain an arc that winds around D_{K_*} $|j - k|$ times (because of the periodicity),

and consequently the index along β is necessarily zero. (The index around an entire diagram will always be zero because the all arrowheads will cancel with their arrowtails in the sum). It follows that $I(c_i) = I(c_{i,j})$ for $j = 0, \dots, q - 1$, and this completes the proof. \square

Corollary 3.6. *If K is a q -periodic almost classical diagram with quotient K_* , then K_* is also almost classical.*

Lemma 3.7. *Suppose K is a q -periodic virtual knot diagram. Then its image $P_f(K)$ under Manturov projection is also q -periodic.*

Proof. At the level of the virtual knot diagram, Manturov projection is the process of replacing all of the odd (real) crossings with virtual crossings. This is because when you remove an odd chord in the Gauss diagram, the chord relates to a crossing in the knot diagram. In order to “get rid of it”, you replace the real crossing with a virtual one. By the previous lemma, if K is q -periodic and has an odd crossing, then so is every other crossing in its \mathbb{Z}/q -orbit. This fact ensures that if K is q -periodic, then so is $P_f(K)$. \square

Theorem 3.8. *If K is a q -periodic virtual knot diagram which represents an almost classical knot, then $\bar{K} = P_f^\infty(K)$ is a q -periodic almost classical diagram representing the same virtual knot.*

Proof. Since K represents an almost classical knot, repeated application of Lemma 3.4 implies that $P_f^\infty(K)$ is virtually isotopic to K . On the other hand, since K is q -periodic, repeated application of Lemma 3.7 ensures that $P_f^\infty(K)$ is also q -periodic. \square

The next proposition is a slightly more general result along the same lines.

Proposition 3.9. *Suppose K is a q -periodic virtual knot diagram. Then its image $P_f^\infty(K)$ under stable Manturov projection is an almost classical q -periodic knot diagram.*

Proof. This follows by repeated application of Lemma 3.7, together with the fact that $P_f^\infty(K')$ is an almost classical diagram for any virtual knot diagram K' . \square

Note that the converse of this proposition is not true. For example, consider the Gauss diagram in Figure 3.4. This diagram does not have period 3. There are only two odd chords (they are the two negative chords), and removing them gives a diagram for the trefoil, which is 3-periodic.

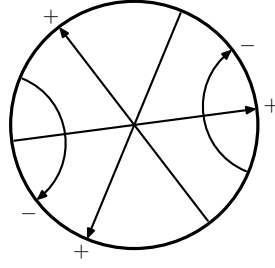


Figure 3.4: A Gauss diagram with two odd chords and projection $P_f(D)$ the trefoil.

Even though Manturov projection P_f is defined at the level of virtual knot diagrams, Proposition 2.10 ensures that it is a well-defined operation on the level of virtual knots. The next corollary will allow us to eliminate periods for a general virtual knot K by applying the Murasugi conditions to the almost classical knot $\bar{K} = P_f^\infty(K)$ obtained by stable projection.

Corollary 3.10. *Let K be a virtual knot, and $\bar{K} = P_f^\infty(K)$ be the associated almost classical knot obtained by stable Manturov projection. If \bar{K} does not admit a q -periodic diagram, then neither does K .*

3.3 Periodic virtual braids

In this section, we will show that every periodic virtual knot diagram K can be realized as the closure of a periodic braid; in other words, $K = \widehat{\beta}^q$ for some virtual braid β . It can be viewed as an equivariant version of Alexander's theorem, and it is proved via an equivariant braiding process. Whenever we have $K = \widehat{\beta}^q$, it is clear that the linking number equals the braid index and that the quotient knot is given by $K_* = \widehat{\beta}$.

Theorem 3.11. (i) *A virtual knot diagram K is q -periodic if and only if there exists a q -periodic Gauss code representing it.*

(ii) *A virtual knot diagram K is q -periodic if and only if it can be realized as the closure of the q -periodic braid, that is, $K = \widehat{\beta}^q$ for some $\beta \in VB_k$.*

Proof. For both (i) and (ii), one direction is clear. For instance, if K is a q -periodic virtual knot diagram, then its corresponding Gauss code is obviously q -periodic. Likewise, if $K = \widehat{\beta}^q$ is the closure of a periodic braid, then obviously K is itself a q -periodic virtual knot diagram.

To show the other directions, we will construct a periodic virtual knot diagram K from a q -periodic Gauss code C . In the construction, we will further arrange that the arcs wind monotonically around the origin, thus it will follow that the virtual knot diagram we construct is in fact the closure of a periodic virtual braid.

Assume then that C is a q -periodic Gauss code, so its Gauss diagram will then have qn chords, which we list

$$c_{1,0}, \dots, c_{n,0}, c_{1,1}, \dots, c_{n,1}, \dots, c_{1,q-1}, \dots, c_{n,q-1}$$

in the order in which their overcrossings are encountered in C . Because C is q -periodic, we can assemble them

$$\begin{array}{ccc} c_{1,0} & \cdots & c_{n,0} \\ \vdots & & \vdots \\ c_{1,q-1} & \cdots & c_{n,q-1} \end{array}$$

so that the periodic action is the vertical shift sending $c_{i,j}$ to $c_{i,j+1}$ for $j = 0, \dots, q-1$, with $j+1$ taken mod q (so if $j = q-1$, then $j+1$ equals 0).

To draw the periodic virtual knot diagram, we draw the crossings $c_{i,j}$ in the plane according to the sign ε_i (which recall by periodicity is independent of j) and such that $c_{i,j}$ goes to $c_{i,j+1}$ under a $2\pi/q$ rotation of the plane.

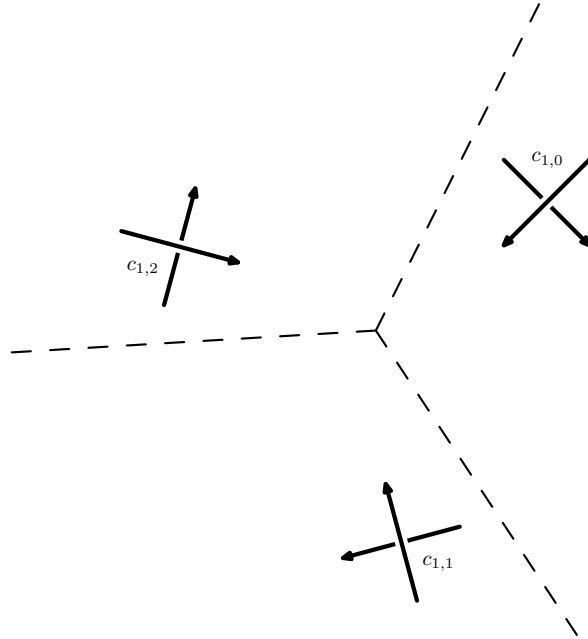
To achieve that, draw the crossings $c_{1,0}, c_{1,1}, \dots, c_{1,q-1}$ equally spaced around a circle, making sure the crossings are all right-handed if $\varepsilon_1 = 1$ and left-handed if $\varepsilon_1 = -1$. Drawing them equally spaced will ensure that each $c_{1,j}$ is sent to $c_{1,j+1}$ under the rotation of $2\pi/q$, see Figure 3.3.

We then do the same for $c_{2,0}, \dots, c_{2,q-1}$, making sure they are equally spaced, then for $c_{3,0}, \dots, c_{3,q-1}$, and so on. To ensure that the virtual knot we construct is the closure of a virtual braid, we draw each crossing so that its arcs are oriented clockwise with respect to the origin. This is easy to arrange, for instance by drawing all of $c_{1,0}, \dots, c_{n,q}$ oriented downwards to the right of the origin, and rotating by an angle of $2\pi j/q$ before drawing the other crossings $c_{i,j}$, see Figure 3.3.

The result is that we have drawn all qn crossings

$$c_{1,0}, \dots, c_{n,0}, c_{1,1}, \dots, c_{n,1}, \dots, c_{1,q-1}, \dots, c_{n,q-1}$$

symmetrically, and we complete the diagram using an equivariant braiding process. For instance, reading the first segment of the Gauss code tells us to connect



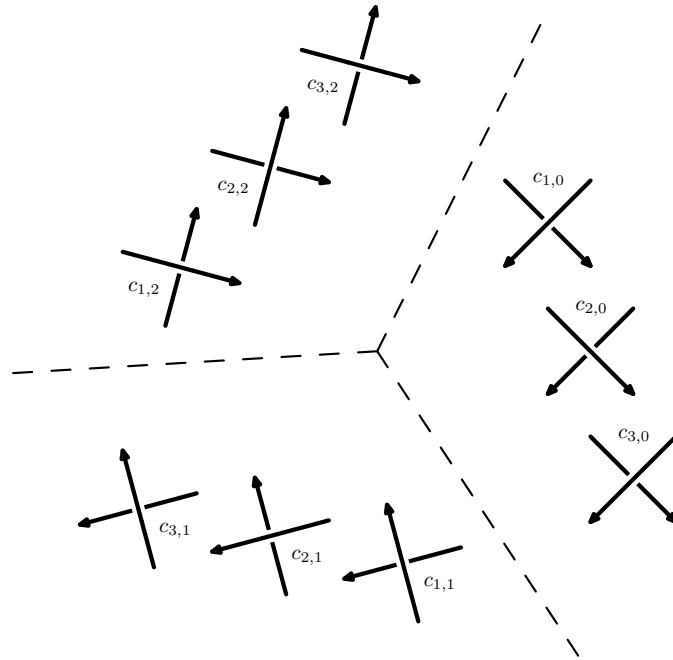
either the over or under-crossing arc from the first crossing $c_{1,0}$ to one of the arcs of another crossing, say $c_{i,j}$. By periodicity, the same arc of $c_{1,\ell}$ will be connected to the corresponding arc of $c_{i,j+\ell}$, with $j + \ell$ taken \pmod{q} . Thus, in total there will be q connecting arcs, and we draw them equivariantly with respect to the \mathbb{Z}/q action. This guarantees the resulting virtual knot diagram will be q -periodic.

To ensure we end up with the closure of a virtual braid, we draw the connecting arcs so they wind monotonically around the origin. This process will typically produce a large number of additional crossings, all of which are taken to be virtual crossings of the resulting periodic virtual knot diagram. See Example 3.13 to see this process carried out for the pretzel knot 9_{35} . \square

The next result is a consequence of Theorems 3.8 and 3.11.

Corollary 3.12. *If the virtual knot K is q -periodic and almost classical, then it can be represented as $K = \widehat{\beta^q}$, the closure of a q -periodic braid $\beta \in VB_k$ that admits an Alexander numbering.*

Example 3.13. Consider the classical pretzel knot 9_{35} , which admits a 3-periodic classical diagram. It is a consequence of the theorem of Edmonds [Edm84] that 9_{35} does not admit a classical q -periodic diagram for any $q > 3$.



Since 9_{35} is a genus one knot, this follows from the general bound $q \leq 2g(K) + 1$ on the possible periods of a classical knot diagram K , where $g(K)$ denotes the Seifert genus of K .

On the other hand, the knot $K = 9_{35}$ has Alexander polynomial $\Delta_K(t) = 7t^2 - 13t + 7$, which satisfies Murasugi's conditions (Theorem 1.5) for $q = 3$ and $k = 2$, see Figure 3.2. However, it is impossible to realize K as the closure of a 3-periodic classical braid β^3 , since β would necessarily be a braid on two strands, and for any braid in $\beta \in B_2$, the closure $\widehat{\beta^3}$ is necessarily a $(2, n)$ torus knot or link. (Any braid on two strands would only have $\sigma_i^{\pm 1}$ terms, so it would reduce to either σ_1^n , which is the $(2, n)$ torus knot; or $\sigma_1 s^{-n}$, which is the $(2, -n)$ torus knot).

However, Theorem 3.11 tells us that K can be realized as the closure of a 3-periodic virtual braid. On the left of Figure 3.5 is a 3-periodic tangle diagram for K as a classical knot that attempts to wind monotonically around the origin. It is not a braid because monotonicity fails along the six dashed arcs in that figure.

On the right of Figure 3.5 is the result of replacing these six arcs with arcs that wind monotonically around the origin. This creates many new crossings, and all of them are virtual. As in the proof of Theorem 3.11, these arcs are added so as to preserve the periodicity of the diagram, and the result is a 3-

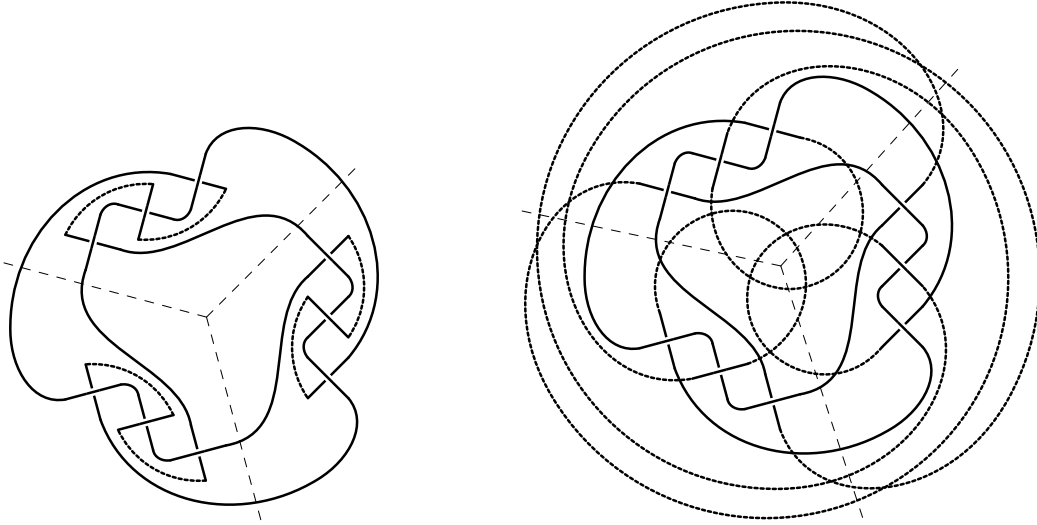


Figure 3.5: Converting a periodic tangle into a periodic virtual braid for 9_{35} .

periodic virtual braid diagram for 9_{35} . One can check that the resulting braid $\beta \in VB_8$ is given by the braid word

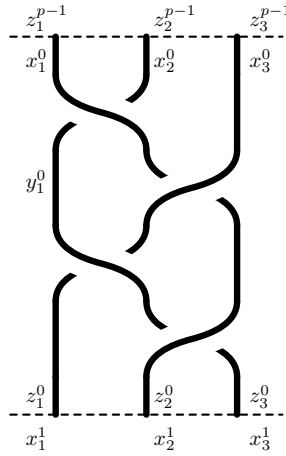
$$\beta = \tau_2 \tau_5 \sigma_4 \tau_1 \tau_3 \sigma_4 \tau_5 \tau_7 \sigma_4 \tau_3 \tau_6.$$

In Theorem 3.3, we applied the construction of Definition 2.6 to determine the Jacobian of any periodic virtual knot K . We now show how to apply Definition 2.15 to give a formula for the Jacobian matrix for a periodic virtual knot K that has been realized as the closure of a periodic virtual braid. As before, the matrix we obtain will be a circulant block matrix, and so completely determined by its first block row. The advantage of using Definition 2.15 here is that, as we shall see, the block matrices A_i vanish for $i \geq 2$.

We label the arcs of K using labels x_i^j, y_i^j, z_i^j as before, with the $j = 0, \dots, q-1$ indicating the period. In particular, in the j -th period, the strands at the top of the braid are labelled x_1^j, \dots, x_k^j , and the strands at the bottom are labelled z_1^j, \dots, z_k^j . The internal arcs are labelled y_1^j, \dots, y_r^j . We assume that there are n crossings in each period, and so we obtain the relations R_1^j, \dots, R_n^j for the internal crossings in the j -th period, and the relations S_1^j, \dots, S_k^j corresponding to setting $z_i^j = x_i^{j+1}$. Notice that $n = k + r$ (since we have an $(n+k) \times (n+k)$ matrix).

The Jacobian B is determined by the first block row, which is obtained by differentiating the relations R_1^0, \dots, R_n^0 and S_1^0, \dots, S_k^0 from the 0-th period. It

follows that this determines the rest of the matrix since the relations R_i^j and S_i^j are obtained from R_i^0 and S_i^0 by adding j to the superscripts of all the occurrences of x_i^0, y_i^0, z_i^0 , and x_i^1 ($j + 1$ is taken mod q here). Recall that the relations R_1^0, \dots, R_n^0 are written in terms of x_i^0, y_i^0, z_i^0 , (for $i \in \{1, \dots, n\}$), and the relation S_i^0 is written in terms of z_i^0 and x_i^1 .



Consider for example the q -periodic braid with first period given as below (here we assume q is relatively prime to 3). It has relations:

$$\begin{aligned}
 R_1^0 &= x_1^0 x_2^0 (x_1^0)^{-1} (y_1^0)^{-1}, \\
 R_2^0 &= (x_3^0)^{-1} x_1^0 x_3^0 (z_2^0)^{-1}, \\
 R_3^0 &= y_1^0 x_3^0 (y_1^0)^{-1} (z_1^0)^{-1}, \\
 R_4^0 &= (z_2^0)^{-1} y_1^0 z_2^0 (z_3^0)^{-1}, \\
 S_1^0 &= z_1^0 (x_1^1)^{-1}, \\
 S_2^0 &= z_2^0 (x_2^1)^{-1}, \\
 S_3^0 &= z_3^0 (x_3^1)^{-1}.
 \end{aligned}$$

The first block row of the Jacobian B therefore has the form:

$$\begin{array}{l}
 R_1^0 \\
 R_2^0 \\
 R_3^0 \\
 R_4^0 \\
 S_1^0 \\
 S_2^0 \\
 S_3^0
 \end{array}
 \left[
 \begin{array}{cccccccc|cccccccc}
 x_1^0 & x_2^0 & x_3^0 & y_1^0 & z_1^0 & z_2^0 & z_3^0 & x_1^1 & x_2^1 & x_3^1 & y_1^1 & z_1^1 & z_2^1 & z_3^1 & x_1^2 & \dots \\
 1-t & t & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 t^{-1} & 0 & 1-t^{-1} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 0 & 0 & t & 1-t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 0 & 0 & 0 & t^{-1} & 0 & 1-t^{-1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 \hline
 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots
 \end{array}
 \right]$$

In general, we will have $A_2 = A_3 = \dots = A_{q-1} = [0]$, and

$$A_0 = \left[\begin{array}{c|c} * & * \\ \hline 0_{k \times n} & I_k \end{array} \right] \quad \text{and} \quad A_1 = \left[\begin{array}{c|c} 0_{n \times k} & 0_{n \times n} \\ \hline -I_k & 0_{k \times n} \end{array} \right]. \quad (3.4)$$

The Jacobian matrix of the quotient knot K_* is the matrix

$$\begin{aligned}
 A = A_0 + A_1 &= \left[\begin{array}{c|c} * & * \\ \hline 0_{k \times n} & I_k \end{array} \right] + \left[\begin{array}{c|c} 0_{k \times k} & 0_{k \times n} \\ \hline -I_k & 0_{k \times n} \end{array} \right] \\
 &= \left[\begin{array}{c|c|c} x_* & y_* & z_* \\ \hline -I_k & 0_{k \times r} & I_k \end{array} \right],
 \end{aligned}$$

where x_* , y_* and z_* represent the matrix entries in the columns corresponding to the differentiating the relations R_1, \dots, R_n with respect to the x_i, y_i and z_i generators, respectively.

Chapter 4

Circulant Matrices

In this chapter we introduce circulant matrices, which arise naturally in computing the Alexander invariants of periodic knots. Circulant matrices have many other applications within mathematics, physics, probability, and statistics, and as a consequence there is a rather extensive literature on them, mostly in characteristic 0 (for instance, see the book [Dav79]).

We are mainly interested in circulant block matrices over a field of prime characteristic $p > 0$, and the two key results we prove are Theorems 4.3 and 4.4.

Let R be a commutative ring with unit and let M be a left R -module. Since R is assumed to be commutative, M is also a right R -module with the right action given by $mr = rm$ for $r \in R$ and $m \in M$.

We will be primarily interested in the case $R = \mathbb{F}_p$, the field of integers modulo a prime p , and M is the vector space of square matrices of a fixed size, say $m \times m$, over an \mathbb{F}_p -algebra of scalars. In that case, circulant matrices over M (defined below) are also called *block circulant matrices*, as the elements of M are themselves matrices. Forgetting the block structure yields an $mn \times mn$ matrix.

If B is an $n \times n$ matrix over M and Y is an $n \times n$ matrix over R then YB is the $n \times n$ matrix over M given by $(YB)_{i,j} = \sum_k Y_{i,k} B_{k,j}$. Note here that the entries in B are actually $m \times m$ matrices, since they are in M . If Y is an elementary matrix then YB is the matrix obtained by performing the corresponding row operation on M . Similarly, BY is the $n \times n$ matrix over M given by $(BY)_{i,j} = \sum_k B_{i,k} Y_{k,j}$ and if Y is an elementary matrix then BY is the matrix obtained by performing the corresponding column operation on M . If $A \in M$ and Y is an $n \times n$ matrix over R then the matrix AY is the $n \times n$ matrix over M with entries $(AY)_{i,j} = AY_{i,j}$. Note that A is a matrix and

this is not matrix multiplication (A is $m \times m$ and Y is $n \times n$, so we couldn't multiply them in the standard way if we wanted to). Since $A \in M$ and Y is a matrix over R (in particular, the entries $Y_{i,j}$ are elements of R), we are allowed to multiply $AY_{i,j}$, since M is a left R -module. We are essentially treating A as a scalar here, although it is in fact a matrix. The matrix YA is the same as AY , since we have that $mr = rm$ for $r \in R$ and $m \in M$.

Given elements $A_0, \dots, A_{n-1} \in M$ the corresponding *circulant matrix* is the $n \times n$ matrix over M given by

$$C(A_0, \dots, A_{n-1}) = \begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & A_1 & \cdots & A_{n-2} \\ A_{n-2} & A_{n-1} & A_0 & \cdots & A_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & A_3 & \cdots & A_0 \end{bmatrix}.$$

Let P be the $n \times n$ matrix over R given by

$$P = C(0, 1, 0, \dots, 0) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix},$$

where $1 \in R$ is the unit element for R .

Observe that for $0 \leq j \leq n-1$, we have $P^j = C(0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears in the $(j+1)$ -st slot. Hence the circulant matrix $C(A_0, \dots, A_{n-1})$ over M can be written as

$$C(A_0, \dots, A_{n-1}) = \sum_{i=0}^{n-1} A_i P^i. \quad (4.1)$$

(Note that $P^0 = I$, the $n \times n$ identity matrix.)

Conventions for binomial coefficients. Let a, b be non-negative integers. We define

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!} & \text{if } a \geq b, \\ 0 & \text{if } a < b. \end{cases}$$

It is also convenient to define

$$\binom{-1}{b} = (-1)^b.$$

The following identities involving binomial coefficients modulo a prime p will be useful.

Lemma 4.1. *Let p be a prime and r a positive integer.*

1. *If $0 \leq \ell \leq p^r - 1$, then*

$$\binom{p^r - 1}{\ell} = (-1)^\ell \pmod{p}.$$

2. *If $1 \leq \ell \leq p^r - 1$, then*

$$\binom{p^r + \ell - 1}{p^r - 1} = 0 \pmod{p}.$$

Proof. The first formula is easily derived from the identity $\binom{p^r - 1}{\ell} + \binom{p^r - 1}{\ell - 1} = \binom{p^r}{\ell} = 0 \pmod{p}$ (which gives $\binom{p^r - 1}{\ell} = -\binom{p^r - 1}{\ell - 1} \pmod{p}$, valid for $0 < \ell \leq p^r - 1$), together with induction on ℓ . $\binom{p^r}{\ell} = 0 \pmod{p}$ comes from the fact that there will be at least one extra factor of p in the numerator that does not cancel with anything in the denominator. For example $\binom{p^r}{p^r - 1} = \frac{p^r!}{(p^r - 1)!1!} = p = 0 \pmod{p}$. For $\ell = 0$, we have that $\binom{p^r}{0} = \frac{p^r!}{0!(p^r)!} = 1 \pmod{p}$. Using this identity, we then have that $\binom{p^r - 1}{1} = -\binom{p^r - 1}{0} = -1 \pmod{p}$. Continuing on, we can easily apply induction to get that $\binom{p^r - 1}{\ell} = (-1)^\ell \pmod{p}$, for $0 \leq \ell \leq p^r - 1$. In order to prove the second formula we apply Lucas's Theorem [Fin47, Theorem 1] which asserts that if M and N are non-negative integers written in base p as $M = \sum_{i=0}^k M_i p^i$ and $N = \sum_{i=0}^k N_i p^i$, with $0 \leq M_i, N_i < p$, then

$$\binom{M}{N} = \prod_{i=0}^k \binom{M_i}{N_i} \pmod{p}.$$

We want to show that $\binom{p^r + \ell - 1}{p^r - 1} = 0 \pmod{p}$. If we can show that one of the $\binom{M_i}{N_i}$ terms is 0 modulo p , then we are done. Our M here is $p^r + \ell - 1 \geq p^r$ (since $1 \leq \ell \leq p^r - 1$), so that $p^r + \ell - 1 = \sum_{i=0}^r M_i p^i$ has $M_r = 1$. Since $1 \leq \ell \leq p^r - 1$, we also have that $p^r + \ell - 1 \leq 2p^r - 2$, and we cannot have $M_i = p - 1$ for all $i < r$, since that would give us $\sum_{i=0}^r M_i p^i = p^r + \sum_{i=0}^{r-1} (p - 1)p^i = p^r + p^r - 1 = 2p^r - 1 > 2p^r - 2$. Therefore, at least one of the numbers M_i for $0 \leq i \leq r - 1$ is strictly less than $p - 1$. Our N here is $p^r - 1$, and we have $p^r - 1 = \sum_{i=0}^{r-1} (p - 1)p^i$, since expanding $\sum_{i=0}^{r-1} (p - 1)p^i = \sum_{i=0}^{r-1} -p^i + p^{i+1}$ gives $-p^0 + p^1 - p^1 + \dots + p^r = p^r - 1$. $p^r - 1 = \sum_{i=0}^{r-1} (p - 1)p^i$ means that all of our N_i terms are $p - 1$. Now, we had

that at least one of the numbers M_i was strictly less than $p - 1$, so for that term, $\binom{M_i}{p-1} = 0 \pmod{p}$, and so by Lucas's Theorem, $\binom{p^r+\ell-1}{p^r-1} = 0 \pmod{p}$. \square

Let \mathbb{F}_p denote the field of integers modulo a prime p . We will show that the matrix $P = C(0, 1, 0, \dots, 0)$ over \mathbb{F}_p of size $p^r \times p^r$ is conjugate, via an explicitly given matrix over \mathbb{F}_p , to an elementary Jordan matrix. Before proving that, we establish a useful lemma.

Lemma 4.2. *Let X be the matrix of size $p^r \times p^r$ over \mathbb{F}_p with*

$$X_{i,j} = \binom{i-2}{j-1}, \quad 1 \leq i, j \leq p^r.$$

Then X is invertible and

$$(X^{-1})_{i,j} = \binom{p^r-j+1}{p^r-i}, \quad 1 \leq i, j \leq p^r.$$

Proof. Let Y be the matrix over \mathbb{F}_p given by $Y_{i,j} = \binom{p^r-j+1}{p^r-i}$, $1 \leq i, j \leq p^r$. We will show that $XY = I$, from which the lemma will follow.

$$(XY)_{i,j} = \sum_{k=1}^{p^r} X_{i,k} Y_{k,j} = \sum_{k=1}^{p^r} \binom{i-2}{k-1} \binom{p^r-j+1}{p^r-k} = \sum_{\ell=0}^{p^r-1} \binom{i-2}{\ell} \binom{p^r-j+1}{p^r-1-\ell}.$$

The well-known *Vandermonde Convolution formula* asserts that for non-negative integers m, n, q ,

$$\sum_{\ell=0}^q \binom{m}{\ell} \binom{n}{q-\ell} = \binom{m+n}{q}.$$

This formula is also valid for $m = -1$ with our convention $\binom{-1}{\ell} = (-1)^\ell$. Applying Vandermonde Convolution to the above expression for $(XY)_{i,j}$ (with $m = i - 2$, $q = p^r - 1$, and $n = p^r - j + 1$) yields

$$(XY)_{i,j} = \binom{p^r+i-j-1}{p^r-1}.$$

We need that $XY = I$, so we need to show that $(XY)_{i,i} = 1$ and $(XY)_{i,j} = 0$ for $i \neq j$. If $i < j$ then $p^r + i - j - 1 < p^r - 1$ and so $(XY)_{i,j} = \binom{p^r+i-j-1}{p^r-1} = 0$ in this case. We have $(XY)_{i,i} = \binom{p^r-1}{p^r-1} = 1$. If $i > j$ then Lemma 4.1(2) applies

(as $1 \leq i - j \leq p^r$), and we have $(XY)_{i,j} = \binom{p^r+i-j-1}{p^r-1} = 0 \pmod p$ in this case. Hence $XY = I$. \square

Theorem 4.3. *Let $P = C(0, 1, 0, \dots, 0)$ be the circulant matrix of size $p^r \times p^r$ over \mathbb{F}_p and let X be the matrix of size $p^r \times p^r$ over \mathbb{F}_p with*

$$X_{i,j} = \binom{i-2}{j-1}, \quad 1 \leq i, j \leq p^r$$

and

$$(X^{-1})_{i,j} = \binom{p^r-j+1}{p^r-i}, \quad 1 \leq i, j \leq p^r.$$

Then

$$X^{-1}PX = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = J.$$

Proof. By Lemma 4.2, we know that X and X^{-1} are indeed inverses, and the theorem follows once we verify that $PX = XJ$.

Since $P_{k,k+1} = 1$ for $k = 1, \dots, p^r - 1$ and $P_{p^r,1} = 1$ and $P_{i,j} = 0$ otherwise,

$$(PX)_{i,j} = \sum_{k=1}^{p^r} P_{i,k} X_{k,j} = \begin{cases} X_{i+1,j} = \binom{i-1}{j-1} & \text{if } i \neq p^r, \\ X_{1,j} = \binom{-1}{j-1} = (-1)^{j-1} & \text{if } i = p^r. \end{cases}$$

Note that this is because if $i = p^r$, $P_{p^r,1} = 1$ and $P_{i,j} = 0$ otherwise, so our sum becomes $1 \cdot X_{1,j} = X_{1,j}$. If $i \neq p^r$, then $P_{i,i+1} = 1$ and $P_{i,j} = 0$ otherwise, so our sum becomes $X_{i+1,j}$. Since $J_{k,k+1} = 1$ for $k = 1, \dots, p^r - 1$ and $J_{k,k} = 1$ for $k = 1, \dots, p^r$ and $J_{i,j} = 0$ otherwise,

$$(XJ)_{i,j} = \sum_{k=1}^{p^r} X_{i,k} J_{k,j} = \begin{cases} X_{i,j-1} + X_{i,j} = \binom{i-2}{j-2} + \binom{i-2}{j-1} = \binom{i-1}{j-1} & \text{if } j \neq 1, \\ X_{i,1} = \binom{i-2}{0} = 1 & \text{if } j = 1. \end{cases}$$

If $j = 1$, $J_{k,1} = 1$ for $k = 1$ and $J_{k,1} = 0$ otherwise, so we have the sum equalling $X_{i,1} J_{1,1} = X_{i,1}$. If $j \neq 1$, $J_{k,j} = 1$ for $k \in \{j-1, j\}$ and $J_{k,j} = 0$ otherwise, so we have the sum equalling $X_{i,j} + X_{i,j-1}$.

It follows immediately that $(PX)_{i,j} = (XJ)_{i,j}$ for $i \neq p^r$, $j \neq 1$. For the

$j = 1$ case, $(PX)_{i,1} = \binom{i-1}{0} = 1$. For the $i = p^r$ case, we have

$$(XJ)_{p^r,j} = \binom{p^r - 1}{j - 1} = (-1)^{j-1} \pmod{p}.$$

with the last equality coming from Lemma 4.1(1). Since $(PX)_{p^r,j} = (-1)^{j-1}$, we obtain $(PX)_{p^r,j} = (XJ)_{p^r,j} \pmod{p}$, completing the proof that $PX = XJ$. \square

We remark that numbers of the form $\binom{p^r+m-1}{p^r-1}$, $1 \leq m \leq p^r - 1$, while divisible by p , need not be divisible by higher powers of p . For example, if $p = 3$, $r = 2$ and $m = 6$ then

$$\binom{p^r + m - 1}{p^r - 1} = \binom{14}{8} = 3 \cdot 7 \cdot 11 \cdot 13$$

which is divisible by 3 but not by 3^2 . This observation obstructs a version of Theorem 4.3 where \mathbb{F}_p would conceivably be replaced by the ring of integers modulo p^r .

Theorem 4.4. *Let M be vector space over \mathbb{F}_p and let $B = C(A_0, \dots, A_{p^r-1})$ be a $p^r \times p^r$ circulant matrix over M . Then $X^{-1}BX = \sum_{i=0}^{p^r-1} A_i J^i$, where X is the matrix of size $p^r \times p^r$ over \mathbb{F}_p with*

$$X_{i,j} = \binom{i-2}{j-1}, \quad 1 \leq i, j \leq p^r,$$

$$(X^{-1})_{i,j} = \binom{p^r - j + 1}{p^r - i}, \quad 1 \leq i, j \leq p^r,$$

and

$$J = X^{-1}PX = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

In particular, $X^{-1}BX$ is upper triangular as a matrix over M .

Proof. By (4.1), $B = \sum_{i=0}^{p^r-1} A_i P^i$ where $P = C(0, 1, 0, \dots, 0)$. Hence

$$\begin{aligned} X^{-1}BX &= X^{-1} \left(\sum_{i=0}^{p^r-1} A_i P^i \right) X \\ &= \sum_{i=0}^{p^r-1} X^{-1} A_i P^i X = \sum_{i=0}^{p^r-1} A_i (X^{-1} P X)^i \\ &= \sum_{i=0}^{p^r-1} A_i J^i \quad \text{by Theorem 4.3.} \end{aligned}$$

Since each $A_i J^i$ is upper triangular, so is $\sum_{i=0}^{p^r-1} A_i J^i$. □

Corollary 4.5. *Let B and X be as Theorem 4.4. Then*

$$(X^{-1}BX)_{j,j+\ell} = \sum_{i=\ell}^{p^r-1} \binom{i}{\ell} A_i.$$

Proof. Let Q be the $p^r \times p^r$ matrix over \mathbb{F}_p given by

$$Q = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & & 0 \end{bmatrix}.$$

Then $J = I + Q$ and so $J^i = (I + Q)^i = \sum_{\ell=0}^i \binom{i}{\ell} Q^\ell$. By Theorem 4.4,

$$X^{-1}BX = \sum_{i=0}^{p^r-1} A_i J^i = \sum_{i=0}^{p^r-1} A_i \left(\sum_{\ell=0}^i \binom{i}{\ell} Q^\ell \right) = \sum_{\ell=0}^{p^r-1} \left(\sum_{i=\ell}^{p^r-1} \binom{i}{\ell} A_i \right) Q^\ell$$

Then $(X^{-1}BX)_{j,j+\ell} = \sum_{\ell=0}^{p^r-1} \left(\sum_{i=\ell}^{p^r-1} \binom{i}{\ell} A_i \right) Q^\ell$. Note that $Q_{j,j+1} = 1$ and in general, $Q^\ell_{j,j+\ell} = 1$ and $Q^k_{j,j+\ell} = 0$, for $k \neq \ell$. So we then get $(X^{-1}BX)_{j,j+\ell} = \sum_{i=\ell}^{p^r-1} \binom{i}{\ell} A_i$. □

The next result is obtained by evaluating the formula in Corollary 4.5 for $\ell = 0, 1$.

Corollary 4.6. *For B and X as Theorem 4.4, we have*

$$X^{-1}BX = \begin{bmatrix} A & D & \cdots & * \\ & \ddots & \ddots & \vdots \\ \vdots & & \ddots & D \\ 0 & \cdots & & A \end{bmatrix},$$

where $A = \sum_{k=0}^{p^r-1} A_k$ and $D = \sum_{k=1}^{p^r-1} kA_k$.

Chapter 5

Murasugi's Theorem for Almost Classical Knots

In this section, we prove Theorem 1.7 from the Introduction, which is the analogue of Murasugi's Theorem 1.5 for periodic almost classical knots. We begin by restating the result. First, recall that for a q -periodic almost classical knot K , Corollary 3.12 allows us to write $K = \widehat{\beta}^q$ for some k -strand virtual braid β that admits an Alexander numbering.

Theorem 5.1. *Let $K = \widehat{\beta}^q$ be a q -periodic almost classical knot diagram, where β a k -strand virtual braid that admits an Alexander numbering, and $q = p^r$ a prime power. Then $K_* = \widehat{\beta}$, and*

1. $\Delta_{K_*}(t)$ divides $\Delta_K(t)$ in $\mathbb{Z}[t^{\pm 1}]$, and
2. $\Delta_K(t) \doteq (\Delta_{K_*}(t))^q (f(t))^{q-1} \pmod{p}$, where $f(t) = \sum_{i=1}^k t^{\lambda_i}$ and λ_i is the Alexander number on the i -th strand of β .

Proof. To prove the first part, notice that in Section 3.1, we showed that there is a surjection $G_K \rightarrow G_{K_*}$ sending meridians of K to meridians of K_* . Now apply Exercise 9 on page 108 of [CF77] to conclude that $\Delta_{K_*}(t)$ divides $\Delta_K(t)$.

We divide the proof of part 2 into several claims.

Claim 5.2. $\Delta_K(t) \doteq (\Delta_{K_*}(t))^q (f(t))^{q-1} \pmod{p}$ for some $f(t) \in \mathbb{Z}/p[t^{\pm 1}]$.

The proof of this claim requires extensive matrix manipulation and spans the next three pages.

Let B be the block circulant Jacobian matrix for K constructed in Definition 2.6 and written out as in Equation (3.3). We will assume it has $n \times n$

blocks, or equivalently that there are n crossings in each period of K . Then Theorem 3.3 and Corollary 4.6 show that

$$B \cong X^{-1}BX = \begin{bmatrix} A & D & \cdots & * \\ & \ddots & \ddots & \vdots \\ \vdots & & \ddots & D \\ 0 & \cdots & & A \end{bmatrix} \pmod{p},$$

where $A = \sum_{k=0}^{q-1} A_k$ is the Jacobian of K_* and $D = \sum_{k=1}^{q-1} kA_k$.

Set

$$C = \begin{bmatrix} A & D & \cdots & * \\ & \ddots & \ddots & \vdots \\ \vdots & & \ddots & D \\ 0 & \cdots & & A \end{bmatrix},$$

which is an $nq \times nq$ matrix written in block form with $n \times n$ blocks.

Let \bar{A} and \bar{C} denote the matrices A and C with their last row and first column removed. So \bar{A} is an $(n-1) \times (n-1)$ matrix and \bar{C} is an $(nq-1) \times (nq-1)$ matrix. Then modulo p , the Alexander polynomials of K and K_* are given by

$$\Delta_K(t) \doteq \det(\bar{C}) \pmod{p} \quad \text{and} \quad \Delta_{K_*}(t) \doteq \det(\bar{A}) \pmod{p}.$$

Let A' be the $(n-1) \times n$ matrix obtained by removing the last row from A , and let A'' be the $n \times (n-1)$ matrix obtained by removing the first column of A . Also let 0_n be the $n \times n$ matrix of zeroes, $0'_n$ the $(n-1) \times n$ matrix of zeroes, and $0''_n$ the $n \times (n-1)$ matrix of zeroes. Note that $\bar{0}_n = 0_{n-1}$.

Using these to rewrite \bar{C} , we get

$$\bar{C} = \begin{bmatrix} A'' & D & * & \cdots & \cdots & * \\ 0''_n & A & D & \ddots & & \vdots \\ \vdots & 0_n & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & * \\ \vdots & & & 0_n & A & D \\ \bar{0}_n & \cdots & \cdots & 0'_n & 0'_n & A' \end{bmatrix}.$$

Notice that \bar{A} is a submatrix of each of the q block terms A , A' , and A'' appearing on the block diagonal of \bar{C} , and our goal is to extract those terms using row and column operations on \bar{C} to reduce it to an upper block triangular matrix with \bar{A} blocks on the diagonal.

For that, we require some additional notation. Let $r_A = [A_{n,2}, \dots, A_{n,n}]$ be the last row of A minus the first entry, let $c_A = [A_{1,1}, \dots, A_{n-1,1}]^t$ be the first column of A minus the last entry, and let $u_A = A_{1,n}$ be the bottom left corner entry for A . We can now rewrite A, A', A'' in terms of c_A, r_A , and u_A as

$$A = \left[\begin{array}{c|c} c_A & \bar{A} \\ \hline u_A & r_A \end{array} \right], \quad A' = \left[\begin{array}{c|c} c_A & \bar{A} \end{array} \right], \quad \text{and} \quad A'' = \left[\begin{array}{c} \bar{A} \\ \hline r_A \end{array} \right].$$

Further, let r_D, c_D , and u_D be the corresponding row, column, and bottom left corner entry of D , which we use to write

$$D = \left[\begin{array}{c|c} c_D & \bar{D} \\ \hline u_D & r_D \end{array} \right],$$

where \bar{D} is the $(n-1) \times (n-1)$ matrix D with its last row and first column removed.

Using these matrices, we can rewrite \bar{C} as:

$$\bar{C} = \left[\begin{array}{cccccccc} \bar{A} & c_D & \bar{D} & * & * & \dots & \dots & * \\ r_A & u_D & r_D & * & * & \dots & \dots & * \\ 0_{n-1} & c_A & \bar{A} & c_D & \bar{D} & \ddots & & \vdots \\ 0_{1 \times (n-1)} & u_A & r_A & u_D & r_D & \ddots & \ddots & \vdots \\ \vdots & \ddots & & & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & & c_A & \bar{A} & c_D & \bar{D} \\ \vdots & & & \ddots & u_A & r_A & u_D & r_D \\ 0_{n-1} & \dots & \dots & \dots & 0_{(n-1) \times 1} & 0_{n-1} & c_A & \bar{A} \end{array} \right].$$

Since A is the Jacobian of the quotient knot K_* , we know that the sum of its columns equals zero, in other words, $\sum_{j=1}^n A_{*,j} = 0$. (Here, $A_{*,j}$ denotes the j -th column of A .) Further, since K_* is almost classical, Proposition 2.11 shows that there is a linear dependence among the rows. More specifically, we have units $\vartheta_1, \dots, \vartheta_n \in \mathbb{Z}[t^{\pm 1}]$ such that $\sum_{i=1}^n \vartheta_i A_{i,*} = 0$. (Here, $A_{i,*}$ denotes the i -th row of A .) Thus, replacing the first column in A by the sum of all its columns, and replacing the last row by the linear combination $\sum_{i=1}^n \vartheta_i A_{i,*}$, we obtain the

matrix

$$\tilde{A} = \left[\begin{array}{c|c} 0'' & \bar{A} \\ \hline 0 & 0' \end{array} \right],$$

where $0' = 0_{1 \times (n-1)}$ is a row of zeros and $0'' = 0_{(n-1) \times 1}$ is a column of zeros.

Performing the same row and column operations to D gives the matrix

$$\tilde{D} = \left[\begin{array}{c|c} \tilde{c}_D & \bar{D} \\ \hline \tilde{u}_D & \tilde{r}_D \end{array} \right],$$

where \tilde{c}_D is the sum of the columns in D minus the last entry, $\tilde{r}_D = \sum_{i=1}^n \vartheta_i D_{i,*}$ is the linear combination of the rows in D minus the first entry, and

$$\tilde{u}_D = \sum_{i,j=1}^n \vartheta_i D_{i,j}.$$

(This is proved in Claim 5.3 below.) The result of performing these operations on each of the blocks of \tilde{C} gives the matrix

$$\tilde{C} = \left[\begin{array}{ccccccc} \boxed{A} & \tilde{c}_D & \bar{D} & & & & \\ 0' & \tilde{u}_D & \tilde{r}_D & & & & * \\ & 0'' & \boxed{A} & \tilde{c}_D & \bar{D} & & \\ & 0 & 0' & \tilde{u}_D & \tilde{r}_D & \ddots & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & \ddots & \\ & & & & & & 0'' & \boxed{A} & \tilde{c}_D & \bar{D} \\ & 0 & & & & & 0 & 0' & \tilde{u}_D & \tilde{r}_D \\ & & & & & & & & 0'' & \boxed{A} \end{array} \right],$$

where $0' = 0_{1 \times (n-1)}$ and $0'' = 0_{(n-1) \times 1}$.

Expanding along the block diagonal of \tilde{C} , we compute that:

$$\begin{aligned} \det(\tilde{C}) &= \det(\bar{A}) \left(\det \begin{bmatrix} \bar{A} & \tilde{c}_D \\ 0 & \tilde{u}_D \end{bmatrix} \right)^{q-1} \pmod{p} \\ &\doteq \det(\bar{A}) (\tilde{u}_D \det(\bar{A}))^{q-1} \pmod{p} \\ &\doteq (\det(\bar{A}))^q (\tilde{u}_D)^{q-1} \pmod{p}. \end{aligned}$$

Since $\Delta_K(t) \doteq \det(\tilde{C}) \pmod{p}$ and $\Delta_{K^*}(t) = \det(\bar{A})$, the equations above imply that

$$\Delta_K(t) \doteq (\Delta_{K^*}(t))^q (f(t))^{q-1} \pmod{p},$$

provided we take $f(t) = \tilde{u}_D$. This completes the proof of Claim 5.2.

The last step in proving Theorem 5.1 is to show that

$$f(t) = \sum_{i=1}^k t^{\lambda_i},$$

where λ_i is the Alexander number on the i -th strand at the top of β .

Before doing that, we shall prove the following

Claim 5.3. $\tilde{u}_D = \sum_{i=1}^n \sum_{j=1}^n \vartheta_i D_{i,j}$

The element \tilde{u}_D is the bottom left corner entry of \tilde{D} , the matrix obtained by performing row and column operations to D . Specifically, \tilde{D} is obtained in two steps. The first step is to replace the first column of D by the sum $\sum_{j=1}^n D_{*,j}$ of all its columns. Here, $D_{*,j}$ denotes the j -th column of D .

Let D' be the matrix obtained after the first step. The second step is to replace the last row of D' by the linear combination $\sum_{i=1}^n \vartheta_i D'_{i,*}$. Here, $D'_{i,*}$ denotes the i -th row of D' , and ϑ_i is the unit in $\mathbb{Z}[t^{\pm 1}]$ whose existence is guaranteed by Proposition 2.11. Therefore, the bottom left entry in the resulting matrix \tilde{D} is given by

$$\tilde{u}_D = \sum_{i=1}^n \vartheta_i D'_{i,1} = \sum_{i=1}^n \sum_{j=1}^n \vartheta_i D_{i,j}.$$

Note that $D'_{i,1} = \sum_{j=1}^n D_{i,j}$ since $D'_{i,j} = D_{i,j}$ for $j \neq 1$. This completes the proof of Claim 5.3.

We are now ready to complete the proof of Theorem 5.1. To that end,

suppose now that B is the block circulant Jacobian matrix obtained by applying Definition 2.15 to the braid $\beta^a \in VB_k$. It follows that

$$B = \begin{bmatrix} A_0 & A_1 & 0 & \cdots & 0 \\ 0 & A_0 & A_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_1 & 0 & \cdots & 0 & A_0 \end{bmatrix},$$

That is, A_ℓ is the zero matrix for $\ell \geq 2$. Now Corollary 4.6 implies that

$$C := X^{-1}BX = \begin{bmatrix} A & D & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ \vdots & & \ddots & D \\ 0 & \cdots & & A \end{bmatrix},$$

where $A = A_0 + A_1$ and $D = A_1$. Note that the blocks in C are $(n+k) \times (n+k)$ matrices. After performing the corresponding row and column operations to \bar{C} , Claim 5.3 implies that

$$f(t) = \tilde{u}_D = \sum_{i=1}^{n+k} \sum_{j=1}^{n+k} \omega_i A_1(i, j).$$

Here, the coefficients $\omega_1, \dots, \omega_{n+k}$ are the units in $\mathbb{Z}[t^{\pm 1}]$ whose existence is guaranteed by Lemma 2.16.

Equation (3.4) implies that

$$A_1 = \left[\begin{array}{c|c} 0_{n \times k} & 0_{n \times n} \\ \hline -I_k & 0_{k \times n} \end{array} \right],$$

so $(A_1)(n+i, i) = -1$ for $1 \leq i \leq k$ and $(A_1)(i, j) = 0$ otherwise.

Thus, we have

$$\begin{aligned} f(t) &= \sum_{i=1}^{n+k} \sum_{j=1}^{n+k} \omega_i A_1(i, j) \\ &= \sum_{i=1}^k -\omega_{n+i} \doteq \sum_{i=1}^k \omega_{n+i}. \end{aligned}$$

(The last step holds because $f(t)$ is only defined up to multiplication by $\pm t^\ell$.) From Lemma 2.16, we have that $\omega_{n+i} = t^{\lambda_i^S}$ for $1 \leq i \leq k$, where recall

that λ_i^S is the Alexander number on the i -th strand of β . (That same Alexander number is denoted simply λ_i here.) Therefore, it follows that

$$f(t) \doteq \sum_{i=1}^k \omega_{n+i} = \sum_{i=1}^k t^{\lambda_i}$$

as desired. \square

By constructing almost classical braids β with a given polynomial $f(t) = \sum_{i=1}^k t^{\lambda_i}$, one can realize q -periodic knots K as the closure of β^q for many different prime powers $q = p^r$. We present an example that illustrates this idea.

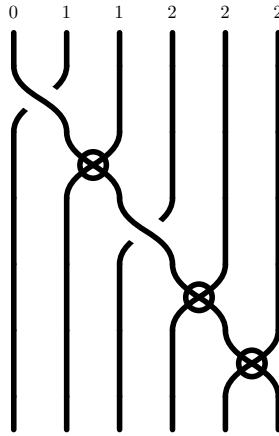


Figure 5.1: The braid $\beta = \sigma_1\tau_2\sigma_3\tau_4\tau_5$ is almost classical.

Example 5.4. Let $f(t) = 1 + 2t + 3t^2$ and set $k = f(1) = 6$ (so β will have 6 strands). We start by labeling the strands at the top of the braid in VB_6 with Alexander numbers, which are 0, 1, 1, 2, 2, 2 going from left to right. (The more general process and reasoning for our choices will be described below). We then form the braid β which drags the left-most strand across all the others with either classical or virtual crossings (the choice of classical or virtual is determined by the Alexander numbering). In this case, we get the braid $\beta = \sigma_1\tau_2\sigma_3\tau_4\tau_5$, see Figure 5.1. Notice that β is almost classical and its closure $\widehat{\beta}$ represents the trivial knot. Notice further that β^q closes up to a virtual knot (as opposed to a virtual link) for any prime power $q = p^r$ as long as $p \neq 2, 3$. For such q , the closure $K = \widehat{\beta^q}$ will be a q -periodic almost classical knot diagram with trivial

quotient. Applying Theorem 5.1, we see that

$$\Delta_K(t) \doteq f(t)^{q-1} = (1 + 2t + 3t^2)^{q-1} \pmod{p}.$$

In this way, we can realize many different Alexander polynomials as arising from q -periodic almost classical knots. The next result gives a general construction along these lines.

Proposition 5.5. *Suppose $f(t) = a_0 + a_1t + \cdots + a_nt^n \in \mathbb{Z}[t^{\pm 1}]$ is a polynomial satisfying $f(1) \not\equiv 0 \pmod{p}$. Assume also that $a_i > 0$ for $i = 0, \dots, n$. Then for $q = p^r$, there exists a q -periodic almost classical knot K with trivial quotient K_* and $\Delta_K(t) \doteq (f(t))^{q-1} \pmod{p}$.*

Proof. We will construct an almost classical braid $\beta \in VB_k$ whose closure $\widehat{\beta}$ is trivial with the property that, for the given polynomial, we have $f(t) = \sum_{i=1}^k t^{\lambda_i} \pmod{p}$, where λ_i denotes the Alexander number on the i -th strand at the top of β . The result will then follow by applying Theorem 5.1 to the closure $K = \widehat{\beta}^q$.

Let $k = f(1)$ and notice that since $f(1) \not\equiv 0 \pmod{p}$, it follows that k is relatively prime to p (otherwise, we could end up with $\widehat{\beta}^q$ being a virtual link, which we explain near the end of the proof). Let $\beta \in VB_k$ be a braid with Alexander numbers along the top strands given by

$$\underbrace{0, \dots, 0}_{a_0}, \underbrace{1, \dots, 1}_{a_1}, \dots, \underbrace{n, \dots, n}_{a_n}$$

as one goes from left to right. We form the braid β by crossing the left-most strand across all the others, using a negative real crossing whenever the Alexander numbers allow it (In other words, whenever the Alexander number on the left is one less than the Alexander number on the right) and a virtual crossing otherwise (that is, whenever the Alexander numbers on the two strands are equal). The condition that each coefficient $a_i > 0$ is positive ensures that, for every Alexander number i with $0 \leq i \leq n$, there is at least one strand labeled i . (Otherwise, $\widehat{\beta}$ would be a virtual link with two or more components.)

In any case, we can write β as the braid word

$$\beta = \theta_1 \theta_2 \cdots \theta_{k-1}, \quad \text{where } \theta_i = \begin{cases} \sigma_i & \text{if } \lambda_{i+1} = \lambda_i + 1, \\ \tau_i & \text{if } \lambda_{i+1} = \lambda_i. \end{cases}$$

It follows that β is almost classical (since it was Alexander numberable), and the permutation induced by β is the k -cycle $(k, k-1, k-2, \dots, 1)$ (since each

θ_i corresponds to the transposition $(i \ i + 1)$, which has order k . Notice, as in Figure 5.1, that the i -th strand at the top goes to the $(i - 1)$ -st at the bottom, for $1 < i \leq k$, and the first strand goes to the k -th. This is what the permutation is reflecting. Thus the closure $K_* = \widehat{\beta}$ is a knot, and in fact an easy argument using real and virtual Reidemeister I moves shows that K_* is trivial. If p is a prime which does not divide k (note that the assumptions of our theorem assumed that $k \not\equiv 0 \pmod{p}$, so p cannot divide k), then the closure $K = \widehat{\beta^q}$ for $q = p^r$ is also a virtual knot diagram (as opposed to being a virtual link). The number of components in a link are determined by the number of closed cycles when you write the permutation mentioned above as a product of closed cycles. For β^q , we have $(k, k - 1, k - 2, \dots, 1)^q$, which will give us a k -cycle as long as k and q are relatively prime, which they are. By construction, it follows that K is q -periodic and almost classical, and its quotient $K_* = \widehat{\beta}$ is trivial. Theorem 5.1 applies to show that $\Delta_K(t) \doteq (f(t))^{q-1} \pmod{p}$ as claimed. \square

Chapter 6

Applications

In this chapter we apply Theorem 5.1 to the problem of determining the possible periods of an almost classical knot.

We recall some facts about Laurent polynomials. Let R be an integral domain. A non-zero Laurent polynomial $f \in R[t^{\pm 1}]$, also written as $f(t)$, can be expressed uniquely in the form $f(t) = \sum_{j=m}^n a_j t^j$ where m, n are integers with $m \leq n$, $a_j \in R$ and where $a_m, a_n \neq 0$. The *degree* of f , denoted $\deg f$, is the non-negative integer $n - m$. The degree is not defined for the zero polynomial. Since, by assumption, R has no zero divisors we have $\deg fg = \deg f + \deg g$. We write $f \doteq g$ if there exists a unit $c \in R$ and an integer j such that $ct^j f(t) = g(t)$. If $f \doteq g$ then $\deg f = \deg g$. A *root* of a non-zero $f \in R[t^{\pm 1}]$ is a non-zero element α in an extension of the field of fractions of R such that $f(\alpha) = 0$. If $f \doteq g$ then f and g have the same roots (with the same multiplicities). Also, $\deg f$ is the number of roots of $f(t)$, counted with multiplicities, in an algebraic closure of the field of fractions of R . We say that a non-zero $f \in R[t^{\pm 1}]$ is *trivial* if $f \doteq 1$, equivalently, f is a unit in $R[t^{\pm 1}]$. If R is a field, then a non-zero $f \in R[t^{\pm 1}]$ is trivial if and only if $\deg f = 0$.

Given an integral Laurent polynomial $f \in \mathbb{Z}[t^{\pm 1}]$, its reduction modulo a prime p , denoted $f \bmod p$, is obtained by reducing the coefficients of $f(t)$ modulo p . We denote the degree of $f \bmod p$, as an element of $\mathbb{F}_p[t^{\pm 1}]$, by $\deg_p f$ (assuming $f \bmod p$ is not zero). Observe that $\deg_p f \leq \deg f$.

We make use of the following fact about the Alexander polynomial, Δ_K , of an almost classical knot K . By [BGH⁺16, Lemma 7.5], $\Delta_K(1) = \pm 1$ and so $\Delta_K \bmod p$ is never the zero polynomial for any prime p .

Proposition 6.1. *Let K be an almost classical knot, p a prime, and r a positive integer. Assume $\Delta_K \bmod p$ is not trivial. If K has period p^r , then $\deg_p \Delta_K \geq$*

$p^r - 1$.

Proof. By Theorem 5.1, there are $f, g \in \mathbb{Z}[t^{\pm 1}]$ such that

$$\Delta_K \doteq g^{p^r} f^{p^r-1} \pmod{p}.$$

It follows that $\deg_p \Delta_K = p^r \deg_p g + (p^r - 1) \deg_p f$. Note that f or g could be trivial. Since, by assumption, $\Delta_K \pmod{p}$ is not trivial, we have $\deg_p \Delta_K > 0$ and so one of the numbers $\deg_p g, \deg_p f$ is positive. At worst, we have $\deg_p f = 1$ and $\deg_p g = 0$. Hence $\deg_p \Delta_K \geq p^r - 1$. \square

The inequality $\deg_p \Delta_K \geq p^r - 1$ immediately gives the following restriction on prime power periods for a given prime p .

Corollary 6.2. *Let K be an almost classical knot and p a prime. If $\Delta_K \pmod{p}$ is not trivial, then p^r is a period for K for at most finitely many r .*

This is because $\deg_p \Delta_K$ is finite and $\deg_p \Delta_K \leq \deg \Delta_K$, so there will only be finitely many r such that $p^r - 1 \leq \deg_p \Delta_K$. The next result is a direct consequence of Proposition 6.1 and Corollary 6.2.

Corollary 6.3 (Finitely many periods). *If K is an almost classical knot such that $\deg_p \Delta_K > 0$ for all primes p , then K admits only finitely many periods.*

Note that assuming $\deg_p \Delta_K > 0$ is equivalent to assuming that $\Delta_K \pmod{p}$ is non-trivial. Corollary 6.3 applies in many circumstances and gives a positive answer to Question 1.3 from the Introduction. For instance, it applies to any classical fibered knot K , and shows that such knots are virtually periodic for only finitely many periods. Table 7.1 lists the Alexander polynomials of the 76 almost classical knots up to 6 crossings, and Corollary 6.3 applies in 44 instances to show the given almost classical knot admits only finitely many periods.

The next result applies more generally, but gives a weaker conclusion.

Theorem 6.4 (Finitely many prime periods). *Let K be an almost classical knot such that Δ_K is not trivial. If K has prime period p , then p divides a non-zero coefficient of Δ_K , or $\deg \Delta_K \geq p - 1$. In particular, there are at most finitely many primes p for which K has period p .*

Proof. Let S be the (possibly empty) set of primes that divide some non-zero coefficient of Δ_K . Since Δ_K has only finitely many coefficients, S is finite. For a prime $p \notin S$, we want to show that, if K has prime period p , $\deg_p K \geq p - 1$.

Since $p \notin S$, p does not divide any coefficients of Δ_K , and we have $\deg \Delta_K = \deg_p \Delta_K$. By assumption, $\deg \Delta_K > 0$ and hence for such p we have that $\Delta_K \bmod p$ is non-trivial. Let p be a prime for which K has period p , and assume $p \notin S$. By Proposition 6.1, $\deg \Delta_K = \deg_p \Delta_K \geq p - 1$. Since $\deg \Delta_K \geq p - 1$, there are at most finitely many such p . \square

The following refinement of Proposition 6.1 will be useful.

Proposition 6.5. *Let K be an almost classical knot, p a prime, and r a positive integer. Assume that $\Delta_K \bmod p$ is divisible by distinct irreducibles u_1, \dots, u_s where $s \geq 1$. If K has period p^r , then $\deg_p \Delta_K \geq (p^r - 1) \sum_{j=1}^s \deg_p u_j$.*

Proof. By Theorem 5.1, there are $f, g \in \mathbb{Z}[t^{\pm 1}]$ such that

$$\Delta_K \doteq g^{p^r} f^{p^r-1} \pmod{p}.$$

Since u_j divides $\Delta_K \bmod p$ and is irreducible, then u_j divides either g or f and hence must appear with multiplicity at least $p^r - 1$. Therefore, $(u_1 \cdots u_s)^{p^r-1}$ divides $\Delta_K \bmod p$, from which the conclusion follows. \square

Corollary 6.6. *Let K be an almost classical knot, p a prime, and r a positive integer. Assume that $\Delta_K = u_1 \cdots u_s \bmod p$ where the u_j 's are distinct irreducible factors and $s \geq 1$. If K has period p^r then $p = 2$ and $r = 1$.*

Proof. The hypothesis on Δ_K implies that $\deg_p \Delta_K > 0$. By Proposition 6.5, $\deg_p \Delta_K \geq (p^r - 1) \deg_p \Delta_K$, and so $p^r - 1 = 1$. It follows that $p = 2$ and $r = 1$. \square

Corollary 6.7. *Let K be an almost classical knot, p a prime, and r a positive integer. Assume that $\Delta_K = u_1 \bmod p$, where u_1 is irreducible. Then K cannot have period p^r for $p \neq 2$.*

Theorem 5.1 is effective for the analysis of the (virtual) periods of torus knots.

Theorem 6.8. *Let m, n be relatively prime integers with $m, n \geq 2$ and let $K_{m,n}$ be the classical (m, n) -torus knot. If p^r is a (virtual) period for $K_{m,n}$ then p divides m or n , or possibly $p = 2$ and $r = 1$.*

Proof. The Alexander polynomial of $K_{m,n}$ is

$$\Delta_{K_{m,n}}(t) = \frac{(t-1)(t^{mn}-1)}{(t^m-1)(t^n-1)},$$

see [Lic97, page 119]. Note that $\Delta_{K_{m,n}}$ has highest order term $t^{(m-1)(n-1)}$ and lowest order term 1. Hence for any prime p , $\deg \Delta_{K_{m,n}} = \deg_p \Delta_{K_{m,n}} = (m-1)(n-1)$, a positive number since we have assumed $m, n \geq 2$. Observe that $\Delta_{K_{m,n}}$ divides $t^{mn} - 1$. Let p be a prime that does not divide m or n . We have $\frac{d}{dt}(t^{mn} - 1) = mnt^{mn-1}$ which is not 0 modulo p because $p \nmid mn$. Hence $t^{mn} - 1$ and mnt^{mn-1} are relatively prime as polynomials over \mathbb{F}_p and so $t^{mn} - 1$ does not have multiple roots. The only root of mnt^{mn-1} is $t = 0$ (with multiplicity $mn - 1$), and this is not a root of $t^{mn} - 1$. Since $\Delta_{K_{m,n}}$ divides $t^{mn} - 1$, it follows that $\Delta_{K_{m,n}}$ does not have multiple roots as a polynomial over \mathbb{F}_p . If p^r is a virtual period for $K_{m,n}$, then by Theorem 5.1

$$\Delta_{K_{m,n}} = (\Delta_{(K_{m,n})_*})^{p^r} f^{p^r-1} \pmod{p}.$$

If $r > 1$ or if $r = 1$ and $p \neq 2$ then the right hand side has multiple roots, a contradiction. This shows that if p^r is a virtual period for $K_{m,n}$ then p divides m or n or possibly $p = 2$ and $r = 1$. \square

We can also use Theorem 5.1 to exclude certain composite periods, as illustrated by the example below.

Example 6.9. The knot 3.6 does not have (virtual) period 6.

Proof. Let $K = 3.6$. For this knot, $\Delta_K = t^2 - t + 1$. Suppose that K has period $6 = 2 \cdot 3$. Then by Theorem 3.8, there exists an almost classical braid β such that $K = \widehat{\beta^6}$. Note that K also has period 2 with quotient $\widehat{\beta^3}$, and period 3 with quotient $\widehat{\beta^2}$. Since the period 3 and period 2 diagrams are the same (coming from the same period 6 diagram), the polynomial $f(t)$ appearing in Theorem 5.1 will be the same, whether we regard K as having period 2 or 3. (This comes from the definition of $f(t)$ as $\sum_{i=1}^k t^{\lambda_i}$ where k is the number of strands in β .) By Theorem 5.1, we have:

$$\Delta_K = t^2 + t + 1 \doteq (\Delta_{\widehat{\beta^3}})^2 f(t) \pmod{2}.$$

Note that $t^2 + t + 1$ is irreducible modulo 2. Hence

$$f(t) \doteq t^2 + t + 1 \pmod{2}$$

and $\Delta_{\widehat{\beta^3}} \doteq 1 \pmod{2}$. Again by Theorem 5.1, we also have:

$$\Delta_K = t^2 + 2t + 1 \doteq (\Delta_{\widehat{\beta^2}})^3 (f(t))^2 \pmod{3}$$

$$\text{implies } (t+1)^2 \doteq (f(t))^2 \pmod{3}$$

$$\text{implies } t+1 \doteq f(t) \pmod{3}$$

and $\Delta_{\widehat{\beta^2}} \doteq 1 \pmod{3}$. Since $\Delta_K = t^2 - t + 1$ is irreducible over \mathbb{Z} and $\Delta_{\widehat{\beta^2}} \mid \Delta_K$, we have $\Delta_{\widehat{\beta^2}} \doteq 1$ or $\Delta_{\widehat{\beta^2}} \doteq \Delta_K$. The latter possibility is excluded because $\Delta_{\widehat{\beta^2}} \doteq 1 \pmod{3}$ and $\Delta_K \not\equiv 1 \pmod{3}$, and so $\Delta_{\widehat{\beta^2}} \doteq 1$. Applying Theorem 5.1 again to $\Delta_{\widehat{\beta^2}}$ as a period 2 knot,

$$1 \doteq \Delta_{\widehat{\beta^2}} \doteq (\Delta_{\widehat{\beta}})^2 f(t) \pmod{2}.$$

Hence $f(t) \doteq 1 \pmod{2}$, a contradiction since $f(t) \doteq t^2 + t + 1 \pmod{2}$. Thus K cannot have period 6. \square

Generalizing Example 6.9, we give some criteria for the exclusion of composite periods of the form $2p$, where p is an odd prime.

Let K be an almost classical knot with period $2p$ where p is an odd prime. By Theorem 3.8, there exists an almost classical braid β such that $K = \widehat{\beta^{2p}}$. Note that K also has period p with quotient $\widehat{\beta^2}$, and period 2 with quotient $\widehat{\beta^p}$. The knot $\widehat{\beta}$ is a quotient of both $\widehat{\beta^2}$ and $\widehat{\beta^p}$. By Theorem 5.1 there is a polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$ such that

$$\Delta_K = \Delta_{\widehat{\beta^{2p}}} \doteq (\Delta_{\widehat{\beta^p}})^2 f \pmod{2} \tag{6.1}$$

$$\Delta_K = \Delta_{\widehat{\beta^{2p}}} \doteq (\Delta_{\widehat{\beta^2}})^p f^{p-1} \pmod{p} \tag{6.2}$$

$$\Delta_{\widehat{\beta^2}} \doteq (\Delta_{\widehat{\beta}})^2 f \pmod{2} \tag{6.3}$$

$$\Delta_{\widehat{\beta^p}} \doteq (\Delta_{\widehat{\beta}})^p f^{p-1} \pmod{p} \tag{6.4}$$

Proposition 6.10. *Let K be an almost classical knot. Assume that $\Delta_K \pmod{p}$ is not trivial, where p is an odd prime. If Δ_K is irreducible over \mathbb{Z} and $\Delta_K \doteq g^2 \pmod{2}$ where g is irreducible modulo 2, then K cannot have period $2p$.*

Proof. Suppose that K has period $2p$. By assumption, $\Delta_K \doteq g^2 \pmod{2}$ for some polynomial g , which is irreducible modulo 2. By (6.1),

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 f \pmod{2}$$

Case 1: $f \doteq 1 \pmod{2}$ and

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 \pmod{2}$$

Since Δ_K is irreducible over \mathbb{Z} , and $\Delta_{\widehat{\beta^p}} \mid \Delta_K$, then $\Delta_{\widehat{\beta^p}} = 1$ or $\Delta_{\widehat{\beta^p}} = \Delta_K$. If $\Delta_{\widehat{\beta^p}} = 1$, then we would have

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 = 1 \pmod{2},$$

which is a contradiction, since $\Delta_K \not\equiv 1 \pmod{2}$. Therefore, we must have $\Delta_{\widehat{\beta^p}} \doteq \Delta_K$. This gives

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 = (\Delta_K)^2 \pmod{2},$$

which is also a contradiction, as $\Delta_K \not\equiv (\Delta_K)^2 \pmod{2}$ for non-trivial Δ_K . Thus Case 1 cannot occur.

Case 2: $f \doteq g^2 \pmod{2}$ and $(\Delta_{\widehat{\beta^p}})^2 \doteq 1 \pmod{2}$. Again, we have that $\Delta_{\widehat{\beta^p}} = 1$ or $\Delta_{\widehat{\beta^p}} = \Delta_K$. Since $(\Delta_{\widehat{\beta^p}})^2 \doteq 1 \pmod{2}$, we must have that $\Delta_{\widehat{\beta^p}} = 1$ over \mathbb{Z} (as $\Delta_K \not\equiv 1 \pmod{2}$). Now we use that $\widehat{\beta^p}$ is a period p knot. Applying (6.4), we get

$$1 = \Delta_{\widehat{\beta^p}} \doteq (\Delta_{\widehat{\beta}})^p f^{p-1} \pmod{p},$$

so that $f \doteq 1 \pmod{p}$. Regarding K as a period p knot, (6.2) yields

$$\Delta_K \doteq (\Delta_{\widehat{\beta^2}})^p f^{p-1} \pmod{p}$$

$$\text{implies } \Delta_K \doteq (\Delta_{\widehat{\beta^2}})^p \pmod{p},$$

since $f \doteq 1 \pmod{p}$. Since Δ_K is irreducible over \mathbb{Z} , and $\Delta_{\widehat{\beta^2}} \mid \Delta_K$, then $\Delta_{\widehat{\beta^2}} = 1$ or $\Delta_{\widehat{\beta^2}} = \Delta_K$. But $\Delta_K \not\equiv 1 \pmod{p}$, so $\Delta_{\widehat{\beta^2}} \neq 1$, and $\Delta_{\widehat{\beta^2}} \neq \Delta_K$ since $\Delta_K \not\equiv (\Delta_K)^p \pmod{p}$. Thus, Case 2 cannot hold. Therefore, since neither case holds, K cannot have period $2p$. \square

Proposition 6.11. *Let K be an almost classical knot. Assume that $\Delta_K \pmod{p}$ is not trivial, where p is an odd prime. If $\Delta_K \doteq g^2$ over \mathbb{Z} for some polynomial g which is irreducible modulo 2, then K cannot have period $2p$.*

Proof. Suppose that K has period $2p$. (6.1) gives

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 f \pmod{2}$$

Case 1: $f \doteq 1 \pmod{2}$ and

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 \pmod{2}$$

Since $\Delta_K = g^2$ over \mathbb{Z} , and $\Delta_{\widehat{\beta^p}} \mid \Delta_K$, then $\Delta_{\widehat{\beta^p}} = 1, g$, or $g^2 = \Delta_K$. If $\Delta_{\widehat{\beta^p}} = 1$,

then we would have

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 = 1 \pmod{2},$$

which is a contradiction, since $\Delta_K \not\equiv 1 \pmod{2}$. If $\Delta_{\widehat{\beta^p}} \doteq \Delta_K$, this gives

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 = (\Delta_K)^2 \pmod{2},$$

which is also a contradiction, as $\Delta_K \not\equiv (\Delta_K)^2 \pmod{2}$ for non-trivial Δ_K . If $\Delta_{\widehat{\beta^p}} \doteq g$, this gives

$$\Delta_K \doteq g^2 \doteq (\Delta_{\widehat{\beta^p}})^2 = g^2 \pmod{2},$$

which holds. Then, since $\widehat{\beta^p}$ is a period p knot, (6.4) gives

$$g = \Delta_{\widehat{\beta^p}} \doteq (\Delta_{\widehat{\beta}})^p f^{p-1} \pmod{p}.$$

But g is irreducible, so this cannot hold by Corollary 6.7. Thus Case 1 cannot hold.

Case 2: $f \doteq g^2 \pmod{2}$ and $(\Delta_{\widehat{\beta^p}})^2 \doteq 1 \pmod{2}$. Again, we have that $\Delta_{\widehat{\beta^p}} = 1, g$, or $g^2 = \Delta_K$. Since $(\Delta_{\widehat{\beta^p}})^2 \doteq 1 \pmod{2}$, we must have that $\Delta_{\widehat{\beta^p}} = 1$ over \mathbb{Z} . Now we use that $\widehat{\beta^p}$ is a period p knot. Applying (6.4), we get

$$1 = \Delta_{\widehat{\beta^p}} \doteq (\Delta_{\widehat{\beta}})^p f^{p-1} \pmod{p},$$

so that $f \doteq 1 \pmod{p}$. Now, looking at K as a period p knot, (6.2) gives

$$\Delta_K \doteq (\Delta_{\widehat{\beta^2}})^p f^{p-1} \pmod{p}$$

$$\text{implies } \Delta_K \doteq (\Delta_{\widehat{\beta^2}})^p \pmod{p},$$

since $f \doteq 1 \pmod{p}$. Since $\Delta_K = g^2$ over \mathbb{Z} , and $\Delta_{\widehat{\beta^2}} \mid \Delta_K$, then $\Delta_{\widehat{\beta^2}} = 1, g$, or $g^2 = \Delta_K$. But $\Delta_K \not\equiv 1 \pmod{p}$, so $\Delta_{\widehat{\beta^2}} \neq 1$, and $\Delta_{\widehat{\beta^2}} \neq \Delta_K$ since $\Delta_K \not\equiv (\Delta_K)^p \pmod{p}$. The last choice is that $\Delta_{\widehat{\beta^2}} = g$, but then we would have $g^2 = \Delta_K \doteq (\Delta_{\widehat{\beta^2}})^p = g^p \pmod{p}$, and $g^2 \not\equiv g^p \pmod{p}$. Thus Case 2 cannot hold. Therefore, since neither case holds, K cannot have period $2p$. \square

We provide some examples of how the above results are used to eliminate periods, and a full list of all known and excluded periods for almost classical knots up to 6 crossings is given in Table 7.2. Proposition 6.1 is particularly

useful in eliminating many possible periods for a given knot. However, if $\Delta_K \doteq 1 \pmod p$ for some prime p , then we cannot eliminate any prime power periods of the form p^r , as Murasugi's condition holds trivially. In particular, for knots K with $\Delta_K \doteq 1$ trivial, we are unable to exclude any periods using Murasugi's conditions. This affects the knots 5.2012, 5.2025, 5.2080, 6.72507, 6.72557, 6.72692, 6.72695, 6.72975, 6.73007, and 6.73583.

The remaining knots all have Δ_K nontrivial, and we make a few general observations about their possible periods. Notice that they all have $\deg \Delta_K \leq 4$, and in each case $\Delta_K \pmod p$ is nontrivial for $p \geq 7$. It follows from Proposition 6.1 that their only possible prime periods are 2, 3 and 5, though they may admit larger periods, either as prime powers of 2, 3, 5, or as composite numbers.

Example 6.12 (Knot 3.6). $\Delta_K \doteq t^2 - t + 1$.

This is the classical knot 3_1 , which admits classical diagrams with periods $p = 2, 3$. Proposition 6.1 shows that if p^r is a prime power period, then $2 = \deg_p \Delta_K \geq p^r - 1$. Thus, $p^r \leq 3$, so our only choices are $r = 1$ and $p = 2$ or $p = 3$. The only other possible period is $q = 6 = 2 \cdot 3$, but this was ruled out by Example 6.9. Thus 3.6 has only 2 and 3 as periods, and we see that virtual knot diagrams of 3_1 do not introduce non-classical periods. This answers Question A in [KLS14] for the trefoil.

Similar considerations apply to almost classical knots K with $\Delta_K(t) \doteq t^2 - t + 1$ to show that they have only 2, 3 as possible periods. This applies to the knots 5.2160, 6.72938, 6.73053, 6.76479, 6.77833, 6.77844, and 6.77985 in Table 7.2.

Example 6.13 (Knot 4.99). $\Delta_K \doteq 2t - 1$.

This knot admits a 2-periodic diagram; in fact $K = \widehat{\beta^2}$ for $\beta = \sigma_1 \tau_2 \sigma_2^{-1} \tau_2$. Proposition 6.1 implies that K cannot have a prime period for any $p \geq 3$, and since $\Delta_K = 1 \pmod 2$, we cannot eliminate prime power periods of the form $2^r, r > 1$. This gives a partial answer to Question A in [KLS14] for 4.99.

Similar considerations apply to almost classical knots K with $\Delta_K(t) \doteq 2t - 1$ to show that they have only $2^r, r \geq 1$ as possible periods. This applies to the knots 5.2133, 6.72944, 6.75341, 6.75348, and 6.89815 in Table 7.2.

Example 6.14 (Knot 4.105). $\Delta_K \doteq 2t^2 - 2t + 1$.

This knot admits a 4-periodic diagram; in fact $K = \widehat{\beta^4}$ for $\beta = \sigma_3 \tau_4 \tau_2 \tau_1 \tau_3 \tau_2$. Proposition 6.1 implies that K cannot have a prime period $p \geq 5$, and period 3 is eliminated by Corollary 6.7, as $\Delta_K = 2t^2 + t + 1 \pmod 3$ is irreducible.

Since $\Delta_K = 1 \pmod{2}$, so we cannot exclude prime power periods of the form $2^r, r > 2$. This gives a partial answer to Question A in [KLS14] for 4.105.

Similar considerations apply to almost classical knots K with $\Delta_K(t) \doteq 2t^2 - 2t + 1$ to show that they have only $2^r, r \geq 1$ as possible periods. This applies to the knots 6.77908, 6.85613, and 6.89623 in Table 7.2.

Example 6.15 (Knot 4.108). $\Delta_K \doteq t^2 - 3t + 1$.

This is the classical knot 4_1 , which admits a classical diagram with period $p = 2$. Proposition 6.1 implies that K cannot have a prime period for $p \geq 5$, and period 3 is excluded by Corollary 6.6, as $\Delta_K = t^2 + 1 \pmod{3}$ is irreducible. As well, the period $4 = 2^2$ is eliminated by Proposition 6.1. Thus 4.108 has only 2 as a period, and virtual knot diagrams of 4_1 do not introduce non-classical periods. This answers Question A in [KLS14] for the figure eight knot.

Similar considerations apply to almost classical knots K with $\Delta_K(t) \doteq t^2 - 3t + 1$ to show that they have only 2 as a possible period. This applies to the knots 6.77905, 6.78358, and 6.79342 in Table 7.2.

Example 6.16 (Knot 5.2426). $\Delta_K \doteq (t^2 - t + 1)^2$.

This knot has no known periods, but notice that $\Delta_K \not\equiv 1 \pmod{p}$ for any prime p . Note that $t^2 - t + 1 \pmod{p}$ is an irreducible factor of $\Delta_K \pmod{p}$ provided $p \neq 3$. Using Proposition 6.5, with $s = 1$ and $u_1 = t^2 - t + 1$, if $\deg_p \Delta_K < (p^r - 1)\deg_p(t^2 - t + 1)$, then K cannot have period p^r (for $p \neq 3$). Now, $\deg_p \Delta_K = 4$, and for prime $p \geq 5$, $(p - 1)\deg_p(t^2 - t + 1) = 2(p - 1) \geq 8$, so $4 = \deg_p \Delta_K < (p - 1)\deg_p(t^2 - t + 1)$, and K cannot have prime period $p \geq 5$. For $p^r = 2^2$, $4 = \deg_2 \Delta_K < (2^2 - 1)\deg_2(t^2 - t + 1) = 6$, so K cannot have period $4 = 2^2$. Prime power periods of the form $3^r, r > 1$, are eliminated by Proposition 6.1 (since $4 = \deg_p \Delta_K < 3^r - 1$), and the composite period $6 = 2 \cdot 3$ is eliminated by Proposition 6.11 as follows. Notice that $\Delta_K \doteq (t^2 - t + 1)^2 \doteq g^2$ over \mathbb{Z} and $\Delta_K \doteq (t^2 + t + 1)^2 \doteq g^2 \pmod{2}$, where g is irreducible in both cases. As well, Δ_K is non-trivial modulo 3, so that Proposition 6.11 applies to show that K cannot have period $q = 6$. The only possible periods for $K = 5.2426$ are therefore 2 and 3.

Similar considerations apply to almost classical knots K with $\Delta_K(t) \doteq (t^2 - t + 1)^2$ to show they have only 2, 3 as possible periods. This applies to the knots 6.87262, 6.89187, and 6.89198 in Table 7.2.

Example 6.17 (Knot 6.87319). $\Delta_K = 3t^2 - 3t + 1$.

This knot admits a 3-periodic diagram; in fact $K = \widehat{\beta^3}$ for $\beta = \sigma_2^{-1}\tau_2\tau_1\sigma_3\tau_2$. Since $\Delta_K \doteq 1 \pmod{3}$, we cannot eliminate prime power periods of the form

$p^r, r > 1$. Proposition 6.1 eliminates prime periods $p \geq 5 (p \neq 3)$ and also 2^2 . The period $p = 2$ cannot be eliminated since $\Delta_K \doteq t^2 + t + 1 \pmod{2}$. Thus, the only remaining possible periods are $2, 3^r, r > 1$, and $2 \cdot 3^r, r > 0$, which cannot be ruled out.

Similar considerations apply to almost classical knots K with $\Delta_K(t) \doteq 3t^2 - 3t + 1$ to show they have only $2, 3^r (r \geq 1)$, and $2 \cdot 3^r (r \geq 1)$ as possible periods. This applies to the knot 6.87859 in Table 7.2, which has known period 2.

Example 6.18 (Knot 6.90099). $\Delta_K = t^4 - t^2 + 1$.

This knot admits a 3-periodic diagram; in fact $K = \widehat{\beta^3}$ for $\beta = \tau_3\sigma_3\sigma_4\tau_1\tau_3\tau_2$. Proposition 6.1 rules out prime periods for $p \geq 7$, and also the prime power 3^2 . Corollary 6.6 excludes $p = 5$ as a period since $\Delta_K = (t^2 + 2t + 4)(t^2 + 3t + 4) \pmod{5}$, with both factors irreducible modulo 5. In addition, Proposition 6.5 can be used to exclude 2^2 as a period, since $\Delta_K = (t^2 + t + 1)^2 \pmod{2}$ and $t^2 + t + 1$ is irreducible modulo 2, so $4 = \deg_2 \Delta_K < (3)(2)$. To exclude the composite period $q = 6 = 2 \cdot 3$, observe that Δ_K is irreducible over \mathbb{Z} , and that $\Delta_K \not\equiv 1 \pmod{3}$ and $\Delta_K = (t^2 + t + 1)^2 = g^2 \pmod{2}$, where g is irreducible modulo 2. Hence, Proposition 6.10 applies to show K cannot have period $q = 6$. Therefore K has the known period 3 and also possibly period 2, but no others.

Example 6.19 (Knot 6.90209). $\Delta_K = t^4 - 3t^3 + 3t^2 - 3t + 1$.

This is the classical knot 6_2 , which admits a classical diagram with period 2. By Proposition 6.1, K cannot have prime power period p for $p \geq 7$. Since $\Delta_K = (t^2 + t + 2)(t^2 + 2t + 2) \pmod{3}$ with both factors irreducible modulo 3, Corollary 6.6 excludes $p = 3$. Further, since $\Delta_K = (t^2 + t + 1)^2 \pmod{5}$, where $t^2 + t + 1$ is irreducible modulo 5, Proposition 6.5 excludes $p = 5$ as well, as $\deg_5 \Delta_K = 4 < (4)(2)$. Note that $\Delta_K = t^4 + t^3 + t^2 + t + 1 \pmod{2}$, which is irreducible modulo 2, thus Proposition 6.5 applies again to show that K cannot have period 2^2 . Thus K has only the known period 2 and no others. In particular, we see that virtual knot diagrams of the classical knot 6_2 do not introduce any non-classical periods.

Chapter 7

Conclusion

Summary

Many authors have studied periodic virtual knots, and they have proven numerous results about the special form that certain invariants of a virtual knot must take when it admits a periodic virtual knot diagram; see [IL12, KLS09, KLS13, BL15, KLS14]. In these papers, the authors apply their techniques to help determine the virtual periods for the table of virtual knots up to four crossings [Gre04]. However, because all of these invariants vanish for classical (and almost classical) knots, their results are ineffective in answering the next question, which is a restatement of Question 1.4, about the virtual periods of classical knots:

Question. Can a classical knot admit a q -periodic virtual knot diagram without admitting a q -periodic classical knot diagram?

In this thesis, we approached this question by extending the Murasugi congruence from classical knots to almost classical knots, and this allowed us to use the Alexander polynomial to restrict the possible virtual periods for classical and almost classical knots. Although the main result, Theorem 5.1, is stated for almost classical knots, it applies and restricts the possible periods of an arbitrary virtual knot as well, as we now explain (cf. Section 3.2, especially Proposition 3.9 and Corollary 3.10). It will be convenient to let $\bar{K} = P_f^\infty(K)$ denote its image under stable Manturov projection, where K is an arbitrary virtual knot, and f is the total Gaussian parity. The next result is a restatement of Corollary 3.10.

Theorem. If K is a virtual knot which admits a q -periodic virtual knot diagram

and $\bar{K} = P_f^\infty(K)$, then \bar{K} admits a q -periodic almost classical knot diagram.

Combining the above result with Corollary 6.3 gives a partial answer to Question 1.3 from the Introduction. In particular, if K is a virtual knot with projection $\bar{K} = P_f^\infty(K)$, and if $\deg_p \Delta_{\bar{K}} > 0$ for all primes p , then Corollary 6.3 implies that \bar{K} can have only finitely many periods, and it follows that K can have only finitely many periods as well. In a similar way, one can combine the above result with Theorem 6.4 to show the following: if K is a virtual knot whose stable projection \bar{K} has non-trivial Alexander polynomial $\Delta_{\bar{K}} \neq 1$, then K can be p -periodic for only finitely many primes p .

We applied our results to 6-crossing almost classical knots and used it to produce a restricted list of possible periods. This allowed us to determine all virtual periods for the six knots 3.6 (having periods 2 and 3), 4.108 (period 2), 6.85091 (period 2), 6.89156 (period 2), 6.90139 (periods 2, 3 and 6), and 6.90219 (period 2). In particular, for the trefoil and figure eight knots, we were able to see that they do not exhibit any new periods as virtual knot diagrams that they did not already display as classical knot diagrams. For 27 of the almost classical knots, we were able to determine all periods except for $p = 2$. In fact, our condition never allows us to rule out period 2 as a possibility, so this is the best we can hope for if period 2 is not already known. For 11 of the knots, we were able to determine all periods except for $p = 2$ and $p = 3$, and for another 16 knots we were able to determine all periods except for some powers of 2. There were 10 knots with trivial Alexander polynomial, in which case we were unable to eliminate any periods, and for the remaining 6 knots, we were able to determine all periods except for a mix of powers of 2 and 3, and composites between them. Since only 10 had trivial Alexander polynomial, we were able to show that the other 66 are p -periodic for only finitely many primes. For 44 of the 76 knots, we have that they are q -periodic for only finitely many $q = p^r$.

In addition to the trefoil (3.6) and figure eight (4.108), there were two other classical knots 6.90209 and 5.2445 that we have shown do not exhibit any new virtual periods that they did not already have classically, but there are 5 other classical knots that we could not show this for. In general, it is an open question as to whether a classical knot admits new q -periodic diagrams virtually that it does not classically.

Table 7.2 provides a list of the known periods and excluded periods for almost classical knots up to six crossings. It was produced by applying the Murasugi conditions to the Alexander polynomials Δ_K of almost classical knots. Note that if we list period q as excluded, then any multiple kq is also automatically

excluded. For simplicity, the table lists only the smallest q as excluded. The “known periods” in Table 7.2 refer to periods that are either immediately obvious from the Gauss diagram, or from the classical periods listed in Table I of [BZH14, Appendix C]. Likewise, if q is a known period, then so is any divisor of q . The table lists only the largest q as known periods. The “–” in the “known periods” column means we do not know of any periods of the knot; the “–” in the “excluded periods” column means that applying the Murasugi conditions did not allow us to eliminate any periods of that type; “N/A” in the “excluded periods - composite” column is because all primes $p \geq 3$ have already been excluded, and hence no composites were left to check; “none” in the “non-excluded periods” column means that all virtual periods are either known or excluded. This table looks only at what the Murasugi conditions will eliminate. The Alexander polynomials were computed in [BGH⁺16] and are listed in Table 7.1, and the Gauss diagrams are illustrated in Figure 7.1. Note that, in the tables, classical knots appear in boldface for easy reference.

Next steps

Studying other knot invariants that extend to virtual knots, such as the Jones polynomial, will help to further restrict the possible virtual periods of virtual and classical knots. For example, in the classical case, the Jones polynomial $V_K(t)$ has been used to restrict the possible periods, see Murasugi [Mur88], Traczyk [Tra90], Yokota [Yok91], and it extends to an invariant of virtual knots in a straightforward way, see Kauffman [Kau99]. In [BKP98], they use Zulli’s trip matrix approach to derive conditions on $V_K(t)$ for q -periodic classical knots, and they show that the trip matrix of K is a block circulant matrix. We intend to extend their argument to periodic virtual knots, and we hope to apply the Jones polynomial of K and its iterated cables to further restrict the possible virtual periods of K .

Another interesting question is whether the real crossing number of a virtual knot bounds the possible periods for the knot.

Conjecture 7.1. *Suppose K is a non-trivial virtual knot with real crossing number n . If K admits a q -periodic diagram, then $q \leq n$.*

In our table, we have examples where $q = n$. For instance, 4.105 has crossing number 4 and period 4, and 5.2433, 5.2445, 6.90139 and 6.90228 are other examples where $q = n$, showing that the inequality in the above conjecture, if

true, is sharp. By [Liv93, Ex. 4.7, p. 173], it follows that the above conjecture holds for the classical periods of a classical knot. Further, it shows that the classical periods satisfy $q \leq n$, with equality only if K is a torus knot. The solution uses Edmonds' condition [Edm84], which is an important result classically, but which has no obvious extension to virtual knots.

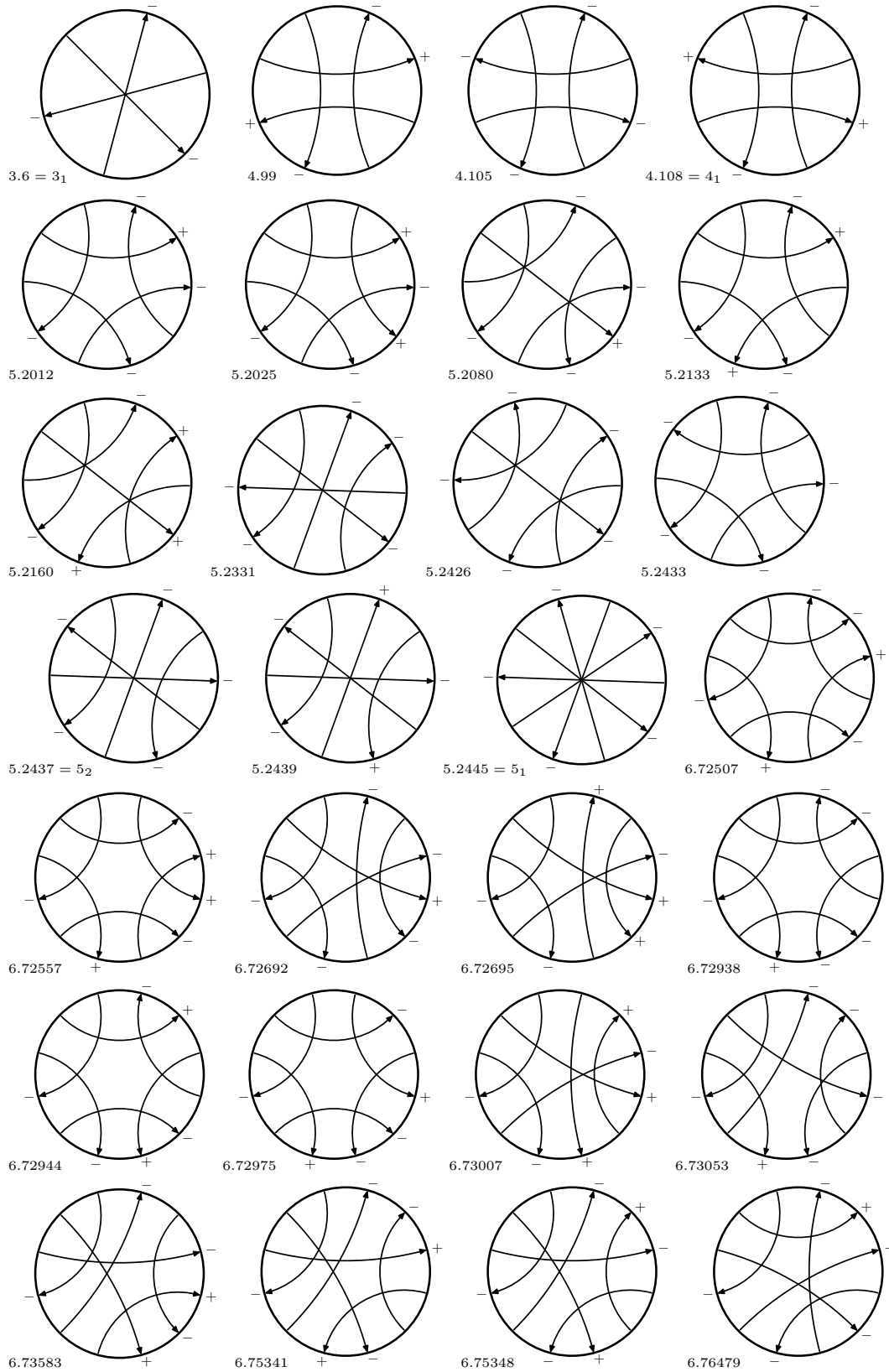
Knot	Alexander polynomial	Knot	Alexander polynomial
3.6	$t^2 - t + 1$	6.85774	$t^3 - t^2 + 1$
4.99	$2t - 1$	6.87188	$(2t - 1)(t^2 - t + 1)$
4.105	$2t^2 - 2t + 1$	6.87262	$(t^2 - t + 1)^2$
4.108	$t^2 - 3t + 1$	6.87269	$(2t - 1)^2$
5.2012	1	6.87310	$t^4 - t^3 + 2t^2 - 2t + 1$
5.2025	1	6.87319	$3t^2 - 3t + 1$
5.2080	1	6.87369	$t^3 - 2t^2 + 3t - 1$
5.2133	$2t - 1$	6.87548	$t^3 - 2t^2 - t + 1$
5.2160	$t^2 - t + 1$	6.87846	$t^3 - t^2 + 2t - 1$
5.2331	$t^3 - t + 1$	6.87857	$t^2 - 4t + 2$
5.2426	$(t^2 - t + 1)^2$	6.87859	$3t^2 - 3t + 1$
5.2433	$t^4 - 2t^3 + 4t^2 - 3t + 1$	6.87875	$t^3 + t^2 - 2t + 1$
5.2437	$2t^2 - 3t + 2$	6.89156	$2t^3 - t^2 - t + 1$
5.2439	$t^3 - 2t^2 + 3t - 1$	6.89187	$(t^2 - t + 1)^2$
5.2445	$t^4 - t^3 + t^2 - t + 1$	6.89198	$(t^2 - t + 1)^2$
6.72507	1	6.89623	$2t^2 - 2t + 1$
6.72557	1	6.89812	$t^3 - 2t + 2$
6.72692	1	6.89815	$2t - 1$
6.72695	1	6.90099	$t^4 - t^2 + 1$
6.72938	$t^2 - t + 1$	6.90109	$(2t^2 - 2t + 1)(t^2 - t + 1)$
6.72944	$2t - 1$	6.90115	$(t^2 - 3t + 1)(t^2 - t + 1)$
6.72975	1	6.90139	$(3t^2 - 3t + 1)(t^2 - t + 1)$
6.73007	1	6.90146	$t^4 - 5t^3 + 9t^2 - 5t + 1$
6.73053	$t^2 - t + 1$	6.90147	$t^4 - 3t^3 + 6t^2 - 5t + 2$
6.73583	1	6.90150	$t^4 - 5t^3 + 6t^2 - 4t + 1$
6.75341	$2t - 1$	6.90167	$t^4 - 2t^3 + 4t^2 - 4t + 2$
6.75348	$t - 2$	6.90172	$t^4 - 3t^3 + 5t^2 - 3t + 1$
6.76479	$t^2 - t + 1$	6.90185	$3t^4 - 6t^3 + 6t^2 - 3t + 1$
6.77833	$t^2 - t + 1$	6.90194	$t^4 - 4t^3 + 8t^2 - 5t + 1$
6.77844	$t^2 - t + 1$	6.90195	$2t^4 - 3t^3 + 3t^2 - 2t + 1$
6.77905	$t^2 - 3t + 1$	6.90209	$t^4 - 3t^3 + 3t^2 - 3t + 1$
6.77908	$2t^2 - 2t + 1$	6.90214	$3t^2 - 4t + 2$
6.77985	$t^2 - t + 1$	6.90217	$t^3 - 4t^2 + 3t - 1$
6.78358	$t^2 - 3t + 1$	6.90219	$2t^3 - 3t^2 + 3t - 1$
6.79342	$t^2 - 3t + 1$	6.90227	$(t - 2)(2t - 1)$
6.85091	$t^2 + t - 1$	6.90228	$4t^2 - 6t + 3$
6.85103	$t^3 - t^2 + 2t - 1$	6.90232	$2t^3 - 6t^2 + 4t - 1$
6.85613	$t^2 - 2t + 2$	6.90235	$t^4 - 3t^3 + 5t^2 - 3t + 1$

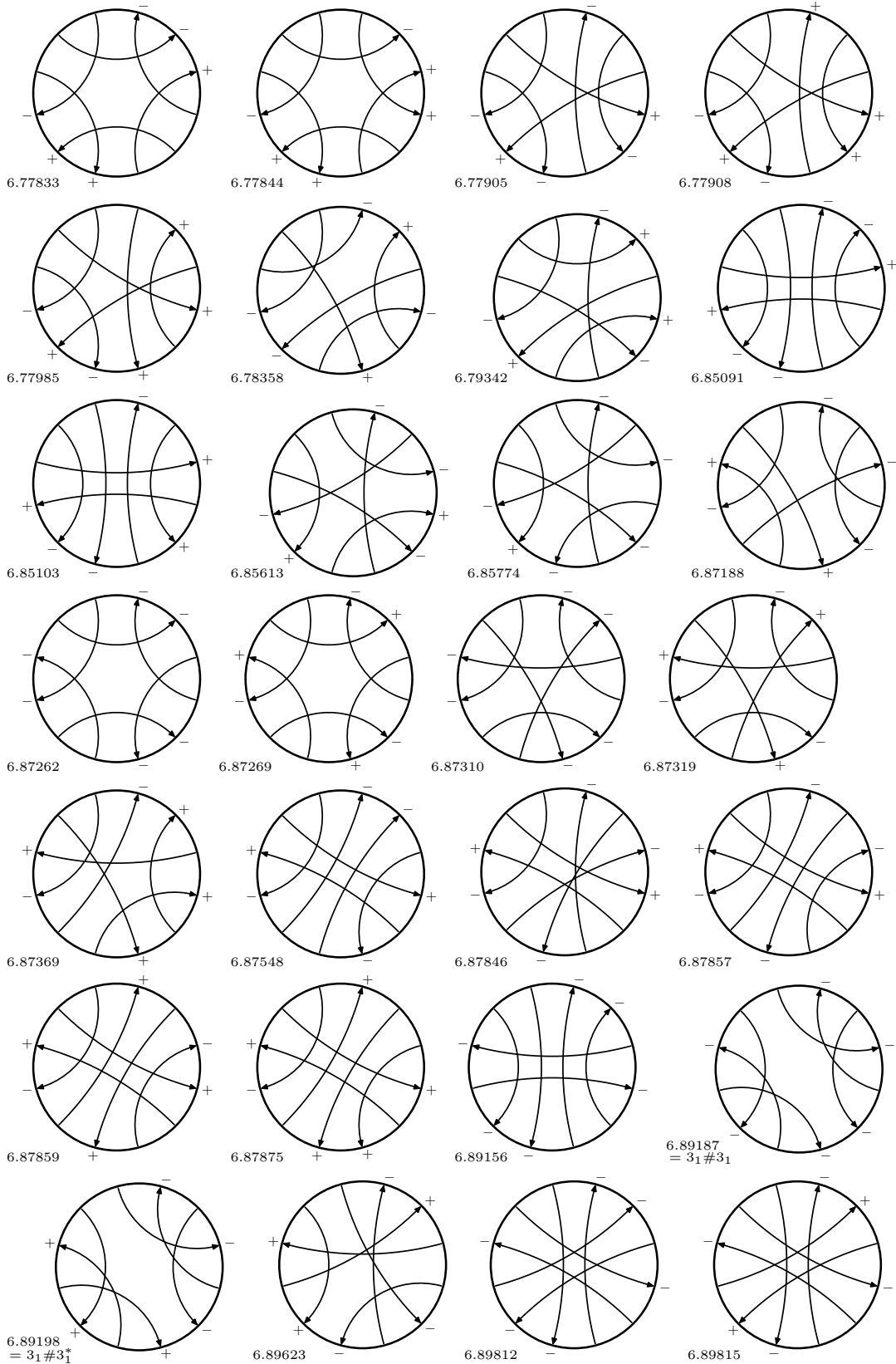
Table 7.1: Alexander polynomials for almost classical knots up to six crossings.

Knot	Known Periods	Excluded Periods			Non-Excluded Periods
		Prime	Prime Powers	Composite	
3.6	2, 3	$p \geq 5$	2^2	$2 \cdot 3$	none
4.99	2	$p \geq 3$	-	N/A	$2^k, k \geq 2$
4.105	4	$p \geq 3$	-	N/A	$2^k, k \geq 3$
4.108	2	$p \geq 3$	2^2	N/A	none
5.2012	-	-	-	-	$q \geq 2$
5.2025	-	-	-	-	$q \geq 2$
5.2080	-	-	-	-	$q \geq 2$
5.2133	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
5.2160	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
5.2331	-	$p \geq 3$	2^2	N/A	2
5.2426	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
5.2433	5	$p \neq 2, 5$	$2^2, 5^2$	$2 \cdot 5$	2
5.2437	2	$p \geq 3$	-	N/A	$2^k, k \geq 2$
5.2439	-	$p \geq 3$	2^2	N/A	none
5.2445	2, 5	$p \neq 2, 5$	$2^2, 5^2$	$2 \cdot 5$	none
6.72507	-	-	-	-	$q \geq 2$
6.72557	-	-	-	-	$q \geq 2$
6.72692	-	-	-	-	$q \geq 2$
6.72695	-	-	-	-	$q \geq 2$
6.72938	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
6.72944	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.72975	-	-	-	-	$q \geq 2$
6.73007	-	-	-	-	$q \geq 2$
6.73053	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
6.73583	-	-	-	-	$q \geq 2$
6.75341	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.75348	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.76479	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
6.77833	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
6.77844	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
6.77905	-	$p \geq 3$	2^2	N/A	2
6.77908	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.77985	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
6.78358	-	$p \geq 3$	2^2	N/A	2
6.79342	-	$p \geq 3$	2^2	N/A	2
6.85091	2	$p \geq 3$	2^2	N/A	none
6.85103	-	$p \geq 3$	2^2	N/A	2
6.85613	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.85774	-	$p \geq 3$	2^2	N/A	2
6.87188	-	$p \geq 3$	2^2	N/A	2
6.87262	3	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2

Knot	Known Periods	Excluded Periods			Non-Excluded Periods
		Prime	Prime Powers	Composite	
6.87269	3	$p \geq 5$	3^2	-	$2^k, k \geq 1$ $3 \cdot 2^k, k \geq 1$
6.87310	3	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2
6.87319	3	$p \geq 5$	2^2	-	$2, 3^k, k \geq 2$ $2 \cdot 3^k, k \geq 1$
6.87369	-	$p \geq 3$	2^2	N/A	2
6.87548	-	$p \geq 3$	2^2	N/A	2
6.87846	-	$p \geq 3$	2^2	N/A	2
6.87857	2	$p \geq 3$	-	N/A	$2^k, k \geq 2$
6.87859	2	$p \geq 5$	2^2	-	$3^k, k \geq 1$ $2 \cdot 3^k, k \geq 1$
6.87875	-	$p \geq 3$	2^2	N/A	2
6.89156	2	$p \geq 3$	2^2	N/A	none
6.89187	-	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2, 3
6.89198	2	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	3
6.89623	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.89812	2	$p \geq 3$	-	N/A	$2^k, k \geq 2$
6.89815	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.90099	3	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2
6.90109	-	$p \geq 3$	2^2	N/A	2
6.90115	-	$p \geq 3$	2^2	N/A	2
6.90139	6	$p \geq 5$	$2^2, 3^2$	-	none
6.90146	3	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2
6.90147	-	$p \geq 3$	2^2	N/A	2
6.90150	-	$p \geq 3$	2^2	N/A	2
6.90167	-	$p \geq 3$	-	N/A	$2^k, k \geq 1$
6.90172	2	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	3
6.90185	3	$p \geq 5$	2^2	-	$2, 3^k, k \geq 2$ $2 \cdot 3^k, k \geq 1$
6.90194	3	$p \geq 5$	2^2	$2 \cdot 3$	2
6.90195	-	$p \geq 3$	2^2	N/A	2
6.90209	2	$p \geq 3$	2^2	N/A	none
6.90214	2	$p \geq 3$	-	N/A	$2^k, k \geq 2$
6.90217	-	$p \geq 3$	2^2	N/A	2
6.90219	2	$p \geq 3$	2^2	N/A	none
6.90227	2	$p \geq 5$	3^2	-	$2^k, k \geq 2$ $3 \cdot 2^k, k \geq 1$
6.90228	6	$p \geq 5$	-	-	$2^j, 3^k, j, k \geq 2$ $2^j \cdot 3^k, j, k \geq 2$
6.90232	2	$p \geq 3$	-	N/A	$2^k, k \geq 2$
6.90235	3	$p \geq 5$	$2^2, 3^2$	$2 \cdot 3$	2

Table 7.2: Known and excluded periods of almost classical knots.





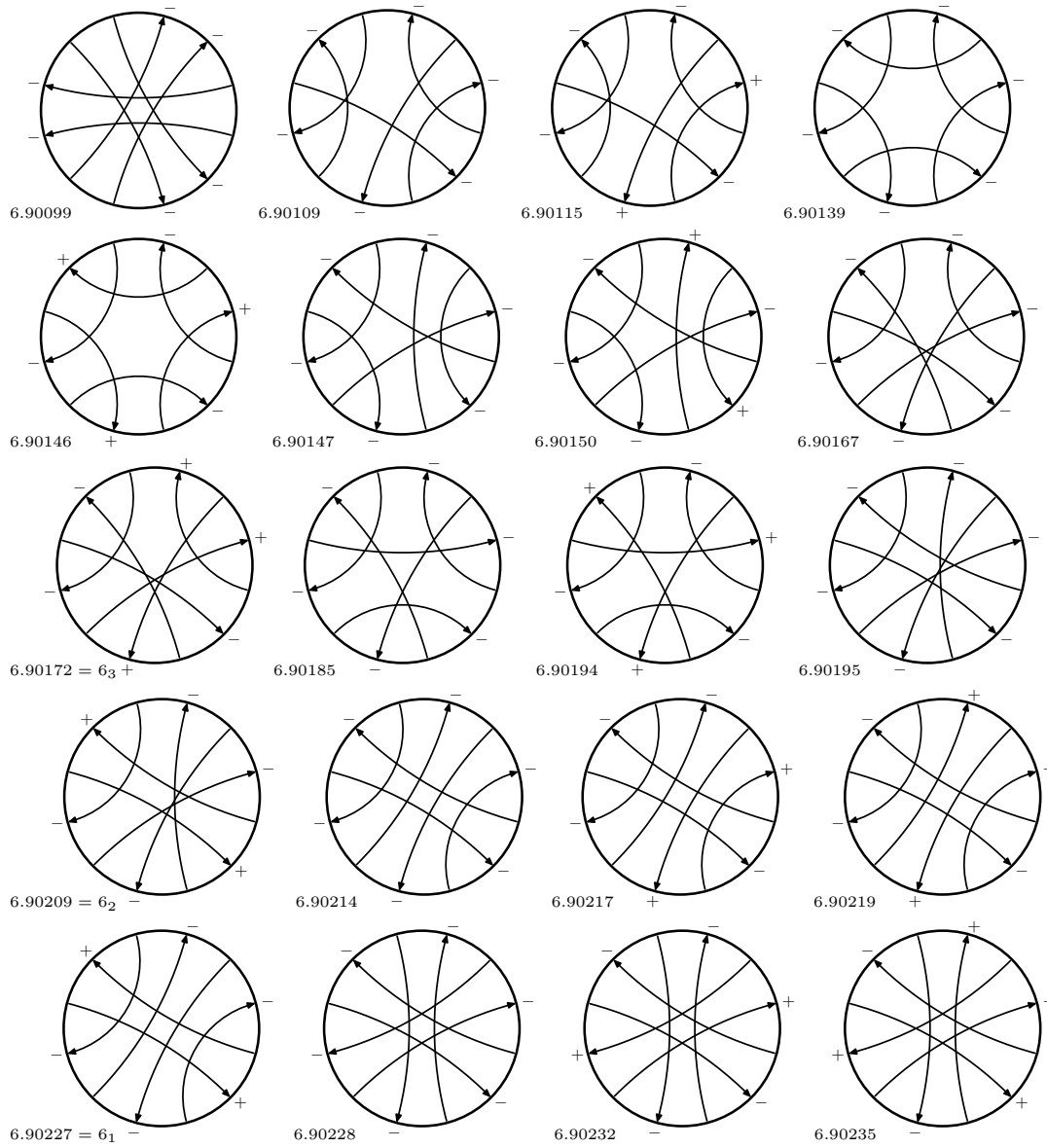


Figure 7.1: Gauss diagrams of almost classical knots with up to six crossings.

Bibliography

- [BGH⁺16] Hans U. Boden, Robin I. Gaudreau, Eric Harper, Andrew J. Nicas, and Lindsay White. Virtual knot groups and almost classical knots. *Fundamenta Mathematicae*, to appear, 2016, arXiv:math.GT/1506.01726.
- [Bir74] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.
- [BKP98] Yongju Bae, Young Kon Kim, and Chan-Young Park. Jones polynomials of periodic knots. *Bull. Austral. Math. Soc.*, 58(2):261–270, 1998.
- [BL15] Yongju Bae and In Sook Lee. On Gauss diagrams of periodic virtual knots. *J. Knot Theory Ramifications*, 24(10):1540008, 17, 2015.
- [BZH14] Gerhard Burde, Heiner Zieschang, and Michael Heusener. *Knots*, volume 5 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, extended edition, 2014.
- [CF77] Richard H. Crowell and Ralph H. Fox. *Introduction to knot theory*. Springer-Verlag, New York-Heidelberg, 1977. Reprint of the 1963 original, Graduate Texts in Mathematics, No. 57.
- [CKS02] J. Scott Carter, Seiichi Kamada, and Masahico Saito. Stable equivalence of knots on surfaces and virtual knot cobordisms. *J. Knot Theory Ramifications*, 11(3):311–322, 2002. Knots 2000 Korea, Vol. 1 (Yongpyong).
- [Dav79] Philip J. Davis. *Circulant matrices*. John Wiley & Sons, New York-Chichester-Brisbane, 1979. A Wiley-Interscience Publication, Pure and Applied Mathematics.

- [DL91a] James F. Davis and Charles Livingston. Alexander polynomials of periodic knots. *Topology*, 30(4):551–564, 1991.
- [DL91b] James F. Davis and Charles Livingston. Periodic knots, Smith theory, and Murasugi’s congruence. *Enseign. Math. (2)*, 37(1-2):1–9, 1991.
- [Edm84] Allan L. Edmonds. Least area Seifert surfaces and periodic knots. *Topology Appl.*, 18(2-3):109–113, 1984.
- [Fin47] Nathan J. Fine. Binomial coefficients modulo a prime. *Amer. Math. Monthly*, 54:589–592, 1947.
- [Fla85] Erica Flapan. Infinitely periodic knots. *Canad. J. Math.*, 37(1):17–28, 1985.
- [GPV00] Mikhael Goussarov, Michael Polyak, and Oleg Viro. Finite-type invariants of classical and virtual knots. *Topology*, 39(5):1045–1068, 2000.
- [Gre04] J. Green. A table of virtual knots, 2004, available online at <http://www.math.toronto.edu/drorbn/Students/GreenJ>.
- [Hil81] Jonathan A. Hillman. New proofs of two theorems on periodic knots. *Arch. Math. (Basel)*, 37(5):457–461, 1981.
- [Hil83] Jonathan A. Hillman. On the Alexander polynomial of a cyclically periodic knot. *Proc. Amer. Math. Soc.*, 89(1):155–156, 1983.
- [Hil84] Jonathan A. Hillman. Links with infinitely many semifree periods are trivial. *Arch. Math. (Basel)*, 42(6):568–572, 1984.
- [HLN06] Jonathan A. Hillman, Charles Livingston, and Swatee Naik. Twisted Alexander polynomials of periodic knots. *Algebr. Geom. Topol.*, 6:145–169 (electronic), 2006.
- [IL12] Young Ho Im and Sang Youl Lee. On the index and arrow polynomials of periodic virtual links. *J. Knot Theory Ramifications*, 21(6):1250058, 21, 2012.
- [IMN11] Denis P. Ilyutko, Vassily O. Manturov, and Igor M. Nikonov. Parity in knot theory and graph links. *Sovrem. Mat. Fundam. Napravl.*, 41:3–163, 2011.

- [Kam07] Seiichi Kamada. Braid presentation of virtual knots and welded knots. *Osaka J. Math.*, 44(2):441–458, 2007.
- [Kau99] Louis H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.
- [Kim00] Se-Goo Kim. Virtual knot groups and their peripheral structure. *J. Knot Theory Ramifications*, 9(6):797–812, 2000.
- [KLS09] Joonoh Kim, Sang Youl Lee, and Myoungsoo Seo. The Miyazawa polynomial of periodic virtual links. *Osaka J. Math.*, 46(3):769–781, 2009.
- [KLS13] Joonoh Kim, Sang Youl Lee, and Myoungsoo Seo. On the VA-polynomial of periodic virtual knots. *J. Knot Theory Ramifications*, 22(5):1350016, 20, 2013.
- [KLS14] Joonoh Kim, Sang Youl Lee, and Myoungsoo Seo. On the virtual Alexander polynomial of periodic virtual knots. *J. Knot Theory Ramifications*, 23(7):1460011, 15, 2014.
- [Kup03] Greg Kuperberg. What is a virtual link? *Algebr. Geom. Topol.*, 3:587–591 (electronic), 2003.
- [Lee12] Sang Youl Lee. Genera and periodicity of virtual knots and links. *J. Knot Theory Ramifications*, 21(4):1250037, 15, 2012.
- [Lic97] W. B. Raymond Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.
- [Liv93] Charles Livingston. *Knot theory*, volume 24 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1993.
- [LP97] Sang Youl Lee and Chan-Young Park. Braid representations of periodic links. *Bull. Austral. Math. Soc.*, 55(1):7–18, 1997.
- [Man13] Vassily O. Manturov. Parity and projection from virtual knots to classical knots. *J. Knot Theory Ramifications*, 22(9):1350044, 20, 2013.

- [MI13] Vassily O. Manturov and Denis P. Ilyutko. *Virtual knots*, volume 51 of *Series on Knots and Everything*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. The state of the art, Translated from the 2010 Russian original, With a preface by Louis H. Kauffman.
- [Mur71] Kunio Murasugi. On periodic knots. *Comment. Math. Helv.*, 46:162–174, 1971.
- [Mur88] Kunio Murasugi. Jones polynomials of periodic links. *Pacific J. Math.*, 131(2):319–329, 1988.
- [Nik16] Igor M. Nikonov. Weak parities and functorial maps. *J. Math. Sciences*, 214(5):699–717, 2016.
- [NNST12] Takuji Nakamura, Yasutaka Nakanishi, Shin Satoh, and Yumi Tomiyama. Twin groups of virtual 2-bridge knots and almost classical knots. *J. Knot Theory Ramifications*, 21(10):1250095, 18, 2012.
- [Pol10] Michael Polyak. Minimal generating sets of Reidemeister moves. *Quantum Topol.*, 1(4):399–411, 2010.
- [Saw99] J. Sawollek. On Alexander-Conway polynomials for virtual knots and links, 1999, arXiv:math.GT/9912173v2.
- [Tra90] Paweł Traczyk. 10_{101} has no period 7: a criterion for periodic links. *Proc. Amer. Math. Soc.*, 108(3):845–846, 1990.
- [Yok91] Yoshiyuki Yokota. The Jones polynomial of periodic knots. *Proc. Amer. Math. Soc.*, 113(3):889–894, 1991.