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Implementation of an adaptive finite-element approximation of the Mumford-Shah functional

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Summary. We present and detail a method for the numerical solving of the Mumford-Shah problem, based on a finite element method and on adaptive meshes. We start with the formulation introduced in [13], detail its numerical implementation and then propose a variant which is proved to converge to the Mumford-Shah problem. A few experiments are illustrated.

Mathematics Subject Classification (1991): 65N30, 26B30, 65K10, 68U10

1. Introduction

In order to solve the image segmentation problem, D. Mumford and J. Shah have proposed in [26] to minimize over u and K the functional

$$\mathcal{G}(u,K) \;=\; \int_\Omega |\nabla u(x)|^2\,dx + \mathcal{H}^1(K) + \int_\Omega |u(x)-g(x)|^2\,dx,$$

where $\Omega \subset \mathbb{R}^2$ is the image domain (a bounded open two-dimensional domain), $g \in L^\infty(\Omega)$ is the original image, that has to be segmented, K is a closed set of Hausdorff one-dimensional measure $\mathcal{H}^1(K)$ and $u \in C^1(\Omega \setminus K)$. The set K is supposed to represent the *edges* of the segmented image u that is regular out of K and can be discontinuous across K (see Appendix A.2 for details).

The actual minimization of $\mathcal{G}$ is a difficult problem, that has been addressed by many authors. Mumford and Shah themselves derived their energy from discrete energies introduced by D. Geman and S. Geman [24]

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and A. Blake and A. Zisserman [7], and these works can be seen as the first attempts to minimize $\mathcal{G}$ by a finite differences approach. In [4], L. Ambrosio and V.-M. Tortorelli proposed an approximation of $\mathcal{G}$, depending on a scale parameter $c > 0$, in which the set K was approximated in some sense by a function v, making thus easier the discretization of the problem. This led G. Bellettini and A. Coscia to propose a finite elements approximation in [6], adapted and implemented by S. Finzi-Vita and P. Perugia [21] and B. Bourdin [10]. In all these approaches, the quality of the approximation is very poor if c, the discretization step ε, and ε/c are not very small, so that the computations must be performed on a very fine mesh.

In [13], A. Chambolle and G. Dal Maso have proposed a different finite elements approach, that is not derived from Ambrosio and Tortorelli's approximation result, and relies only on one scale parameter (the discretization step ε). On the other hand, unlike the previous methods, it requires an adaption of the triangulation in order to approximate correctly the theoretical Mumford-Shah energy. We show in this paper how to deal with this difficulty.

Although this method seems very complicated, it is a promising approach and has been successfully implemented for a brittle fracture formulation proposed by G. Francfort and J.-J. Marigo in [22], that is similar to the Mumford-Shah problem. In that particular case, a finite elements method is natural. Furthermore, one wants to localize the cracks as well as possible, and the mesh adaption method described in this paper is a real advantage since is doesn't require the use of a very thin mesh to do so.

In what follows, we recall the results of [13]. We then propose a variant whose interest will be discussed in Sect. 2, where the numerical implementation of both formulations are described. Then, in Sect. 3, we prove that our variant actually approximates the Mumford-Shah functional, in the sense of the Γ–convergence, a notion of variational convergence introduced by De Giorgi (see for instance [15] and Appendix B).

In the whole paper, The set Ω is a bounded domain of $\mathbb{R}^2$, with Lipschitz-regular boundary. A *triangulation* of Ω is a finite family of (closed) triangles covering Ω, and such that the intersection of any two such triangles, if not empty, is either an edge or a vertex common to both triangles. Following [13], we fix some angle $\theta_0 > 0$ ($\theta_0 \leq 60°$), a constant $c \geq 6$, and let for any $\varepsilon > 0$ $\mathcal{T}_\varepsilon(\Omega) = \mathcal{T}_\varepsilon(\Omega, c, \theta_0)$ be the set of all triangulations of Ω whose triangles T have the following characteristics

- the length of all three edges of T is between ε and $c\varepsilon$,
- the three angles of T are greater than or equal to θ_0.

We call $V_\varepsilon(\Omega)$ the set of all continuous functions $u : \Omega \to \mathbb{R}$ such that u is affine on any triangle $T \in \boldsymbol{T}$ (more precisely, on $T \cap \Omega$) of some triangulation $\boldsymbol{T} \in \mathcal{T}_\varepsilon(\Omega)$, and given such a u, $\mathcal{T}_\varepsilon(u) \subseteq \mathcal{T}_\varepsilon(\Omega)$ is the set of all triangulations *adapted* to u, i.e., such that this property is satisfied (for "most" u, $\mathcal{T}_\varepsilon(u)$ has just one element, if u = constant, $\mathcal{T}_\varepsilon(u) = \mathcal{T}_\varepsilon(\Omega)$). Given a triangle T we denote by h_T its smallest height. If T belongs to some triangulation of $\mathcal{T}_\varepsilon(\Omega)$, then $\varepsilon \sin\theta_0 \le h_T \le \varepsilon c\sqrt{3}/2$.

Throughout the whole paper, we fix, as in [11,13], a non-decreasing continuous function $f : [0,+\infty) \to [0,+\infty)$ such that

$$\text{(1)} \qquad \lim_{t\downarrow 0} \frac{f(t)}{t} = 1 \qquad \text{and} \qquad \lim_{t\to+\infty} f(t) = f_\infty.$$

The simplest case is $f(t) = \min(t, f_\infty)$. For the sake of simplicity, we will also assume that

$$\text{(2)} \qquad f(t) \le \min(t, f_\infty) \text{ for all } t \ge 0$$

(in the practical applications f is concave and this condition is obviously satisfied).

Fix $p \in [1,+\infty)$. In [13], the following functional $G_\varepsilon(u,\boldsymbol{T})$ is introduced, for any $u \in L^p(\Omega)$ and $\boldsymbol{T} \in \mathcal{T}_\varepsilon(\Omega)$:

$$G_\varepsilon(u,\boldsymbol{T}) = \begin{cases} \displaystyle\sum_{T\in\boldsymbol{T}} |T\cap\Omega| \frac{1}{h_T} f(h_T|\nabla u_T|^2), & \text{if } u \in V_\varepsilon(\Omega), \boldsymbol{T} \in \mathcal{T}_\varepsilon(u), \\ +\infty, & \text{otherwise,} \end{cases} \tag{3}$$

where ∇u_T denotes the (constant) gradient of u on the triangle T. Then, if for any u we set

$$\text{(4)} \qquad G_\varepsilon(u) = \min_{\boldsymbol{T}\in\mathcal{T}_\varepsilon(\Omega)} G_\varepsilon(u,\boldsymbol{T}),$$

(which means, practically, that the "best" triangulation adapted to u is chosen) it is proved that, as ε goes to zero and provided θ_0 is less than some $\Theta > 0$, G_ε Γ-converges to the Mumford-Shah functional

$$G(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^2\,dx + f_\infty \mathcal{H}^1(S_u), & \text{if } u \in L^p(\Omega)\cap GSBV(\Omega), \\ +\infty, & \text{if } u \in L^p(\Omega)\setminus GSBV(\Omega), \end{cases} \tag{5}$$

where the space $GSBV(\Omega)$ and the essential jumps set S_u are defined in Appendix A.1. The definition and basic properties of the Γ-convergence are reviewed in Appendix B, we just recall that what we are mainly interested in is the fact that the minimizers of G_ε will be, as ε becomes infinitesimal, good approximations of minimizers of G.

In the next Sect. 2, we describe a way to implement numerically the minimization of G_ε. The results are quite good, but the method is subject to numerical instabilities. We introduce therefore a "stabilized" version in the following way, that is inspired by an approximation of A. Braides and G. Dal Maso. In [11], they introduce the non-local functional

$$\frac{1}{\varepsilon}\int_\Omega f\left(\varepsilon\frac{1}{|B_\varepsilon(x)\cap\Omega|}\int_{B_\varepsilon(x)\cap\Omega}|\nabla u(y)|^2\,dy\right)dx \tag{6}$$

and show that it Γ-converges, as ε goes to 0, to $\int_\Omega |\nabla u|^2\,dx + 2f_\infty\mathcal{H}^{N-1}(S_u)$, provided f satisfies conditions (1). Unfortunately, this formulation doesn't fit easily into a finite elements implementation. We follow a slightly different approach, replacing the mean on the ball $B_\varepsilon(x)$ in (6) by an averaging operator that depends on the triangulation.

Given a triangulation $\boldsymbol{T}$, we define on the Euclidean space $\mathbb{R}^{\boldsymbol{T}}$ the operator $M : \mathbb{R}^{\boldsymbol{T}} \to \mathbb{R}^{\boldsymbol{T}}$ such that for any $v = (v_T)_{T\in\boldsymbol{T}} \in \mathbb{R}^{\boldsymbol{T}}$,

$$(Mv)_T \;=\; M_T(v) \;=\; \frac{\displaystyle\sum_{T'\in\boldsymbol{T},T'\cap T\neq\emptyset}|T'\cap\Omega|v_{T'}}{\displaystyle\sum_{T'\in\boldsymbol{T},T'\cap T\neq\emptyset}|T'\cap\Omega|}. \tag{7}$$

If v is considered as a piecewise constant function on Ω, such that $v\equiv v_T$ on each triangle $T\in\boldsymbol{T}$, $M_T(v)$ is therefore the mean of v over T and all the neighboring triangles T'. We introduce on $\mathbb{R}^{\boldsymbol{T}}$ the scalar product

$$(u,v) \;=\; \sum_{T\in\boldsymbol{T}}|T\cap\Omega|\,u_T\cdot v_T.$$

It is simple to check that, with respect to this scalar product, the adjoint of M is given by

$$(M^*v)_T \;=\; M_T^*(v) \;=\; \sum_{T'\in\boldsymbol{T},T'\cap T\neq\emptyset}\frac{|T'\cap\Omega|}{S_{T'}}\,v_{T'},$$

where $S_T = \sum_{T'\cap T\neq\emptyset}|T'\cap\Omega|$.

For $u\in L^p(\Omega)$ and $\boldsymbol{T}\in\mathcal{T}_\varepsilon(\Omega)$, the new functional is

$$F_\varepsilon(u,\boldsymbol{T})=\begin{cases}\displaystyle\sum_{T\in\boldsymbol{T}}|T\cap\Omega|\frac{1}{h_T}f(h_T M_T^*(|\nabla u|^2)), & \text{if } u\in V_\varepsilon(\Omega), \boldsymbol{T}\in\mathcal{T}_\varepsilon(u),\\ +\infty, & \text{otherwise,}\end{cases} \tag{8}$$

and $F_\varepsilon(u)$ is defined on $L^p(\Omega)$ by a formula similar to (4). Then, if we let

$$F(u)=\begin{cases}\displaystyle\int_\Omega|\nabla u(x)|^2\,dx+3f_\infty\mathcal{H}^1(S_u), & \text{if } u\in L^p(\Omega)\cap GSBV(\Omega),\\ +\infty, & \text{if } u\in L^p(\Omega)\setminus GSBV(\Omega),\end{cases} \tag{9}$$

we have the following theorem, that holds for any $p\in[1,+\infty)$.

Theorem 1. *There exists $\Theta > 0$ such that if $\theta_0 \leq \Theta$, F_ε Γ-converges to F in $L^p(\Omega)$ as ε goes to zero.*

We do not know whether the upper bound Θ is the same as in the previous result of [13], however, both are larger than $18°$.

Remark. For technical reasons, the adjoint M^* of M has to be used in the definition (8) of $F_\varepsilon(u, \boldsymbol{T})$. If M were used instead, the Γ-limit of F_ε would be strictly below F, although the exact form is not clear. However, the example in Appendix C shows that the optimal triangulation one would have to use in this case in order to get a good estimate of the limiting energy is a very complex, "oscillating" triangulation, that it would be absurd, if not impossible, to try to build.

We finally state the following compactness result, that ensures that functional F can be approximated by means of functional F_ε in a "practical" sense.

Theorem 2. *Let $p \in [1, +\infty)$ and $(u^\varepsilon)_{\varepsilon>0}$ be a family of functions such that $u^\varepsilon \in V_\varepsilon(\Omega)$ for all ε and*

$$\sup_{\varepsilon>0} F_\varepsilon(u^\varepsilon) + \|u^\varepsilon\|_{L^p(\Omega)} < +\infty.$$

Then there exists $u \in GSBV(\Omega)$ and a subsequence u^{ε_j} converging to u a.e. in Ω, such that

$$F(u) \leq \liminf_{j\to\infty} F_{\varepsilon_j}(u^{\varepsilon_j}). \tag{10}$$

In particular, if $g \in L^p(\Omega)$ and for each $\varepsilon > 0$, u^ε is a solution of the problem

$$\min_{v\in L^p(\Omega)} F_\varepsilon(v) + \int_\Omega |v(x) - g(x)|^p \, dx, \tag{11}$$

then the limit u solves

$$\min_{v\in L^p(\Omega)} F(v) + \int_\Omega |v(x) - g(x)|^p \, dx, \tag{12}$$

and, if $p > 1$, the sequence u^{ε_j} strongly converges to u.

2. Numerical implementation

In this section, we describe the scheme we propose for minimizing

$$G(u) + \beta \int_\Omega |u(x) - g(x)|^2 \, dx, \tag{13}$$

where g is the original image and $\beta > 0$ a fixed parameter. Since G_ε Γ-converges to G [13], an approximation of the solution can be computed by minimizing the functional

$$G_\varepsilon(u) + \beta \int_\Omega |u(x) - g(x)|^2\, dx \tag{14}$$

for a "small enough" ε. Since G_ε depends on u but also on the triangulation on which u is defined, this problem is achieved by finding both a minimizing function u *and* an optimal triangulation $\boldsymbol{T}$, adapted to u, such that $(u, \boldsymbol{T})$ minimizes $G_\varepsilon(u, \boldsymbol{T}) + \beta \int_\Omega |u - g|^2\, dx$. A huge difficulty, in view of a finite element implementation, is the fact that the optimal mesh depends on the unknown solution that is to be computed.

2.1. Minimization method

When estimating the Γ-lim sup of G_ε, one has to build, given a function u, an optimal sequence of functions and their associated meshes $(u_\varepsilon, \boldsymbol{T}_\varepsilon)$ such that

$$\limsup_{\varepsilon \downarrow 0} G_\varepsilon(u_\varepsilon, \boldsymbol{T}_\varepsilon) \le G(u).$$

This is done in [13, Sect. 4], but the sequence built in Sect. 3.2 for the functionals F_ε, F could also be used for G_ε and G. If we knew in advance a minimizer u for (13) and its jump set S_u, these constructions would show us how to build the optimal triangulation for the approximated problem (14). This minimizer u being obviously unknown (since it is exactly what we are looking for), we propose to deduce some nearly optimal triangulation from a previously computed approximation u_ε, assuming that it is "close", in some sense, to u. The following iterative algorithm, that can also be seen as a relaxation algorithm between both unknown for (3) is then natural.

- *initialization* (background mesh generation):
  given ε_0, choose an arbitrary (regular) triangulation $\boldsymbol{T}_{\varepsilon_0}$.
- *iteration* i (minimization process):
  i. find u_i solving $\min_{u \in V_{\varepsilon_i}(\Omega)} G_{\varepsilon_i}(u, \boldsymbol{T}_{\varepsilon_i}) + \beta \int_\Omega |u - g|^2\, dx$
  ii. mesh adaption: build the mesh $\boldsymbol{T}_{\varepsilon_{i+1}}$, according to the function u_i and the choice of ε_{i+1} (that can be the same as ε_i).

In Sects. 2.1.1 and 2.1.2, we detail the method we use to achieve points *i* and *ii*. Note however that we do not know how to *really minimize* (14) with respect to the triangulation, and just *estimate* some triangulation that seems optimal, according to the construction in Sect. 3.2.

2.1.1. Minimization of (8) for a fixed $\boldsymbol{T}_\varepsilon$ In this section, we assume that a triangulation $\boldsymbol{T}_\varepsilon$ is given and show how to minimize $G_\varepsilon(u, \boldsymbol{T}_\varepsilon)$ with respect to u, for u, piecewise linear on each element $T \in \boldsymbol{T}_\varepsilon$ and continuous on Ω. Of course, the energies we are dealing with, that are strongly non convex, may have many local minimizers. We can never be sure that we will not compute one of these. However, the iterative algorithm we propose, which is classical in image reconstruction methods (see [23], and for instance [5]) ensures that the energy decreases at each iteration and converges to some critical value. It has been successfully used for other similar problems (see for instance [12]).

In what follows, we suppose that the function f is concave and differentiable and that $f(0) = 0$, which is a consequence of (2). Thus, extending f with the value $-\infty$ on $]-\infty, 0]$, $-f$ is convex and lower semi-continuous. Let

$$\psi(-v) \;=\; \sup_{t\in\mathbb{R}} tv - (-f)(t) \;=\; (-f)^*(v).$$

be the Legendre-Fenchel transform of f. By a classical result (see for example [18]), $(-f)^{**} = -f$, so that

$$-f(t) = \sup_{v\in\mathbb{R}} -\psi(-v) = \inf_{v\in\mathbb{R}} tv + \psi(v)$$

It is well known that the first sup in this equation is attained at v such that $t \in \partial(-f)^*(v)$ (the subdifferential of $(-f)^*$ at t), and that it is equivalent to $v \in \partial(-f)(t)$. Since $\partial(-f)(t) = \{-f'(t)\}$ for $t > 0$ and $]-\infty, -1]$ for $t = 0$, we deduce that the sup is reached at some $v \in [-1, 0]$ (since for $t = 0$ we check that $(-f)^*(-1) = 0$ and thus the sup is reached at $v = -1$). Hence,

$$f(t) = \min_{v\in[0,1]} tv + \psi(v)$$

and the min is reached for $v = f'(t)$. Given $\boldsymbol{T}_\varepsilon$, the minimization of (3) is then equivalent to that of

$$G'_\varepsilon(u, v, \boldsymbol{T}_\varepsilon) = \sum_{T\in\boldsymbol{T}_\varepsilon} |T \cap \Omega| \left(v_T|\nabla u_T|^2 + \frac{\psi(v_T)}{h_T} \right) \tag{15}$$

over all $u \in V_\varepsilon(\Omega)$ and $v = (v_T)_{T\in\boldsymbol{T}_\varepsilon}$, piecewise constant on each $T \in \boldsymbol{T}_\varepsilon$. This problem is still non linear and non convex, but for fixed u, the minimizer over each v is explicitly given by

$$v_T = f'(h_T|\nabla u_T|^2) \tag{16}$$

and the optimal u for fixed v solves an elliptic equation.

The use of an iterative method for the solving of (15) is then natural and our algorithm is:

i. initialization
Choose u_0 and v_0,
ii. iteration
fixed v, find $u \in V_\varepsilon(\Omega)$, minimizing

$$\sum_{T \in \boldsymbol{T}_\varepsilon} \int_{T \cap \Omega} v_T |\nabla u_T|^2 \, dx + \beta \int_\Omega |u - g|^2 \, dx. \tag{17}$$

Then, fixed u, compute the new v using equation (16).

In particular, note that we do not need to compute the Legendre-Fenchel transform of function f, neither for the minimization of (3) nor for its computation. The minimization with respect to v is explicit while the minimization with respect to u is a simple (linear) problem, since the energy is convex and quadratic. Its solving is achieved by the use of a standard finite element method, since the triangulation used is usually unstructured (i.e. not a grid), due to the mesh adaption process, described in the following section. Since the v field is piecewise constant, the solving of (17) is very efficient and doesn't require a complicated assembly procedure for the finite element matrix, compared to the method described in [10].

2.1.2. Mesh adaption For the generation of the adapted triangulation, we use the automatic mesh generator $\mathrm{BL2D}$, developed at the INRIA[1] (see [9] and [8] for details about anisotropic mesh generation).

Prior to describing the mesh adaption method, one has to introduce a few notions. The *background mesh* is an existing mesh that one wishes to adapt to a *foreground mesh*. The foreground mesh is built from the background mesh by the use of an *estimator* which consists in giving a *metric* at each point of the background mesh. This metric is defined by a symmetric definite positive 2×2 matrix A that identifies the points (x, y) at distance 1 from the reference point with the ellipsis $(x, y)A^{\mathrm{T}}(x, y) = 1$. Up to a rotation of angle θ and a translation, this ellipsis is described by the equation $x^2/h_1^2 + y^2/h_2^2 = 1$ (h_1^{-2}, h_2^{-2} being thus the eigenvalues of A). The three quantities (θ, h_1, h_2) are related to the orientation and anisotropy factor of the elements in the adapted triangulation. The foreground mesh is then built as a Delaunay triangulation, with respect to the metric, given at each point of the background triangulation. A complete description of the algorithms used for the building of such adapted meshes and of the theories involved is to be found in [9].

The *anisotropy ratio* of an element, defines the ratio between its smallest and its largest height, i.e. $R(T) = \min_{i \in \{1,2,3\}} (h_T/h_i)$, h_i , being the i^{th} height of T, the *orientation* of an element is that of its longest edge.

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[1] available at http://www-rocq.inria.fr/gamma/cdrom/www/bl2d/eng.htm

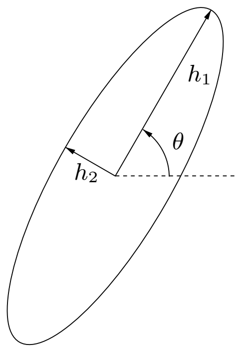
The optimal triangulation for problem (3) as described in Sect. 3.2 is such that “close” to the edge set the elements have an high anisotropy ratio and an orientation parallel to the edge, while “far” from the edges the elements may have an anisotropy ratio close to 1. This description is intentionally unprecise, since in the minimization process, we don’t have any description of the theoretical edge set. Thus, one has to build the estimator by the use of the functions u and v, computed on the background mesh.

The first idea is then to use the value of v, so as to set the anisotropy (h_1 and h_2) and the gradient of u for the angle θ, since v is supposed to be close to 0, near the edges and close to 1, otherwise, while the gradient of u, on the set where $v = 0$ should represent the normal to S_u. Unfortunately, this method causes several problems:

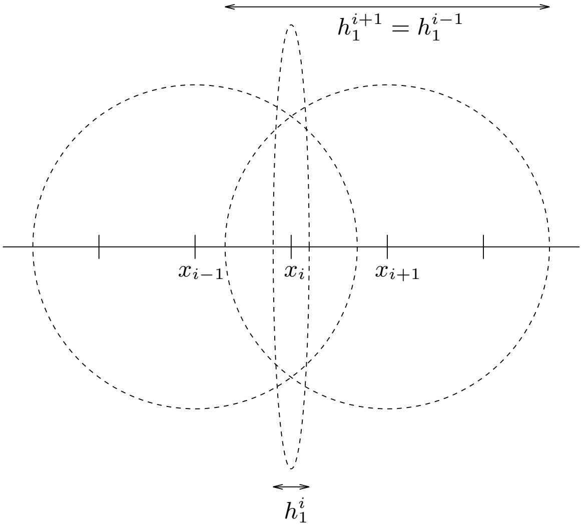
- A first technical problem is that one has to build the estimator at each node of the background mesh, while both v and ∇u are piecewise constant on $\boldsymbol{T}_\varepsilon$ and then not uniquely defined on the nodes.
- Another problem is the regularity of the fields v and ∇u: if (u, v) are minimizers for (3), then across the area where $v \simeq 0$, the gradient of u is oscillating and its direction is related more to the orientation of the elements rather than to the real orientation of the jump set. Indeed, u being nearly constant on each side of its jump, its gradient inside each triangle is perpendicular to the edge along which u is constant. If we use this information without care, the adapted triangulation will thus be too sensitive to the background mesh.
- Then, one needs the h_1 and h_2 values of the estimator to be smooth enough, to ensure that the adaption is feasible. This need can be easily illustrated in an unidimensional problem. Set $x^i = i.h$, the coordinate of the nodes of the 1D mesh, h, the mesh size. In that particular case, the metric for the estimator is defined by only one parameter, denoted by h_1^i. If $h_1^{i-1} = h_1^{i+1} \gg h$ and $h_1^i \ll h$ then one cannot build a triangulation with respect to the estimator. This problem is illustrated in Fig. 1(b).

The technique we use for preventing such effects is a regularization of the minimizers u and v. A possible method is to regularize v by iterating n times the operator M_T defined in equation (7), and to truncate it at an arbitrary lower value $V_{\mathrm{R}}^{\min}$: $v_{\mathrm{R}} = \big((M_T)^n(v) \vee V_{\mathrm{R}}^{\min}\big)$. Then, the regularized function u_{R} is computed by replacing v by v_{R} in the problem in u (17). With a good choice for n and $V_{\mathrm{R}}^{\min}$, it is then possible to deduce usable h_1 and h_2 from v_{R} and θ from ∇u_{R}. A typical choice for both parameter is $n \simeq 10$ and $V_{\mathrm{R}}^{\min} \simeq 0.5$.

The algorithm for the minimization of G_{ε_i} for a fixed mesh $\boldsymbol{T}_{\varepsilon_i}$ and the generation of $\boldsymbol{T}_{\varepsilon_{i+1}}$ is then:

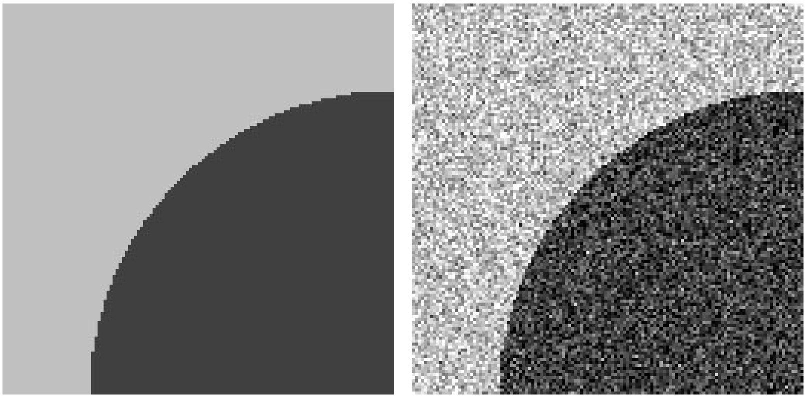


(a) The unit ball of the metric (θ, h_1, h_2) in the reference metric



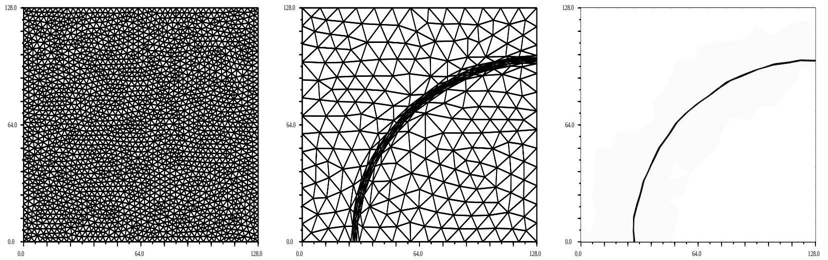
(b) An uncorrectly defined metric

Fig. 1. Estimators



(a) The original image (128×128) (b) Same image with noise

Fig. 2. Two artificial images to be segmented



(a) Background mesh (b) Adapted Mesh (c) Edge set v

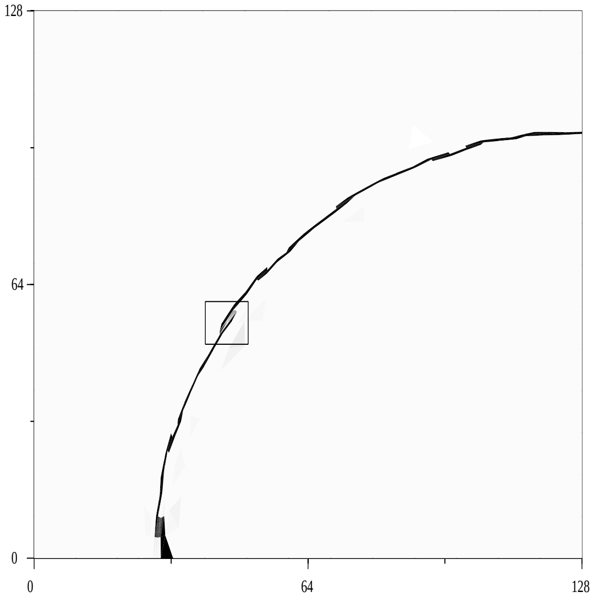
Fig. 3. Segmentation of Figure 2(a)

i. *initialization,*
set u_0 and v_0, possibly using the results of a previous fixed triangulation problem.

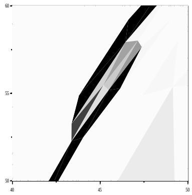
ii. *minimization,*
minimize (8), by solving iteratively the problems (17) and (16) until numerical convergence.

iii. *regularization,*
set u_{R} and v_{R}, as described above.

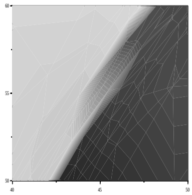
iv. *estimation,*
choose ε_{i+1} and deduce from ε_{i+1}, v_{R}, ∇u_{R}, and the direction of ∇u_{R} "good" values for (h_1, h_2, θ).



(a) The v field, given by equation (16)



(b) Detail from the previous figure



(c) Detail of the u field

Fig. 4. Segmentation of Figure 2(b)

v. *adaption,*
   run program BL2D with input values (h_1, h_2, θ) to build $\boldsymbol{T}_{\varepsilon_{i+1}}$, and restart from *i*.

2.2. Examples

In the following tests, we use $f(x) = \frac{2\alpha}{\pi} \arctan(\frac{\pi x}{2\alpha})$, so that the weight on the edge set is $f_\infty = \alpha > 0$.

The initial mesh used for all experiments is shown in Fig. 3(a).

In Fig. 3, we present the result of the segmentation of the image shown in Fig. 2(a) after 2 mesh adaption processes. The successive values (in pixels unit) for h_1 and h_2 are: (3.0,3.0) for the background mesh; (1.0,7.5) close to the edges and (10.0,10.0) far from the edges for the first adaption; (1.0,10.0) and (10.0,10.0) for the second adaption. The other parameters are $\alpha = 400$, $\beta = 0.05$. The edges are well focussed and at their right position. The theoretical surface energy (i.e., $\alpha\times$ the length of the jump) is 6.28×10^4 and the computed one is 6.46×10^4. The adapted mesh is shown on Fig. 3(b) and the field v on Fig. 3(c). Note that the number of nodes in the successive meshes are 2298, 523 and 341. Since the mesh can be coarse far from the edges, the mesh adaption process permits to reduce the number of nodes used at each iteration.

Figure 4 shows the results for g as in Fig. 2(b), with $\alpha = 200$ and $\beta = 0.05$, after 3 adaptions. The computed surface energy is 4.30×10^4 while the theoretical one should be 3.14×10^4. The edge set is broken at some points as shown in the detail 4(b). This is due to the very low noise sturdiness of the approximation G_ε. In some sense, the representation of the jump set in G_ε is very local. This could be seen as an advantage, since it should provide a more accurate estimate of the length of the jump set. Unfortunately, it also makes the edge detection more sensitive to noise so that the jump set that is detected is deformed. It also seems that the scheme gets easily stuck in local minima, and is very sensitive to the initial guesses u_0, v_0 and to mesh effects.

In order to reduce this numerical instability, we introduce in (8) the functional F_ε, which is a variant of G_ε in which a smoothing operator has been inserted.

2.3. *The "stabilized" version*

The algorithm for minimizing

$$F_\varepsilon(u, \boldsymbol{T}_\varepsilon) + \beta \int_\Omega |u(x) - g(x)|^2 \, dx \tag{18}$$

is similar to the one described above, with a few modifications. Indeed, we also introduce a piecewise constant field $v = (v_T)_{T\in\boldsymbol{T}_\varepsilon}$ and introduce the functional

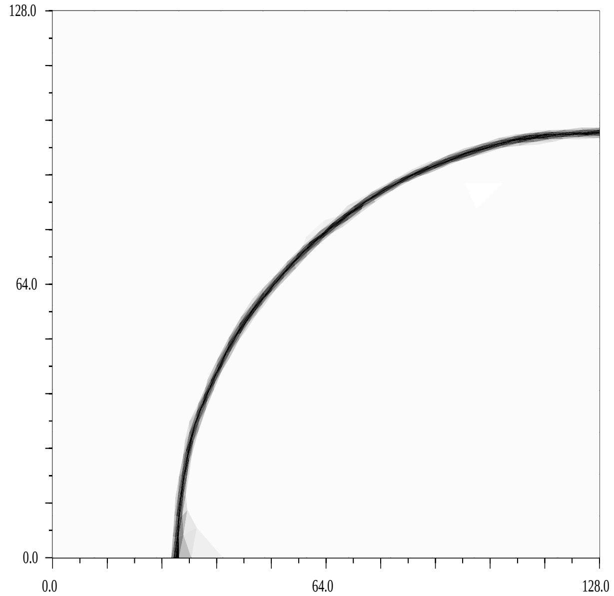
$$F'_\varepsilon(u, v, \boldsymbol{T}_\varepsilon) = \sum_{T\in\boldsymbol{T}_\varepsilon} |T \cap \Omega| \left(v_T M^*_T(|\nabla u|^2) + \frac{\psi(v_T)}{h_T} \right).$$

The minimization of $F'_\varepsilon(u, v, \boldsymbol{T}_\varepsilon) + \beta \int_\Omega |u - g|^2 \, dx$ over v is explicit and is given by $v_T = f'(h_T M^*_T(|\nabla u|^2))$. In order to perform the minimization with respect to u, we rewrite F'_ε in the following way

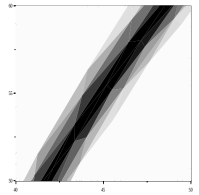
$$F'_\varepsilon(u, v, \boldsymbol{T}_\varepsilon) = \sum_{T\in\boldsymbol{T}_\varepsilon} |T \cap \Omega| \left(M_T(v)(|\nabla u_T|^2) + \frac{\psi(v_T)}{h_T} \right),$$

so that the problem is the same as minimizing (17), with the v field being replaced by the field $w = (w_T)_{T\in\boldsymbol{T}_\varepsilon}$, given by

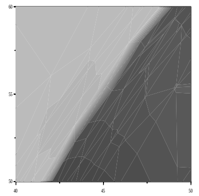
$$w_T = M_T(v) = M_T(f'(hM^*(|\nabla u|^2)). \tag{19}$$



(a) The $w_T = M_T(v)$ field, given by formula (19)

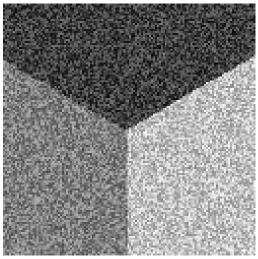


(b) Detail from the previous figure

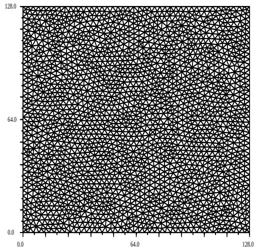


(c) Detail of the u field

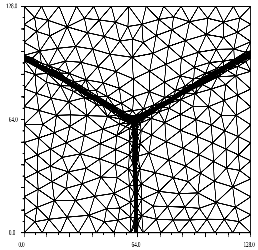
Fig. 5. Segmentation of Figure 2(b) with the approximation F_ε.



(a) Original image

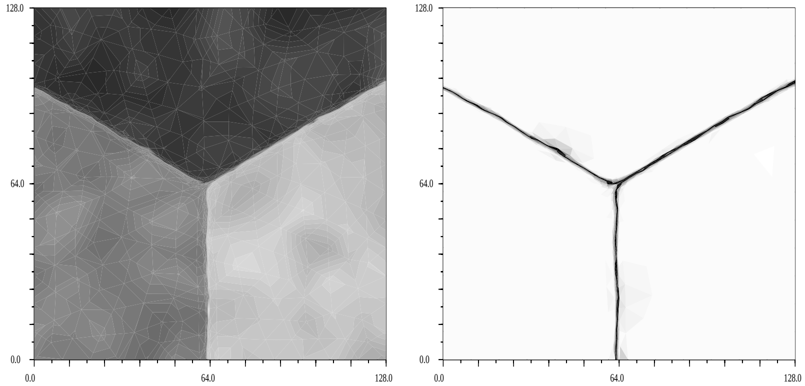


(b) Background mesh $\boldsymbol{T}_{\varepsilon_0}$



(c) Final mesh, $\boldsymbol{T}_{\varepsilon_3}$

Fig. 6. Illustration of the mesh adaption process



(a) Optimal u for $\boldsymbol{T}_{\varepsilon_0}$ (b) Optimal $w_T = M_T(v)$ for $\boldsymbol{T}_{\varepsilon_3}$

Fig. 7. The segmented image and its jump set

In Fig. 5, we show the results (for the same problem of Fig. 4) given by our "stabilized" functional, with the same parameter set. The computed surface energy, 3.30×10^4 is closer to the theoretical one than that computed with G_ε.

Remark. The use of the filters M and M^* in the computation of w can be compared to the filtering used in [28] for the stabilization of topology optimization algorithms.

Remark. It is to notice that one can combine both formulation into one by setting

$$E_\varepsilon(u, \boldsymbol{T}) = \sum_{T \in \boldsymbol{T}} \frac{|T \cap \Omega|}{h_T} \left\{ (1-\theta) f(h_T |\nabla u_T|^2) + \frac{\theta}{3} f\left(3 h_T M_T^*(|\nabla u|^2)\right) \right\} \tag{20}$$

if $u \in V_\varepsilon(\Omega)$, $\boldsymbol{T} \in \mathcal{T}_\varepsilon(u)$, and $E_\varepsilon(u, \boldsymbol{T}) = +\infty$ otherwise. (We then define $E_\varepsilon(u)$ as in (4).) The proof of the Γ-convergence of (20) to $G(u)$ is a simple adaption of the proof given for F_ε, since the sequence built in Sect. 3.2 for the estimate from above of the Γ-lim sup of F_ε also suits to G_ε (and thus to E_ε). With this third functional, one can, in essence, control the width of the regularization operator M_T.

This functional E_ε seems to give better results when used with a parameter θ close to 0.5 (i.e., when E_ε is "halfway" between G_ε and F_ε). In the following Table 1, we compare the values of the total energy $E_\varepsilon(u) +$

Table 1. Computed energies for various choices of v_0

v_0	θ	E_ε	Surface energy	$\beta\|u-g\|^2_{L^2(\Omega)}$	Total energy
$v_0 \equiv 1$	1.00	5.84×10^4	3.30×10^4	2.13×10^6	2.19×10^6
$v_0 \equiv 1$	0.75	5.97×10^4	3.34×10^4	2.13×10^6	2.18×10^6
$v_0 \equiv 1$	0.50	5.89×10^4	3.27×10^4	2.10×10^6	2.16×10^6
$v_0 \equiv 1$	0.25	6.20×10^4	3.65×10^4	2.14×10^6	2.20×10^6
$v_0 \equiv 1$	0.00	6.83×10^4	4.30×10^4	2.24×10^6	2.31×10^6
$v_0 \in [0.5,1]$	1.00	5.86×10^4	3.24×10^4	2.14×10^6	2.19×10^6
$v_0 \in [0.5,1]$	0.75	5.98×10^4	3.34×10^4	2.11×10^6	2.17×10^6
$v_0 \in [0.5,1]$	0.50	5.86×10^4	3.24×10^4	2.10×10^6	2.16×10^6
$v_0 \in [0.5,1]$	0.25	6.87×10^4	4.18×10^4	2.15×10^6	2.22×10^6
$v_0 \in [0.5,1]$	0.00	6.72×10^4	4.15×10^4	2.22×10^6	2.29×10^6
$v_0 \in [0,1]$	1.00	6.13×10^4	3.62×10^4	2.15×10^6	2.21×10^6
$v_0 \in [0,1]$	0.75	6.08×10^4	3.41×10^4	2.16×10^6	2.23×10^6
$v_0 \in [0,1]$	0.50	6.23×10^4	3.62×10^4	2.19×10^6	2.25×10^6
$v_0 \in [0,1]$	0.25	6.53×10^4	3.97×10^4	2.19×10^6	2.25×10^6
$v_0 \in [0,1]$	0.00	7.52×10^4	4.82×10^4	2.28×10^6	2.36×10^6

$\beta\int_\Omega |u-g|^2\,dx$ and of various other energies for different values of θ. The same computation has been made for various choices of the initial guess v_0 for v, respectively $v_0 \equiv 1$, v_0 randomly chosen in $[0.5,1]$ and v_0 randomly chosen in $[0,1]$. The problem is the segmentation of the image shown in Figure 2(b). The mesh is the same for all experiments, it is the mesh generated after three adaptions with $\theta = 1$. The idea is to test the sensitivity of the algorithm with respect to the initial values. In order to give an estimate of the length $\mathcal{H}^1(S_u)$ of the approximated solution we compute a "surface energy" as the sum of $|T|/h_T$ over all the triangles T where $h_T|\nabla u_T|^2 \geq \alpha$. This is exactly the surface energy that would be measured by energy G_ε in the case where $f(x) = \min(|x|,\alpha)$, and can be compared with the theoretical expected value $\alpha\times 50\pi \simeq 3.14\times10^4$. In every case, the stabilized functional F_ε gives better results than the original one G_ε. By introducing a diffusion operator, we decreased the sensitiveness to the initial guess, without losing accuracy either on the surface energy or on the focalization of the edges. Functional E_ε is a good compromise between F_ε and G_ε since it can give a better approximation of the energies than F_ε (see the case $v_0 \equiv 1$), even if it is more sensitive on the initial guess (see the case $v_0 \in [0,1]$).

The last example is the segmentation of a "T junction" with 20 percent of additive noise, shown in Fig. 6(a). To adapt the mesh at the junction is difficult: if the anisotropy ratio is high, the set of "flat" elements cannot be curved enough so as to follow the edge while a smaller ratio prevents a good approximation of the surface energy. An improvement to the way we adapt the mesh would then be to add in some way the local curvature of the edge set. The first and last mesh are shown in Fig. 6(b) and 6(c). The segmented image and its edge set are in Fig. 7(a) and 7(b). The parameters are $\alpha = 75$, $\beta = 0.075$, $\theta = 0.5$; and the successive values of h_1 and h_2 are (1,5), (0.75, 3.75), (0.5, 3.75), close to the edges and (10,10), (7.5,7.5), (10,10) far from the edges.

The next section is devoted to the mathematical proofs of the convergence results.

3. Proof of the convergence results

In Sects. 3.1 and 3.2 we first show Theorem 1. We will show, in a standard way, that the Γ-$\liminf$ F' of F_ε satisfies $F' \geq F$, while the Γ-$\limsup$ F'' is less than F. Then, in Sect. 3.3, we deduce Theorem 2.

3.1. Estimate from below of the Γ-lim inf

For every open set $A \subseteq \Omega$ and every $\varepsilon > 0$, we define

$$F_\varepsilon(u,\boldsymbol{T},A)=\begin{cases}\displaystyle\sum_{T\in\boldsymbol{T}} |T\cap A|\frac{1}{h_T}f(h_T M_T^*(|\nabla u|^2)), & \text{if } u\in V_\varepsilon(\Omega), \boldsymbol{T}\in\mathcal{T}_\varepsilon(u),\\ +\infty, & \text{otherwise,}\end{cases} \tag{21}$$

and we let $F_\varepsilon(u,A) = \inf_{\boldsymbol{T}\in\mathcal{T}_\varepsilon(u)} F_\varepsilon(u,\boldsymbol{T},A)$. We choose a sequence $(\varepsilon_j)_{j\geq 1}$ with $\varepsilon_j \downarrow 0$ as $j \to \infty$, and set for every $u \in L^p(\Omega)$ and every open set A

$$F'(u,A) = \Gamma - \liminf_{j\to\infty} F_{\varepsilon_j}(u,A).$$

In order to prove that $F'(u,\Omega)$ is finite if and only if $u \in L^p(\Omega) \cap GSBV(\Omega)$, and that in this case

$$F'(u,\Omega) \geq \int_\Omega |\nabla u(x)|^2\,dx + 3f_\infty\mathcal{H}^1(S_u), \tag{22}$$

we will use the same localization technique as in [4,11,13]. The result will be a consequence of the following lemma.

Lemma 1. *Let $A \subseteq \Omega$ be an open set and $u \in L^p(\Omega)$, and assume $F'(u,A) < +\infty$. Then, $u \in GSBV(A)$ and*

$$F'(u,A) \geq \int_A |\nabla u(x)|^2\,dx, \tag{23}$$

$$F'(u,A) \geq 3f_\infty\mathcal{H}^1(A\cap S_u). \tag{24}$$

Arguing for instance as in [11, Prop. 6.5], we immediately deduce inequality (22) from Lemma 1, since it can easily be shown that given $u \in L^p(\Omega)$, the set functions $A \mapsto F'(u,A)$ are increasing and superadditive.

3.1.1. Proof of (23) Choose an open set $A \subseteq \Omega$ and a sequence $(u^{\varepsilon_j}, \boldsymbol{T}^{\varepsilon_j})$ such that $u^{\varepsilon_j} \to u$ in $L^p(\Omega)$ and $\liminf_{j\to\infty} F_{\varepsilon_j}(u^{\varepsilon_j}, \boldsymbol{T}^{\varepsilon_j}, A) < +\infty$. For simplicity's sake we will drop the subscript j and write $\varepsilon \downarrow 0$ instead of $j \to \infty$. We fix a positive constant κ, and write, taking into account the fact that f is nondecreasing,

$$F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A) \geq \sum_{T\subset A} \frac{|T|}{h_T} f\left(h_T M_T^*\left(|\nabla u^\varepsilon|^2 \wedge \frac{\kappa}{\varepsilon}\right)\right). \tag{25}$$

For every $T \in \boldsymbol{T}^\varepsilon$, we have the following estimates:

$$\frac{1}{2}\varepsilon^2 \sin\theta_0 \leq |T| \leq \frac{\sqrt{3}}{2}c^2\varepsilon^2, \tag{26}$$

and

$$\frac{1}{2}\varepsilon^2 \sin 2\theta_0 \leq S_T \leq \pi\left(1+\frac{\sqrt{3}}{3}\right)^2 c^2\varepsilon^2 \tag{27}$$

as soon as T or some T' with $T'\cap T \neq \emptyset$ is included in Ω; moreover,

$$\varepsilon \sin\theta_0 \leq h_T \leq \frac{\sqrt{3}}{2}c\varepsilon. \tag{28}$$

In particular, if $T \subset A \subseteq \Omega$,

$$\begin{aligned} M_T^*\left(|\nabla u^\varepsilon|^2 \wedge \frac{\kappa}{\varepsilon}\right) &\leq \frac{1}{S_T}\sum_{T'\cap T\neq\emptyset} \frac{S_T}{S_{T'}}|T'\cap\Omega|\,\frac{\kappa}{\varepsilon} \\ &\leq \frac{2\pi c^2}{\sin 2\theta_0}\left(1+\frac{\sqrt{3}}{3}\right)^2 \frac{\kappa}{\varepsilon}, \end{aligned}$$

so that there exists some constant K depending only on θ_0, c such that

$$h_T M_T^*\left(|\nabla u^\varepsilon|^2 \wedge \frac{\kappa}{\varepsilon}\right) \leq K\kappa. \tag{29}$$

Let $b_\kappa = \inf_{0<t\leq K\kappa} f(t)/t$, by (1) we know that $b_\kappa \to 1$ as κ goes to zero, and from (25), (29) we get

$$F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A) \geq b_\kappa \sum_{T\subset A} |T| M_T^*\left(|\nabla u^\varepsilon|^2 \wedge \frac{\kappa}{\varepsilon}\right). \tag{30}$$

Let now $\boldsymbol{T}_A^\varepsilon$ be the set of all triangles $T \in \boldsymbol{T}^\varepsilon$ such that every triangle $T' \in \boldsymbol{T}^\varepsilon$, $T'\cap T \neq \emptyset$, lies in A. We have from (30)

$$F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A) \geq b_\kappa \sum_{T\subset A} |T| \sum_{T'\cap T\neq\emptyset} \frac{|T'\cap\Omega|}{S_{T'}}\left(|\nabla u^\varepsilon_{T'}|^2 \wedge \frac{\kappa}{\varepsilon}\right)$$

$$
\begin{aligned}
&= b_\kappa \sum_{T' \in \boldsymbol{T}^\varepsilon} |T' \cap \Omega| \left(|\nabla u^\varepsilon_{T'}|^2 \wedge \frac{\kappa}{\varepsilon} \right) \frac{1}{S_{T'}} \sum_{T \cap T' \neq \emptyset, T \subset A} |T| \\
&\geq b_\kappa \sum_{T' \in \boldsymbol{T}^\varepsilon_A} |T'| \left(|\nabla u^\varepsilon_{T'}|^2 \wedge \frac{\kappa}{\varepsilon} \right) \frac{1}{S_{T'}} \sum_{T \cap T' \neq \emptyset} |T| \\
&= b_\kappa \sum_{T \in \boldsymbol{T}^\varepsilon_A} |T| \left(|\nabla u^\varepsilon_T|^2 \wedge \frac{\kappa}{\varepsilon} \right),
\end{aligned}
$$

so that, letting $\boldsymbol{T}^\varepsilon_\kappa = \{T \in \boldsymbol{T}^\varepsilon \ : \ |\nabla u^\varepsilon_T|^2 > \kappa/\varepsilon\}$,

$$
F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A) \geq b_\kappa \sum_{T \in \boldsymbol{T}^\varepsilon_A \setminus \boldsymbol{T}^\varepsilon_\kappa} |T||\nabla u^\varepsilon_T|^2 + b_\kappa \kappa \sum_{T \in \boldsymbol{T}^\varepsilon_A \cap \boldsymbol{T}^\varepsilon_\kappa} \frac{|T|}{\varepsilon} \tag{31}
$$

Let $A(\varepsilon) = \bigcup_{T \in \boldsymbol{T}^\varepsilon_A \cap \boldsymbol{T}^\varepsilon_\kappa} T \subset A$ and set $v^\varepsilon(x) = (1 - \chi_{A(\varepsilon)}(x))u^\varepsilon(x)$ for all $x \in \Omega$. We can assume that $u^\varepsilon(x) \to u(x)$ for almost every $x \in \Omega$ as $\varepsilon \downarrow 0$. Since by (31), $|A(\varepsilon)| \leq (F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A)/b_\kappa \kappa)\varepsilon$, we can also assume (up to a subsequence) that $\chi_{A(\varepsilon)}(x) \to 0$ for almost every $x \in \Omega$. Then, $v^\varepsilon \to u$ a.e. in Ω. The function v^ε belongs to $SBV(\Omega)$, and its jumps set satisfies $S_{v^\varepsilon} \subseteq \bigcup_{T \in \boldsymbol{T}^\varepsilon_A \cap \boldsymbol{T}^\varepsilon_\kappa} \partial T$. Since for any $T \in \boldsymbol{T}^\varepsilon$,

$$
\mathcal{H}^1(\partial T) \leq 6 \frac{|T|}{h_T} \leq \frac{6}{\sin \theta_0} \frac{|T|}{\varepsilon}, \tag{32}
$$

we deduce from (31) that

$$
\mathcal{H}^1(S_{v^\varepsilon}) \leq \frac{6}{b_\kappa \kappa \sin \theta_0} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A), \tag{33}
$$

and is thus bounded. If we fix now $\delta > 0$ and let $A^\delta = \{x \in A \ : \ \mathrm{dist}(x, A) > \delta\}$, it is clear that if ε is small enough, $A^\delta \subseteq \bigcup_{T \in \boldsymbol{T}^\varepsilon_A} T$, so that we also have from (31)

$$
b_\kappa \int_{A^\delta} |\nabla v^\varepsilon(x)|^2 \, dx \leq F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A). \tag{34}
$$

Since $\|v^\varepsilon\|_{L^p(\Omega)} \leq \|u^\varepsilon\|_{L^p(\Omega)}$ is also bounded, we can invoke Ambrosio's Theorem 3 to conclude that $u \in GSBV(A^\delta)$, with

$$
\mathcal{H}^1(S_u \cap A^\delta) \leq \frac{6}{b_\kappa \kappa \sin \theta_0} \liminf_{\varepsilon \downarrow 0} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A),
$$

and

$$
b_\kappa \int_{A^\delta} |\nabla u(x)|^2 \, dx \leq \liminf_{\varepsilon \downarrow 0} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A). \tag{35}
$$

Sending δ to zero, we deduce that $u \in GSBV(A)$, with $\mathcal{H}^1(S_u \cap A) < +\infty$, and sending then κ to zero in (35) we get

$$\int_A |\nabla u(x)|^2\, dx \le \liminf_{\varepsilon\downarrow 0} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A). \tag{36}$$

Since the sequence $(u^\varepsilon, \boldsymbol{T}^\varepsilon)$ was arbitrary, we deduce (23).

3.1.2. Proof of (24) The proof of inequality (24) is similar. We choose again A, u^ε, $\boldsymbol{T}^\varepsilon$ as in the previous section. $\boldsymbol{T}^\varepsilon_A$ and $\boldsymbol{T}^\varepsilon_\kappa$ are defined in the same way, and we also let $\widehat{\boldsymbol{T}}^\varepsilon_\kappa \supset \boldsymbol{T}^\varepsilon_\kappa$ be the set of triangles $T \in \boldsymbol{T}^\varepsilon$ such that some $T' \in \boldsymbol{T}^\varepsilon_\kappa$ satisfies $T' \cap T \neq \emptyset$. We now define $\tilde{\boldsymbol{T}}^\varepsilon_\kappa$ as the set of triangles $T \in \boldsymbol{T}^\varepsilon$ such that, along at least two edges of T, the slope of u^ε is (strictly) greater than $\sqrt{\frac{\kappa}{\varepsilon}}$. Clearly, for such a triangle, $|\nabla u^\varepsilon_T|^2 > \frac{\kappa}{\varepsilon}$, so that $\tilde{\boldsymbol{T}}^\varepsilon_\kappa \subset \boldsymbol{T}^\varepsilon_\kappa \subset \widehat{\boldsymbol{T}}^\varepsilon_\kappa$. If $T \in \widehat{\boldsymbol{T}}^\varepsilon_\kappa$, and if for all T', $T' \cap T \neq \emptyset \Rightarrow T' \subset \Omega$, we have, using (26), (27) and (28),

$$h_T M^*_T(|\nabla u^\varepsilon|^2) \ge \frac{\sin\theta_0 \sin 2\theta_0}{\left(1+\frac{\sqrt{3}}{3}\right)^2} \frac{\kappa}{2\pi c^2} = K'\kappa,$$

so that, for such a T,

$$f(h_T M^*_T(|\nabla u^\varepsilon|^2)) \ge a_\kappa$$

for some constant $a_\kappa = f(K'\kappa)$ that goes to f_∞ as κ goes to infinity. Thus,

$$F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A) \ge a_\kappa \sum_{T \in \widehat{\boldsymbol{T}}^\varepsilon_\kappa \cap \boldsymbol{T}^\varepsilon_A} \frac{|T|}{h_T} \tag{37}$$

In the same way as in the previous section, we let

$$B(\varepsilon) = \bigcup_{T \in \boldsymbol{T}^\varepsilon_A \cap \tilde{\boldsymbol{T}}^\varepsilon_\kappa} T \qquad \text{and} \qquad w^\varepsilon(x) = (1-\chi_{B(\varepsilon)}(x))u^\varepsilon(x),$$

$$D(\varepsilon) = \bigcup_{T \in \boldsymbol{T}^\varepsilon_A \cap \widehat{\boldsymbol{T}}^\varepsilon_\kappa} T \qquad \text{and} \qquad r^\varepsilon(x) = (1-\chi_{D(\varepsilon)}(x))u^\varepsilon(x).$$

We also let $C(\varepsilon)$ be the union of all the triangles in $\boldsymbol{T}^\varepsilon_A \cap \boldsymbol{T}^\varepsilon_\kappa$, plus the triangles in $\boldsymbol{T}^\varepsilon_A \cap \widehat{\boldsymbol{T}}^\varepsilon_\kappa \setminus \boldsymbol{T}^\varepsilon_k$ that have at least two edges that belong to triangles of $\boldsymbol{T}^\varepsilon_\kappa$. This set $C(\varepsilon)$ is thus slightly larger than the set $A(\varepsilon)$ of the previous section. We let $q^\varepsilon(x) = (1-\chi_{C(\varepsilon)}(x))u^\varepsilon(x)$ for all $x \in \Omega$. We have $B(\varepsilon) \subset C(\varepsilon) \subset D(\varepsilon)$ and by (37), $|D(\varepsilon)| \to 0$ as $\varepsilon \downarrow 0$, so that we may assume that w^ε, q^ε and r^ε go to u a.e. in Ω.

If $T \notin \tilde{\boldsymbol{T}}^{\varepsilon}_{\kappa}$, the slope of u^{ε} along at least two edges of T is less than $\sqrt{\frac{\kappa}{\varepsilon}}$, so that it is not difficult to check (see [13, Remark 3.5]) that

$$|\nabla u^{\varepsilon}_{T}|^2 \le \frac{5}{\sin\theta_0}\frac{\kappa}{\varepsilon}.$$

Letting $\kappa' = 5\kappa/\sin\theta_0$, we deduce, as in the previous section, that for any $\delta > 0$ and ε small enough,

$$b_{\kappa'}\int_{A^{\delta}} |\nabla w^{\varepsilon}(x)|^2\,dx \le F_{\varepsilon}(u^{\varepsilon}, \boldsymbol{T}^{\varepsilon}, A) \tag{38}$$

for some constant $b_{\kappa'} > 0$ (not depending on δ nor ε), and obviously the same inequality holds for q^{ε} and r^{ε}.

We now estimate the length of $\partial B(\varepsilon)\cap A^{\delta}$, $\partial C(\varepsilon)\cap A^{\delta}$ and $\partial D(\varepsilon)\cap A^{\delta}$, for ε small enough.

Consider first a triangle $T \in \tilde{\boldsymbol{T}}^{\varepsilon}_{\kappa}$. If (part of) an edge L of T belongs to $\partial B(\varepsilon)\cap A^{\delta}$, then, if the slope of u^{ε} along L is smaller than $\sqrt{\frac{\kappa}{\varepsilon}}$, we use the estimate $\mathcal{H}^1(L) \le 2|T|/h_T$, otherwise, L is the edge common to T an another triangle T' such that $T' \notin \tilde{\boldsymbol{T}}^{\varepsilon}_{\kappa}$, but since the slope of u^{ε} along L is larger than $\sqrt{\frac{\kappa}{\varepsilon}}$, $T' \in \boldsymbol{T}^{\varepsilon}_{\kappa}$, and we estimate $\mathcal{H}^1(L) \le 2|T'|/h_{T'}$. Since the first case occurs for at most one edge of T (by definition of $\tilde{\boldsymbol{T}}^{\varepsilon}_{\kappa}$), and since the triangle T' in the second case can only be considered once (otherwise the slope of u^{ε} along at least two edges of T' would be larger than $\sqrt{\frac{\kappa}{\varepsilon}}$ and we would have $T' \in \tilde{\boldsymbol{T}}^{\varepsilon}_{\kappa}$), we conclude that (if ε is small enough, so that $T\cap A^{\delta} \neq \emptyset \Rightarrow T \in \boldsymbol{T}^{\varepsilon}_{A}$)

$$\mathcal{H}^1(\partial B(\varepsilon)\cap A^{\delta}) \le 2\sum_{T\in \boldsymbol{T}^{\varepsilon}_{A}\cap \boldsymbol{T}^{\varepsilon}_{k}} \frac{|T|}{h_T}. \tag{39}$$

Consider now an edge L which is part of the boundary of $C(\varepsilon)$ (inside A^{δ}). Then, there are two possible cases:

i. L is the edge of a triangle $T \subset C(\varepsilon)$, $T \in \widehat{\boldsymbol{T}}^{\varepsilon}_{\kappa} \setminus \boldsymbol{T}^{\varepsilon}_{\kappa}$, and each of the two other edges of T is the edge of another triangle that belongs to $\boldsymbol{T}^{\varepsilon}_{\kappa}$. In this case we estimate the length of L with $2|T|/h_T$, and the two other edges of T are inside $C(\varepsilon)$ so that they can not be part of the boundary of neither $C(\varepsilon)$ nor $D(\varepsilon)$.

ii. L is an edge common to a triangle $T \in \widehat{\boldsymbol{T}}^{\varepsilon}_{\kappa} \setminus \boldsymbol{T}^{\varepsilon}_{\kappa}$ and a triangle $T' \in \boldsymbol{T}^{\varepsilon}_{\kappa}$, with therefore $T' \subset C(\varepsilon)$ and $\mathring{T}\cap C(\varepsilon) = \emptyset$. In this case, the two other edges of T can not be on the boundary of $D(\varepsilon)$ (since they share a vertex with T' and thus each one is common to T and another triangle of $\widehat{\boldsymbol{T}}^{\varepsilon}_{\kappa}$), and can not either be common to T and another triangle of $\boldsymbol{T}^{\varepsilon}_{\kappa}$, otherwise, by definition of $C(\varepsilon)$, we would have $T\cup T' \subset C(\varepsilon)$ and L would not be on the boundary. Therefore, if one of these edges is part of the boundary of $C(\varepsilon)$, it must enter the previous case (*i*) and its length can already be estimated by the surface of the adjacent triangle included in $C(\varepsilon)$. We estimate $\mathcal{H}^1(L) \le 2|T|/h_T$.

We deduce that $\mathcal{H}^1(\partial C(\varepsilon) \cap A^\delta)$ can be bounded by a sum $2\sum_T |T|/h_T$ that is taken on triangles T of $\widehat{\boldsymbol{T}}^\varepsilon_\kappa \setminus \boldsymbol{T}^\varepsilon_\kappa$ such that none of the three edges of T coincides with some part of $\partial D(\varepsilon)$ (or, more precisely, such that $\mathcal{H}^1(\partial T \cap \partial D(\varepsilon) \cap A^\delta) = 0$).

On the other hand, if L is the edge of some triangle $T \in \widehat{\boldsymbol{T}}^\varepsilon_\kappa \setminus \boldsymbol{T}^\varepsilon_\kappa$ such that (part of) L belongs to the boundary of $D(\varepsilon)$, we easily show that the two other edges of T are not part of $\partial D(\varepsilon) \cap A^\delta$. Indeed, if $T = [x, y, z]$ with $L = [x, y]$, then z must also be a vertex of some triangle of $\boldsymbol{T}^\varepsilon_k$ (otherwise it would be the case for either x or y and L could not be on the boundary of $D(\varepsilon)$), therefore neither $[x, z]$ nor $[y, z]$ can be on the boundary of $D(\varepsilon)$ (nor on the boundary of $C(\varepsilon)$, in fact). We still estimate $\mathcal{H}^1(L) \le 2|T|/h_T$. We deduce, if ε is small enough, that

$$\mathcal{H}^1(\partial C(\varepsilon) \cap A^\delta) + \mathcal{H}^1(\partial D(\varepsilon) \cap A^\delta) \le 2 \sum_{T \in \boldsymbol{T}^\varepsilon_A \cap \widehat{\boldsymbol{T}}^\varepsilon_\kappa \setminus \boldsymbol{T}^\varepsilon_k} \frac{|T|}{h_T}. \tag{40}$$

Thus, with (37), (39) and (40),

$$\begin{aligned} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A) \ge \frac{a_\kappa}{2} \ & \Big\{ \mathcal{H}^1(\partial B(\varepsilon) \cap A^\delta) \\ & + \mathcal{H}^1(\partial C(\varepsilon) \cap A^\delta) + \mathcal{H}^1(\partial D(\varepsilon) \cap A^\delta) \Big\}. \end{aligned} \tag{41}$$

Since $\|w^\varepsilon\|_{L^p(\Omega)} \le \|q^\varepsilon\|_{L^p(\Omega)} \le \|r^\varepsilon\|_{L^p(\Omega)} \le \|u^\varepsilon\|_{L^p(\Omega)}$, which is uniformly bounded, we may invoke Lemma 2 (see Appendix 3.3) to get that

$$\mathcal{H}^1(S_u \cap A^\delta) \le \frac{1}{2} \liminf_{\varepsilon \downarrow 0} \mathcal{H}^1(\partial X(\varepsilon) \cap A^\delta),$$

where $X(\varepsilon)$ is anyone of the sets $B(\varepsilon)$, $C(\varepsilon)$ or $D(\varepsilon)$. We conclude from (41) that

$$3a_\kappa \mathcal{H}^1(S_u \cap A^\delta) \le \liminf_{\varepsilon \downarrow 0} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A).$$

Sending δ to zero, and then κ to infinity, we deduce

$$3f_\infty \mathcal{H}^1(S_u \cap A) \le \liminf_{\varepsilon \downarrow 0} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon, A), \tag{42}$$

and by the arbitrariness of the sequence $(u^\varepsilon, \boldsymbol{T}^\varepsilon)$ we get (24). The proof of Lemma 1 is thus achieved.

Remark. Inequalities (36) and (42) were proved using only the assumptions that u^ε is bounded in $L^p(\Omega)$ and that it goes to u a.e. in Ω. They still hold if

u^ε does not converge strongly to u, but only weakly in $L^p(\Omega)$, and satisfy these assumptions. We deduce that, in the same way as (22) is deduced from (23) and (24), $u \in GSBV(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^2\,dx + 3f_\infty \mathcal{H}^1(S_u) \le \liminf_{\varepsilon\downarrow 0} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon) \tag{43}$$

as soon as u^ε is bounded in $L^1(\Omega)$ and goes to u a.e.in Ω. This will be useful in the proof of Theorem 2.

It remains to estimate the Γ-$\limsup$ F'' of F_ε.

3.2. *Estimate from above of the Γ-lim sup*

We now wish to prove that for any $u \in L^p(\Omega)$, there exists $(u^\varepsilon)_{\varepsilon>0}$ converging to u in $L^p(\Omega)$ with

$$\limsup_{\varepsilon\downarrow 0} F_\varepsilon(u^\varepsilon) \le F(u) \tag{44}$$

(more exactly, we should find for any sequence (ε_j) with $\varepsilon_j \downarrow 0$ as $j \to \infty$ a sequence $(u^{\varepsilon_j})_{j\ge 1}$ with $\limsup_{j\to\infty} F^{\varepsilon_j}(u^{\varepsilon_j}) \le F(u)$).

Arguing like in [14], where the approximation result of Dibos and Séré [17] (see Appendix 3.3) is generalized, we can find, given any $u \in L^p(\Omega)$ with $F(u) < +\infty$, a sequence $(u_n)_{n\ge 1}$ of bounded functions such that

- each u^n is defined on some $\Omega'_n \supset\supset \Omega$;
- for every n, there exist disjoint closed segments L_i^n, $i = 1, \dots, N_n$, such that $u_n \in W^{2,\infty}(\Omega'_n \setminus \cup_{i=1}^{N_n} L_i^n)$;
- for every n, $\|u_n\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$;
- as n goes to infinity, $u_n \to u$ in $L^p(\Omega)$ and

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n(x)|^2\,dx + 3f_\infty \sum_{i=1}^{N_n} \mathcal{H}^1(L_i^n) \le F(u).$$

The assumption that $\partial\Omega$ is Lipschitz is crucial in order to establish the existence of the functions u_n in a larger domain Ω'_n. Now, by a standard diagonalization argument, if for each n we find a sequence $(u_n^\varepsilon)_{\varepsilon>0}$, converging to u_n in $L^p(\Omega)$ as ε goes to zero, such that

$$\limsup_{\varepsilon\downarrow 0} F_\varepsilon(u_n^\varepsilon) \le \int_\Omega |\nabla u_n(x)|^2\,dx + 3f_\infty \sum_{i=1}^{N_n} \mathcal{H}^1(L_i^n) \tag{45}$$

then we will be able to build the sequence (u^ε) satisfying (44).

We fix $n \ge 1$ and now describe how to build the sequence $(u_n^\varepsilon)_{\varepsilon>0}$. In order to simplify the notations we will drop all subscripts (superscripts)

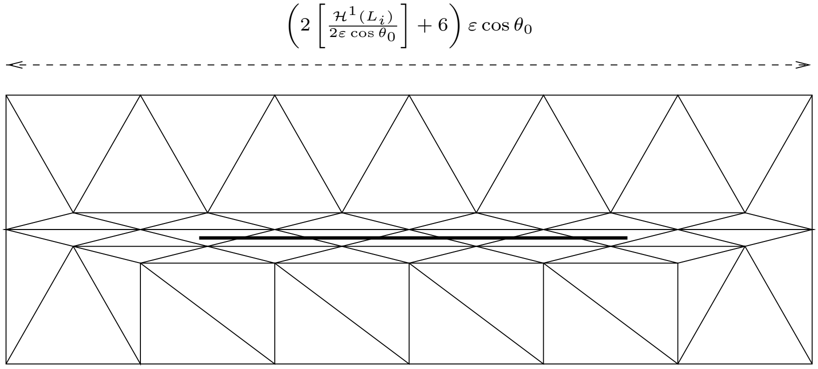


Fig. 8. Each discontinuity L_i (thick line) is covered by a strip of $6N_i + 13$ "minimal" triangles ($N_i = [\mathcal{H}^1(L_i)/\varepsilon']$), the whole strip is included in a rectangle whose external nodes have to be connected to a "background triangulation"

n. The function u is thus a piecewise regular function, defined on an open domain $\Omega' \supset\supset \Omega$, and such that there exist N disjoint closed segments $(L_i)_{i=1}^N$ with $u \in W^{2,\infty}(\Omega' \setminus \cup_{i=1}^N L_i)$.

The construction we use in order to build the sequence $(u^\varepsilon)_{\varepsilon>0}$ satisfying (45) is almost identical to the construction in [13, Appendix], so that we will not enter too much into the details. The idea is to build and connect together local triangulations designed around each discontinuity L_i, in such a way that the energy F_ε around L_i gives a good estimate of $3f_\infty\mathcal{H}^1(L_i)$, as ε goes to zero. This is obtained by recovering L_i with a strip of "minimal" triangles, i.e., triangles with two edges of length ε and one of length $\varepsilon' = 2\varepsilon\cos\theta_0$, as shown in Fig. 8. Then, these triangulations are connected in some way to a uniform "background triangulation" (made of the squares $[k\varepsilon', (k+1)\varepsilon'] \times [l\varepsilon', (l+1)\varepsilon']$, $k, l \in \mathbb{Z}$, cut into two triangles along a diagonal), in order to obtain a global triangulation $\boldsymbol{T}^\varepsilon$ over all Ω.

Let us give some details about this construction. Calling $\boldsymbol{T}_i^\varepsilon$ the union of all triangles that touch L_i and $\widehat{\boldsymbol{T}}_i^\varepsilon$ the union of all triangles T such that either T or some neighbor of T (i.e., a triangle T' with $T' \cap T \neq \emptyset$) touches L_i, we want to design this strip in such a way that

$$\sum_{T \in \widehat{\boldsymbol{T}}_i^\varepsilon} \frac{|T|}{h_T} \sim 3\mathcal{H}^1(L_i)$$

as ε goes to zero. Figure 8 shows how to do this: it suffices to include L_i first in $2N_i + 5$ minimal triangles, with $N_i = [\mathcal{H}^1(L_i)/\varepsilon']$ ($[\cdot]$ denoting the integer part), and then juxtapose along this thin strip two other strips in such a way that the height h_T of each triangle in $\widehat{\boldsymbol{T}}_i^\varepsilon$ is perpendicular to L_i. For this particular construction, $\#\widehat{\boldsymbol{T}}_i^\varepsilon = 6N_i + 13$, and, since for every $T \in \widehat{\boldsymbol{T}}_i^\varepsilon$,

$|T|/h_T = \varepsilon\cos\theta_0 = \varepsilon'/2$, we have

$$3\mathcal{H}^1(L_i) + 7\varepsilon\cos\theta_0 \le \sum_{T\in\widehat{\boldsymbol{T}}_i^\varepsilon} \frac{|T|}{h_T} \le 3\mathcal{H}^1(L_i) + 13\varepsilon\cos\theta_0. \tag{46}$$

The strip $\widehat{\boldsymbol{T}}_i^\varepsilon$ is then included into a larger rectangle of size $2\varepsilon' \times (\mathbb{N}_i + 3)\varepsilon'$. The algorithm described in [13] shows that it is always possible to connect the rectangle of Fig. 8 to the background triangulation, provided $\varepsilon << \min_{i\neq j}\operatorname{dist}(L_i, L_j) > 0$ (so that for $i \neq j$, $\widehat{\boldsymbol{T}}_i^\varepsilon$ and $\widehat{\boldsymbol{T}}_j^\varepsilon$ are not too close), θ_0 is not too large, and c is not too small ($\theta_0 \le 18°$ and $c \ge 6$ suit).

Now, $\boldsymbol{T}^\varepsilon$ being constructed, we simply let u^ε be the piecewise constant function equal to u at each node of $\boldsymbol{T}^\varepsilon$ (including the vertices that are not in Ω of triangles that meet Ω without being included in it — we assume ε is small enough, so that each triangle $T \in \boldsymbol{T}^\varepsilon$ with $|T\cap\Omega| > 0$ is contained in Ω'). It is a classical fact that u^ε converges to u in $L^p(\Omega)$ as $\varepsilon \downarrow 0$. Let $\boldsymbol{T}_S^\varepsilon = \bigcup_{i=1}^N \boldsymbol{T}_i^\varepsilon$ and $\widehat{\boldsymbol{T}}_S^\varepsilon = \bigcup_{i=1}^N \widehat{\boldsymbol{T}}_i^\varepsilon$, we have, using (2) and (46),

$$\begin{aligned} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon) &\le \sum_{T\in\boldsymbol{T}^\varepsilon} \frac{|T\cap\Omega|}{h_T}\left(h_T M_T^*(|\nabla u^\varepsilon|^2)\wedge f_\infty\right) \\ &\le \sum_{T\in\boldsymbol{T}^\varepsilon\setminus\widehat{\boldsymbol{T}}_S^\varepsilon} |T\cap\Omega| M_T^*(|\nabla u^\varepsilon|^2) + 3f_\infty\sum_{i=1}^N \mathcal{H}^1(L_i) \\ &\quad + 13Nf_\infty\varepsilon\cos\theta_0, \end{aligned} \tag{47}$$

We write the first sum in the last line of (47) as follows:

$$\begin{aligned} &\sum_{T\in\boldsymbol{T}^\varepsilon\setminus\widehat{\boldsymbol{T}}_S^\varepsilon} |T\cap\Omega| M_T^*(|\nabla u^\varepsilon|^2) \\ &\quad = \sum_{T\in\boldsymbol{T}^\varepsilon\setminus\widehat{\boldsymbol{T}}_S^\varepsilon} |T\cap\Omega| \sum_{T'\cap T\neq\emptyset} \frac{|T'\cap\Omega|}{S_{T'}} |\nabla u_{T'}^\varepsilon|^2 \\ &\quad = \sum_{T'\in\boldsymbol{T}^\varepsilon} |T'\cap\Omega||\nabla u_{T'}^\varepsilon|^2 \left\{\frac{1}{S_{T'}} \sum_{T\cap\boldsymbol{T}'\neq\emptyset, T\notin\widehat{\boldsymbol{T}}_S^\varepsilon} |T\cap\Omega|\right\}. \end{aligned}$$

Since the last term in the parentheses is always less than 1, and is zero as soon as $T \in \boldsymbol{T}_S^\varepsilon$, we deduce

$$\begin{aligned} F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon) &\le \sum_{T\in\boldsymbol{T}^\varepsilon\setminus\boldsymbol{T}_S^\varepsilon} |T\cap\Omega||\nabla u_T^\varepsilon|^2 + 3f_\infty\sum_{i=1}^N \mathcal{H}^1(L_i) \\ &\quad + 13Nf_\infty\varepsilon\cos\theta_0. \end{aligned} \tag{48}$$

Again, we proceed now as in [13]: using [27, Theorem 4.4-3], we have for any $T \notin \boldsymbol{T}^\varepsilon_S$,

$$\|u - u^\varepsilon\|_{H^1(T)} \le c(\theta_0)\, \varepsilon\, \|D^2 u\|_{L^2(T,\mathbb{R}^4)}$$

where $c(\theta_0) > 0$ is some constant (depending only on θ_0) and $D^2 u$ is the Hessian of u, so that, if we let

$$U^\varepsilon(x) = \sum_{T \in \boldsymbol{T}^\varepsilon \setminus \boldsymbol{T}^\varepsilon_S} \nabla u^\varepsilon_T \cdot \chi_T(x)$$

for all $x \in \Omega$,

$$\|\nabla u - U^\varepsilon\|^2_{L^2(\Omega,\mathbb{R}^2)} \le c(\theta_0)^2\, \varepsilon^2\, \|D^2 u\|^2_{L^2(\Omega,\mathbb{R}^4)} + \int_{A(\varepsilon)} |\nabla u(x)|^2\, dx$$

where $A(\varepsilon) = \bigcup_{T \in \boldsymbol{T}^\varepsilon_S} T$ satisfies $|A(\varepsilon)| \to 0$ as $\varepsilon \downarrow 0$. Thus, U^ε goes to ∇u strongly in $L^2(\Omega, \mathbb{R}^2)$, and sending ε to zero in (48) we get (45). The proof of Theorem 1 is thus achieved.

Remark. If the function u is bounded, we notice that the functions u^ε that we build satisfy $\|u^\varepsilon\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$ for every $\varepsilon > 0$. In particular, we deduce that the functionals F^ε Γ-converge to F as $\varepsilon \downarrow 0$ also in the space

$$\left\{u \in L^p(\Omega)\ :\ \|u\|_{L^\infty(\Omega)} \le M\right\}$$

for any $M > 0$.

3.3. *Proof of Theorem 2*

We consider $(u^\varepsilon)_{\varepsilon>0}$ satisfying the assumptions of Theorem 2, and for each ε an adapted triangulation $\boldsymbol{T}^\varepsilon$ such that $F_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon) = F_\varepsilon(u^\varepsilon)$. Like in Sect. 3.1.1 we fix $\kappa > 0$ and introduce a set $\boldsymbol{T}^\varepsilon_\kappa = \{T \in \boldsymbol{T}^\varepsilon\ :\ |\nabla u^\varepsilon_T|^2 > \kappa/\varepsilon\}$. Following the proof in Sect. 3.1.1 with $A = \Omega$, we can build functions $v^\varepsilon \in SBV(\Omega)$ such that for every $\varepsilon > 0$, $\|v^\varepsilon\|_{L^p(\Omega)} \le \|u^\varepsilon\|_{L^p(\Omega)}$,

$$\lim_{\varepsilon \downarrow 0} |\{x \in \Omega\ :\ u^\varepsilon(x) \neq v^\varepsilon(x)\}| = 0 \tag{49}$$

and for every $\delta > 0$,

$$\sup_{\varepsilon>0} \int_{\Omega^\delta} |\nabla v^\varepsilon(x)|^2\, dx + \mathcal{H}^1(S_{v^\varepsilon} \cap \Omega^\delta)\ <\ +\infty.$$

We can thus invoke Ambrosio's Theorem 3, and follow a diagonal procedure in order to build a function $u \in GSBV(\Omega) \cap L^p(\Omega)$ and a subsequence v^{ε_j} such that $v^{\varepsilon_j} \to u$ a.e. in Ω. From (49) we deduce, possibly extracting another subsequence, that u^{ε_j} goes to u a.e. in Ω. The Remark at the end of Sect. 3.1 yields (10).

Assume now that for each $\varepsilon > 0$, the function u^ε is a solution of (11). For any $w \in L^p(\Omega)$ we proved in Sect. 3.2 that we can build a sequence $(w^{\varepsilon_j})_{j\geq 1}$ converging to w in $L^p(\Omega)$ such that

$$\limsup_{j\to\infty} F_{\varepsilon_j}(w^{\varepsilon_j}) + \int_\Omega |w^{\varepsilon_j}(x) - g(x)|^p \, dx \leq F(w) + \int_\Omega |w(x) - g(x)|^p \, dx.$$

Since for every j, $F_{\varepsilon_j}(w^{\varepsilon_j}) + \int_\Omega |w^{\varepsilon_j} - g|^p dx \geq F_{\varepsilon_j}(u^{\varepsilon_j}) + \int_\Omega |u^{\varepsilon_j} - g|^p dx$, we deduce from (10) and Fatou's Lemma that

$$F(u) + \int_\Omega |u(x) - g(x)|^p \, dx \leq F(w) + \int_\Omega |w(x) - g(x)|^p \, dx,$$

hence u solves (12). Taking $w = u$, we also get that

$$\lim_{j\to\infty} F_{\varepsilon_j}(u^{\varepsilon_j}) + \int_\Omega |u^{\varepsilon_j}(x) - g(x)|^p \, dx = F(u) + \int_\Omega |u(x) - g(x)|^p \, dx,$$

which yields, if $p > 1$, the strong convergence of u^{ε_j} to u.

Remark. If $g \in L^\infty(\Omega)$, it is standard that any solution u of (12) satisfies $\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$. This might not be true for the approximated problem (11), however, the Remark at the end of Sect. 3.2 shows that Theorem (2) still holds if we add in the minimization problems (11) and (12) the additional constraint $\|v\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$. In this case, u^{ε_j} converges strongly to u for any $p \in [1, +\infty)$.

A. Special functions of bounded variation

A.1. The spaces SBV and $GSBV$: definitions and main properties

In this section we define briefly the "special functions of bounded variation" and state a few properties. See for instance [3] or [2] for further details. Given $\Omega \subseteq \mathbb{R}^N$ and $u : \Omega \to [-\infty, +\infty]$ a measurable function, we first define the *approximate upper limit* of u at $x \in \Omega$ as

$$u_+(x) = \inf \left\{ t \in [-\infty, +\infty] \ : \ \lim_{\rho\downarrow 0} \frac{|\{y \ : \ u(y) > t\} \cap B_\rho(x)|}{\rho^N} = 0 \right\},$$

where $B_\rho(x)$ is the ball of radius ρ centered at x and $|E|$ denotes the Lebesgue measure of the set E. The *approximate lower limit* $u_-(x)$ is defined in the same way (i.e., $u_-(x) = -(-u)_+(x)$). The set

$$S_u = \{x \in \Omega \ : \ u_-(x) < u_+(x)\},$$

is the set of essential discontinuities of u, it is a (Lebesgue-)negligible Borel set. If $x \notin S_u$, we say that u is *approximately continuous* at x and we write $\tilde{u}(x) = u_-(x) = u_+(x) = \text{ap } \lim_{y\to x} u(y)$.

A function $u \in L^1(\Omega)$ is a *function of bounded variation* if its distributional derivative Du is a vector-valued measure with finite total variation in Ω (equivalently, if the partial distributional derivatives $D_i u$, $i = 1, \ldots, N$, are real-valued measures with finite total variation in Ω). The space of functions of bounded variation is denoted by $BV(\Omega)$. For the general theory we refer to [19], [20], [25] and [29]. If $u \in BV(\Omega)$, the set S_u is *countably* $(\mathcal{H}^{N-1}, N-1)$*-rectifiable*, i.e,

$$S_u = \bigcup_{i=1}^{\infty} K_i \cup \mathcal{N}$$

where $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ and each K_i is a compact subset of a C^1-hypersurface Γ_i. There exists a Borel function $\nu_u : S_u \to \mathbb{S}^{N-1}$ such that $\mathcal{H}^{N-1}$-a.e. in S_u the vector $\nu_u(x)$ is normal to S_u at x in the sense that it is normal to Γ_i if $x \in K_i$. For every $u, v \in BV(\Omega)$, we must therefore have $\nu_u = \pm\nu_v$ $\mathcal{H}^{N-1}$-a.e. in $S_u \cap S_v$.

For every $u \in BV(\Omega)$ the measure Du can be decomposed as follows:

$$Du = \nabla u(x)dx + (u_+ - u_-)\nu_u \mathcal{H}^{N-1} \llcorner S_u + Cu$$

where ∇u is the *approximate gradient* of u, defined a.e. in Ω by

$$\text{ap} \lim_{y\to x} \frac{u(y) - u(x) - \langle \nabla u(x), y - x\rangle}{|y - x|} = 0,$$

$\mathcal{H}^{N-1} \llcorner S_u$ is the restriction of the $N-1$ dimensional Hausdorff measure to the set S_u, and Cu is the *Cantor part* of the measure Du, which is singular with respect to the Lebesgue measure and such that $|Cu|(E) = 0$ for any E with $\mathcal{H}^{N-1}(E) < +\infty$.

We say that a function $u \in BV(\Omega)$ is a *special function of bounded variation* if $Cu = 0$, which means that the singular part of the distributional derivative Du is concentrated on the jumps set S_u. We denote by $SBV(\Omega)$ the space of such functions. We also define the space $GSBV(\Omega)$ of *generalized* SBV functions as the set of all measurable functions $u : \Omega \to [-\infty, +\infty]$ such that for any $\Omega' \subset\subset \Omega$ and any $k > 0$, $u^k = (-k \vee u) \wedge k \in SBV(\Omega')$ (where $X \wedge Y = \min(X, Y)$ and $X \vee Y = \max(X, Y)$).

If $u \in GSBV(\Omega) \cap L^1_{\text{loc}}(\Omega)$, u has an approximate gradient a.e. in Ω, moreover, as $k \uparrow \infty$,

(50) $\nabla u^k \to \nabla u$ a.e. in Ω, and $|\nabla u^k| \uparrow |\nabla u|$ a.e. in Ω;

$$S_{u^k} \subseteq S_u, \mathcal{H}^{N-1}(S_{u^k}) \to \mathcal{H}^{N-1}(S_u) \qquad \text{and}$$
$$\nu_{u^k} = \nu_u \ \mathcal{H}^{N-1}\text{-a.e. in } S_{u^k}. \tag{51}$$

Slicing. We consider now for $\xi \in \mathbb{S}^{N-1}$ the sets $\xi^\perp = \{x \in \mathbb{R}^N : \langle \xi, x\rangle = 0\}$ and for any $z \in \xi^\perp$, $\Omega_{z,\xi} = \{t \in \mathbb{R} : z + t\xi \in \Omega\}$. On $\Omega_{z,\xi}$ we define a function $u_{z,\xi} : \Omega_{z,\xi} \to [-\infty, +\infty]$ by $u_{z,\xi}(s) = u(z+s\xi)$. If $u \in BV(\Omega)$, we have the following classical representation (see for instance [1], [4]): for $\mathcal{H}^{N-1}$-a.e. $z \in \xi^\perp$, $u_{z,\xi} \in BV(\Omega_{z,\xi})$ and for any Borel set $B \subseteq \Omega$

$$\langle Du, \xi\rangle(B) = \int_{\xi^\perp} d\mathcal{H}^{N-1}(z) Du_{z,\xi}(B_{z,\xi})$$

where $B_{z,\xi}$ is defined in the same way as $\Omega_{z,\xi}$; conversely if $u_{z,\xi} \in BV(\Omega_{z,\xi})$ for at least N independent vectors $\xi \in \mathbb{S}^{N-1}$ and $\mathcal{H}^{N-1}$-a.e. $z \in \xi^\perp$, and if

$$\int_{\xi^\perp} d\mathcal{H}^{N-1}(z)|Du_{z,\xi}|(\Omega_{z,\xi}) \ < \ +\infty$$

then $u \in BV(\Omega)$. Now (see [2], [1]), if $u \in SBV_{\text{loc}}(\Omega)$, then for almost every $z \in \xi^\perp$, $u_{z,\xi} \in SBV_{\text{loc}}(\Omega_{z,\xi})$ (the converse is true provided this property is satisfied for at least N independent vectors ξ and u has locally bounded variation), the approximate derivative satisfies

$$u'_{z,\xi}(s) = \langle \nabla u(z + s\xi), \xi\rangle$$

for a.e. $s \in \Omega_{z,\xi}$, moreover

$$S_{u_{z,\xi}} = \{s \in \Omega_{z,\xi} \ : \ z + s\xi \in S_u\},$$

and for all $s \in S_{u_{z,\xi}}$,

$$(u_{z,\xi})_\pm(s) = u_\pm(z + s\xi).$$

Eventually, for any Borel set $B \subseteq \Omega$

$$\int_{\xi^\perp} d\mathcal{H}^{N-1}(z)\mathcal{H}^0(B_{z,\xi} \cap S_{u_{z,\xi}}) = \int_B |\langle \nu_u(x), \xi\rangle|\, d\mathcal{H}^{N-1}(x).$$

Compactness. We finally mention two compactness result. The first one is the classical compactness and lower semi-continuity result proved in [1] (see also [2], [3]). The lemma that follows is a variant that is useful in the proof of our Γ-convergence theorems.

Theorem 3 (Ambrosio). *Let Ω be an open subset of $\mathbb{R}^N$ and let (u_j) be a sequence in $GSBV(\Omega)$. Suppose that there exist $p \in [1, \infty]$ and a constant C such that*

$$\int_\Omega |\nabla u_j|^2 \, dx + \mathcal{H}^{N-1}(S_{u_j}) + \|u_j\|_{L^p(\Omega)} \le C < +\infty$$

for every j. Then there exist a subsequence (still denoted by (u_j)) and a function $u \in GSBV(\Omega) \cap L^p(\Omega)$ such that

$$u_j(x) \to u(x) \text{ a.e. in } \Omega,$$
$$\nabla u_j \rightharpoonup \nabla u \text{ weakly in } L^2(\Omega, \mathbb{R}^N),$$
$$\mathcal{H}^{N-1}(S_u) \le \liminf_{j\to\infty} \mathcal{H}^{N-1}(S_{u_j}).$$

Moreover

$$\int_{S_u} |\langle \nu_u, \xi \rangle| \, d\mathcal{H}^{N-1} \le \liminf_{j\to\infty} \int_{S_{u_j}} |\langle \nu_{u_j}, \xi \rangle| \, d\mathcal{H}^{N-1}$$

for every $\xi \in \mathbb{S}^{N-1}$.

Lemma 2. *Let Ω be an open subset of $\mathbb{R}^N$, $p \in [1, +\infty]$, and let (A_j) be a sequence of open subsets of Ω and and (u_j) a sequence of functions such that*

- *for all j, $u_j \in C(\Omega \setminus A_j) \cap H^1(\Omega \setminus \overline{A_j})$,*
- $\sup_j \|u_j\|_{L^p(\Omega)} + \int_\Omega |\nabla u_j(x)|^2 \, dx \, < \, +\infty$*,*
- $\sup_j \mathcal{H}^{N-1}(\partial A_j) \, < \, +\infty$.
- $\lim_{j\to\infty} |A_j| = 0$.

Then there exist a subsequence (not relabelled) and a function $u \in GSBV(\Omega) \cap L^p(\Omega)$ such that (each u_j being extended with the value zero in A_j)

$$u_j(x) \to u(x) \text{ a.e. in } \Omega,$$
$$\nabla u_j \rightharpoonup \nabla u \text{ weakly in } L^2(\Omega, \mathbb{R}^N),$$
$$\mathcal{H}^{N-1}(S_u) \le \frac{1}{2} \liminf_{j\to\infty} \mathcal{H}^{N-1}(\partial A_j \cap \Omega).$$

Proof of the lemma. The only property to check is the inequality

$$\mathcal{H}^{N-1}(S_u) \le \frac{1}{2} \liminf_{j\to\infty} \mathcal{H}^{N-1}(\partial A_j \cap \Omega), \tag{52}$$

since the other statements easily follow from Ambrosio's previous theorem.

We can assume $p = \infty$: indeed if we replace the functions and their limits by the truncations at any level $k > 0$, $u_j^k = (-k \vee u_j) \wedge k$ and $u^k = (-k \vee u) \wedge k$, and if the result holds for the sequence (u_j^k) and the function u^k, then from (51) we'll deduce the general result.

We call $M = \sup_j \|u_j\|_{L^\infty(\Omega \setminus \overline{A_j})}$, and on each A_j we set $u_j \equiv M+1$ (the value of u_j inside A_j does not affect the limit).

Since u_j is continuous on $\Omega \setminus A_j$,

$$S_{u_j} \subseteq \{x \in \partial A_j \ : \ 0 < D^+(x, A_j),\ D^-(x, A_j) < 1\} = S\chi_{A_j} \subseteq \partial A_j$$

where $D^+(x, A_j) = \limsup_{\rho \downarrow 0} |B(x,\rho) \cap A_j| / \omega_N \rho^N$ and $D^-(x, A_j) = \liminf_{\rho \downarrow 0} |B(x,\rho) \cap A_j| / \omega_N \rho^N \in [0,1]$ are the N-dimensional upper and lower densities of A_j at x; and for all $x \in S_{u_j}$, $(u_j)_+(x) = M+1$ and $(u_j)_-(x) \leq M$. We denote by $\partial^* A_j$ the set $S\chi_{A_j}$.

We first choose $\xi \in \mathbb{S}^{N-1}$ and an open set $B \subset\subset \Omega$, and we will show that

$$\int_{S_u \cap B} |\langle \nu_u, \xi \rangle| d\mathcal{H}^{N-1} \leq \frac{1}{2} \liminf_{j \to \infty} \mathcal{H}^{N-1}(\partial^* A_j \cap B). \tag{53}$$

For any $z \in \xi^\perp$ we denote by B^z the set $\{t \in \mathbb{R} \ : \ z + t\xi \in B\}$ and respectively by $u^z(t)$ and $u_j^z(t)$ the functions $u(z + t\xi)$ and $u_j(z + t\xi)$.

We have that

$$\int_B |u_j - u| \, dx = \int_{\xi^\perp} d\mathcal{H}^{N-1}(z) \int_{B^z} |u_j^z(t) - u^z(t)| \, dt \to 0$$

as $j \to \infty$, so that we may assume (up to a subsequence) that for $\mathcal{H}^{N-1}$-a.e. $z \in \xi^\perp$,

$$\lim_{j \to \infty} \int_{B^z} |u_j^z - u^z| \, dt = 0. \tag{54}$$

Moreover, for $\mathcal{H}^{N-1}$-a.e. $z \in \xi^\perp$,

$$\int_{B^z} |(u^z)'(t)|^2 \, dt + \mathcal{H}^0(B^z \cap S_{u^z}) \ < \ +\infty, \tag{55}$$

since

$$\int_{\xi^\perp} d\mathcal{H}^{N-1}(z) \left(\int_{B^z} |(u^z)'(t)|^2 \, dt + \mathcal{H}^0(B^z \cap S_{u^z}) \right) = \int_B |\langle \nabla u(x), \xi \rangle|^2 \, dx + \int_{S_u} |\langle \nu_u(x), \xi \rangle| \, d\mathcal{H}^{N-1}(x) \ < \ +\infty;$$

and u_j^z is in $SBV(B^z)$, with

$$\liminf_{j \to \infty} \int_{B^z} |(u_j^z)'(t)|^2 \, dt + \mathcal{H}^0(B^z \cap S_{u_j^z}) < \ +\infty, \tag{56}$$

since, by Fatou's lemma,

$$\int_{\xi^\perp} d\mathcal{H}^{N-1}(z)\left(\liminf_{j\to\infty}\int_{B^z}|(u_j^z)'(t)|^2\,dt+\mathcal{H}^0(B^z\cap S_{u_j^z})\right)$$
$$\leq \liminf_{j\to\infty}\int_B|\langle\nabla u_j(x),\xi\rangle|^2\,dx+\int_{B\cap S_{u_j}}|\langle\nu_{u_j}(x),\xi\rangle|d\mathcal{H}^{N-1}(x)<+\infty.$$

We also know that for $\mathcal{H}^{N-1}$-a.e. $z\in\xi^\perp$, the jumps set of $u_j^z\in SBV(B^z)$ is

$$B^z\cap S_{u_j^z}=\{t\in B^z\ :\ z+t\xi\in S_{u_j}\}$$

and $(u_j^z)_+(t)=M+1$, $(u_j^z)_-(t)\leq M$ for any $t\in S_{u_j^z}$.

We may therefore choose a z such that this holds for all j, as well as (54), (55) and (56). By (55), the function u^z is piecewise continuous on B^z with a finite number of jumps. We let $S_{u^z}\cap B^z=\{t_1,\dots,t_n\}$, with $t_1<t_2<\cdots<t_n$. Let $\varepsilon>0$ be chosen such that $(t_i-\varepsilon,t_i+\varepsilon)\subset\subset B_z$ and $t_i+\varepsilon<t_{i+1}-\varepsilon$ for all i. We will show that for an infinity of indices j, each interval $(t_i-\varepsilon,t_i+\varepsilon)$ contains at least two jumps of u_j^z.

Consider a subsequence $(u_{j_k}^z)$ of (u_j^z) such that $u_{j_k}^z\to u^z$ a.e. on B^z and

$$\eta\ell=\lim_{k\to\infty}\eta\int_{B^z}|(u_{j_k}^z)'(t)|^2\,dt+\mathcal{H}^0(B^z\cap S_{u_{j_k}^z})$$
$$=\liminf_{j\to\infty}\eta\int_{B^z}|(u_j^z)'(t)|^2\,dt+\mathcal{H}^0(B^z\cap S_{u_j^z})\ <\ +\infty,$$

where $\eta>0$ is a fixed small parameter. For every $i=1,\dots,n$, let $\sigma_i=u_+^z(t_i)-u_-^z(t_i)$ and choose $\delta_i<\min(\varepsilon,\sigma_i^2/8\ell)$. We choose α_i, β_i such that

- $t_i-\delta_i<\alpha_i<t_i<\beta_i<t_i+\delta_i$,
- $\lim_{k\to\infty}u_{j_k}^z(\alpha_i)=u^z(\alpha_i)$ and $\lim_{k\to\infty}u_{j_k}^z(\beta_i)=u^z(\beta_i)$,
- and $|u^z(\beta_i)-u^z(\alpha_i)|>\frac{1}{2}\sigma_i$ (by continuity).

For k large enough, $S_{u_{j_k}^z}\cap(\alpha_i,\beta_i)\neq\emptyset$, otherwise we would have, for all indices k such that this is not true,

$$|u_{j_k}^z(\beta_i-0)-u_{j_k}^z(\alpha_i+0)|\leq\int_{\alpha_i}^{\beta_i}|(u_{j_k}^z)'(t)|\,dt$$
$$\leq\left\{\int_{\alpha_i}^{\beta_i}|(u_{j_k}^z)'(t)|^2\,dt\right\}^{\frac{1}{2}}\sqrt{\beta_i-\alpha_i}.$$

As k goes to infinity, the limit of the left-hand term would be greater than $\sigma_i/2$, while the limit of the right-hand term would be smaller than $\sqrt{\ell}\sqrt{2\delta_i}<\sigma_i/2$, a contradiction.

Now, for k large enough, both $u^z_{j_k}(\alpha_i)$ and $u^z_{j_k}(\beta_i)$ are less than $\sup_{B^z} u^z$ $+1/2 \leq M + 1/2$ (thus than M, since u_{j_k} takes its values in $[-M, M] \cup \{M+1\}$), so that if $S_{u^z_{j_k}} \cap (\alpha_i, \beta_i) \neq \emptyset$ it must contain at least two points (since any jump of $u^z_{j_k}$ occurs between a value lower than M and the value $M+1$).

We deduce that if k is large enough,

$$\mathcal{H}^0\left(B^z \cap S_{u^z_{j_k}}\right) \geq \sum_{i=1}^{n} \mathcal{H}^0\left((\alpha_i, \beta_i) \cap S_{u^z_{j_k}}\right) \geq 2n,$$

therefore

$$2\,\mathcal{H}^0\left(B^z \cap S_{u^z}\right) \leq \liminf_{k\to\infty} \mathcal{H}^0\left(B^z \cap S_{u^z_{j_k}}\right).$$

Notice now that applying Ambrosio's Theorem 3 to the sequence $(u^z_{j_k})_{k\geq 1}$ of $SBV(B^z)$, we deduce that $(u^z_{j_k})'$ goes weakly to $(u^z)'$ in $L^2(B^z)$ as k goes to infinity so that

$$\int_{B^z} |(u^z)'(t)|^2\,dt \,\leq\, \liminf_{k\to\infty} \int_{B^z} |(u^z_{j_k})'(t)|^2\,dt.$$

Combining the last two inequalities, we deduce

$$\begin{aligned}
&\eta \int_{B^z} |(u^z)'(t)|^2\,dt \,+\, 2\,\mathcal{H}^0(B^z \cap S_{u^z}) \\
&\quad\leq \lim_{k\to\infty} \eta \int_{B^z} |(u^z_{j_k})'(t)|^2\,dt \,+\, \mathcal{H}^0(B^z \cap S_{u^z_{j_k}}) \\
&\quad= \liminf_{j\to\infty} \eta \int_{B^z} |(u^z_j)'(t)|^2\,dt \,+\, \mathcal{H}^0(B^z \cap S_{u^z_j})
\end{aligned}$$

This inequality being true for a.e. $z \in \xi^\perp$, we can integrate over z, and Fatou's lemma yields

$$\begin{aligned}
&\eta \int_B |\langle \nabla u(x), \xi\rangle|^2\,dx \,+\, 2\int_{B\cap S_u} |\langle \nu_u(x), \xi\rangle|\,d\mathcal{H}^{N-1}(x) \\
&\quad\leq \liminf_{j\to\infty} \eta \int_B |\nabla u_j(x)|^2\,dx \,+\, \mathcal{H}^{N-1}(B \cap S_{u_j}).
\end{aligned}$$

Since $\sup_j \int_B |\nabla u_j(x)|^2\,dx < +\infty$, we can send η to zero and we get (53). Finally, by a standard localization argument (used for instance in the proof of Theorem 3, see [2]), we deduce from (53) that

$$2\,\mathcal{H}^{N-1}(S_u) \leq \liminf_{j\to\infty} \mathcal{H}^{N-1}(\partial^*\!A_j \cap \Omega),$$

hence (52).

Remark. The same result holds if, instead of assuming u_j continuous on $\Omega \setminus A_j$, we assume that the boundary of A_j is "regular" in the sense that $\mathcal{H}^{N-1}(\partial A_j \setminus \partial^*\!A_j) = 0$.

A.2. An application: the Mumford-Shah functional

The functional originally introduced by D. Mumford and J. Shah, in order to modelize the image segmentation problem in a continuous setting, is the following

$$\begin{aligned} \mathcal{G}(u,K) = \int_{\Omega\setminus K} |\nabla u(x)|^2\,dx + \mathcal{H}^{N-1}(K) \\ + \int_{\Omega} |u(x)-g(x)|^2\,dx, \end{aligned} \tag{57}$$

where $g \in L^\infty(\Omega)$ is a given "original image", K is a closed set and $u \in C^1(\Omega\setminus K)$. L. Ambrosio and E. De Giorgi introduced the weak formulation in $GSBV(\Omega)$

$$G(u) = \int_{\Omega} |\nabla u(x)|^2\,dx + \mathcal{H}^{N-1}(S_u) + \int_{\Omega} |u(x)-g(x)|^2\,dx, \tag{58}$$

and proved the existence of a minimizer for G using Theorem 3. Then, E. De Giorgi, M. Carriero and A. Leaci established the existence of a minimizer for $\mathcal{G}$ by proving that if u minimizes G, then $\mathcal{H}^{N-1}(\Omega\cap\overline{S_u}\setminus S_u) = 0$ and $u \in C^1(\Omega\setminus\overline{S_u})$, so that $(u,\overline{S_u})$ minimizes $\mathcal{G}$ [16].

In [17], Dibos and Séré showed that any minimizer u of G may be approximated by a sequence $(u_\varepsilon)_{\varepsilon>0}$ of piecewise regular functions such that the jumps set S_{u_ε} of each u_ε is contained in a finite union of parallelipedic subsets of hyperplanes $(K_\varepsilon^1,\dots,K_\varepsilon^{n_\varepsilon})$, $u_\varepsilon \to u$ a.e. as ε goes to zero, and

$$\lim_{\varepsilon\downarrow 0} \int_{\Omega} |\nabla u_\varepsilon(x)|^2\,dx + \sum_{i=1}^{n_\varepsilon} \mathcal{H}^{N-1}(K_\varepsilon^i) = \int_{\Omega} |\nabla u(x)|^2\,dx + \mathcal{H}^{N-1}(S_u).$$

This result is generalized in [14]. In order to establish inequality (44) (Sect. 3.2) we need a variant of [14, Co 3.11], whose proof we do not give since it is easily derived from the proofs in [17] and [14].

B. The Γ-convergence

We shortly define the Γ-convergence of functionals (in metric spaces) and its main properties. For more details we refer to [15].

Given a metric space (X,d) and $F_k : X \to [-\infty,+\infty]$ a sequence of functions, we define for every $u \in X$ the Γ-lim inf of F

$$F'(u) = \Gamma - \liminf_{k\to\infty} F_k(u) = \inf_{u_k\to u} \liminf_{k\to\infty} F_k(u_k)$$

and the Γ-lim sup of F

$$F''(u) = \Gamma - \limsup_{k\to\infty} F_k(u) = \inf_{u_k \to u} \limsup_{k\to\infty} F_k(u_k),$$

and we say that F_k Γ-converges to $F : X \to [-\infty, +\infty]$ if $F' = F'' = F$. F', F'', and F (if it exists) are lower semi-continuous on X. We have the following two properties:

1. *F_k Γ-converges to F if and only if for every $u \in X$,*

(i) for every sequence u_k converging to u, $F(u) \leq \liminf_{k\to\infty} F_k(u_k)$;
(ii) there exists a sequence u_k that converges to u and such that $\limsup_{k\to\infty} F_k(u_k) \leq F(u)$;

2. *If $G : X \to \mathbb{R}$ is continuous and F_k Γ-converges to F, then $F_k + G$ Γ-converges to $F + G$.*

The following result makes clear the interest of the notion of Γ-convergence:

Theorem 4. *Assume F_k Γ-converges to F and for every k let u_k be a minimizer of F_k over X. Then, if the sequence (or a subsequence) u_k converges to some $u \in X$, u is a minimizer for F and $F_k(u_k)$ converges to $F(u)$.*

Finally, we give the following definition of Γ-convergence in the case where $(F_h)_{h>0}$ is a family of functionals on X indexed by a continuous parameter h: we say that F_h Γ-converges to F in X as $h \downarrow 0$ if and only if for every sequence (h_j) that converges to zero as $j \to \infty$, F_{h_j} Γ-converges to F.

C. A strange triangulation

In this section we show why M^* has to be used instead of M in the definition (8) of F_ε. Actually, let $\Omega = (0,1) \times (0,1)$, and for each $u^\varepsilon \in V_\varepsilon(\Omega)$, $\boldsymbol{T}^\varepsilon \in \mathcal{T}_\varepsilon(u^\varepsilon)$, let

$$I_\varepsilon(u^\varepsilon, \boldsymbol{T}^\varepsilon) = \sum_{T \in \boldsymbol{T}^\varepsilon} |T| M_T(|\nabla u^\varepsilon|^2).$$

We will construct a sequence $u_n, \boldsymbol{T}_n$, with $u_n \in V_{\frac{1}{n}}(\Omega)$ and $\boldsymbol{T}_n \in \mathcal{T}_\varepsilon(u_n)$, such that u_n converges to $u(x,y) \equiv y$ and $\lim_{n\to\infty} I_{\frac{1}{n}}(u_n, \boldsymbol{T}_n) < 1$. Fix $\lambda \in (0,1)$ and, for $n \geq 1$, consider in $\mathbb{R}^2$ the dots $x_{k,l} = \left(\frac{k}{n}, \frac{l}{n}\right)$, $k,l \in \mathbb{Z}$, and $y_{k,l} = \left(\frac{k}{n}, \frac{l+\lambda}{n}\right)$, $k,l \in \mathbb{Z}$. Let $\boldsymbol{T}_n$ be the triangulation of Ω made of the triangles $(x_{k,l}, x_{k+1,l}, y_{k,l})$, $(x_{k+1,l}, y_{k,l}, y_{k+1,l})$, of surface $\lambda/2n^2$, and of the triangles $(y_{k,l}, y_{k+1,l}, x_{k+1,l+1})$, $(y_{k,l}, x_{k,l+1}, x_{k+1,l+1})$, of surface

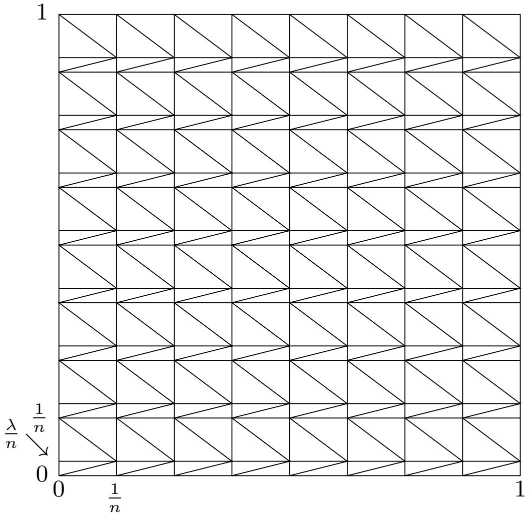


Fig. 9. The triangulation $\boldsymbol{T}_n$

$(1-\lambda)/2n^2$, contained in $\overline{\Omega}$ (Fig. 9). We assume n is large and restrict our attention to the triangles included in $\left[\frac{1}{n}, 1-\frac{1}{n}\right]^2$. We will call "small" triangles the triangles of surface $\lambda/2n^2$ and "large" triangles the other triangles. For each triangle T of one kind (i.e., "small" or "large") there are 5 triangles T' of the same kind (including T itself) such that $T' \cap T \neq \emptyset$ and 8 triangles of the other kind satisfying the same property. We fix $\alpha, \beta \in \mathbb{R}$ and define a function u_n, with $u_n(\cdot, 0) \equiv 0$ and $u_n(\cdot, 1) \equiv 1$, such that $\nabla u_n = (0, 1+\alpha)$ on the small triangles and $\nabla u_n = (0, 1+\beta)$ on the large triangles. We must have $\lambda\alpha + (1-\lambda)\beta = 0$, so that u_n goes to u (uniformly, and weakly in $H^1(\Omega)$) as $n \to \infty$.

If T is a small triangle, $S_T = (8-3\lambda)/2n^2$, while if T is large, $S_T = (5+3\lambda)/2n^2$, so that for T small,

$$M_T(|\nabla u|^2) = \frac{5\lambda(1+\alpha)^2 + 8(1-\lambda)(1+\beta)^2}{8-3\lambda}$$

and if T is large,

$$M_T(|\nabla u|^2) = \frac{5(1-\lambda)(1+\beta)^2 + 8\lambda(1+\alpha)^2}{5+3\lambda}.$$

Since there are $2n^2$ triangles of each kind, we easily deduce that

$$\begin{aligned} I(\alpha, \beta, \lambda) &= \lim_{n\to\infty} I_{\frac{1}{n}}(u_n, \boldsymbol{T}_n) \\ &= \lambda \frac{5\lambda(1+\alpha)^2 + 8(1-\lambda)(1+\beta)^2}{8-3\lambda} \end{aligned}$$

$$+(1-\lambda)\frac{5(1-\lambda)(1+\beta)^2+8\lambda(1+\alpha)^2}{5+3\lambda}.$$

If λ is small, this expression is less than 1 for admissible values of α, β, λ, for instance,

$$I\left(-\frac{1}{5}, \frac{1}{45}, \frac{1}{10}\right) \simeq 0.9873$$

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