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Past and Present of Variational Fracture

By Blaise Bourdin and Gilles A. Francfort

In 1920, Alan Griffith laid the foundations of brittle fracture [10]. His basic postulate was simple; the current state of a crack in an otherwise-elastic material is a cost-benefit analysis between (i) the increment of elastic energy δW_e recovered through the crack's putative infinitesimal advance from its current state, and (ii) the increment of surface energy δW_e spent to achieve such an extension. In short, Griffith's premise was a local minimality principle for the sum $\mathcal{E} := W_e + W_s$ of the elastic and surface energies. He complemented this with the intuitive idea that the surface energy W_s should be proportional to the number of broken bonds, and hence to the length (in two dimensions) or surface area (in three dimensions) of the crack. In two dimensions, $W_s = G_c \ell$, where ℓ denotes the crack length and G_c (the critical energy release rate) is a macroscopic material property.

Griffith's next of kin soon focused on a mere statement of first-order optimality vis-à-vis ℓ for a preordained crack path in two dimensions, i.e., $G := -\partial W_e / \partial \ell \leq G_c$. Because of the connection that George R. Irwin later made between and the coefficients in front of the singular part of the kinematic field at the crack tip [11], computation of G became the fracture mantra and the sole object of mechanical desire. Three decades later, the research community had forgotten the original variational attitude displayed by its forebearer.

Griffith had remained silent on the topic of the crack path. Path identification was proposed at a cost: the introduction of ad-hoc criteria both for smooth crack paths and for a sudden change of direction (kink) in the crack. With further refinements, this became the field of linear elastic fracture mechanics (LEFM), which held its ground from an engineering standpoint.

The Variational Stance

The variational viewpoint reemerged in 1998 with inspiration from the work of the Ennio De Giorgi school on image segmentation [9]. The idea was to combine minimality with evolution through an energy conservation statement.

Non-interpenetration notwithstanding, the fracture energy associated with a displacement field $u : \Omega \mapsto \mathbb{R}^2$ and a crack Γ in a homogeneous material occupying a two-dimensional domain Ω is

$$\mathcal{E}(u, \Gamma) := W_e(u, \Gamma) + W_s(\Gamma) = \int_{\Omega \setminus \Gamma} \frac{1}{2} \mathrm{Ae}(u) \cdot \mathrm{e}(u)\, dx + \int_{\Omega \setminus \Gamma} G_c \mathcal{H}^1(\Gamma),$$

where A denotes the elasticity of the material $\mathrm{e}(u) := \frac{\nabla u + \nabla u^t}{2}$ and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

Throughout this evolution, the only transfer of energy is an exchange between elastic and surface energies. Minimality is now global; the cracks are simply sets of co-dimension 1 and finite length (or area). They are only constrained by their past, and add-cracks at each time are topologically free. Cracks are free agents empowered to choose their future paths without a priori imposition of an external criterion.

For scalar-valued kinematic fields—as in the antiplane shear case—existence proofs of a well-posed evolution ensued; this first occurred for closed, connected cracks [7] in a topological setting, then for general cracks [8] in the weak *SBV* setting of De Giorgi and Luigi Ambrosio for the Mumford-Shah image segmentation problem.[1]

A variational phase-field model for fracture [3] soon followed as an adaptation of the Ambrosio-Tortorelli approximation of the Mumford-Shah functional [1] in the sense of Γ-convergence. The corresponding functional is

$$\mathcal{E}_\varepsilon(u, \alpha) := \int_\Omega \frac{1}{2}(a(\alpha) + \eta_\varepsilon)\, \mathrm{Ae}(u) \cdot \mathrm{e}(u)\, dx + \frac{G_c}{4c_w} \int_\Omega \left(\frac{w(\alpha)}{\varepsilon} + \varepsilon |\nabla \alpha|^2 \right) dx,$$

with $\alpha : \Omega \mapsto [0, 1]$ as the phase-field variable, ε and $\eta_\varepsilon \ll \varepsilon$ as regularization parameters, and a and w as two continuous strictly-monotonic functions, such that $w(0) = a(1) = 0,\ w(1) = a(0) = 1$, and $c_w := \int_0^1 \sqrt{w(s)} ds$. The approximate evolution proved numerically palatable and offered practitioners a computational tool for complex crack evolutions.

The mathematical activity surrounding variational and phase-field models of fracture initially went unnoticed by the larger fracture community, even as it flourished in the mathematical literature. Meanwhile, a growing interest in complex fracture problems was challenging practitioners to put forth tools that exceeded the capabilities of LEFM. A decade after the introduction of the aforementioned variational model, the fracture community experienced a sudden outburst of phase-field models (and paternity claims). The phase-field approach to fracture was on its way to its current quasi-hegemonic status.

Knowns and Unknowns of the Variational Model

The variational approach dispels the misbegotten notion that crack path prediction is subordinate to specific criteria. In particular, local stability combined with energy preservation adjudicates the rivalry between the two main kinking standards: maximality of the elastic energy release rate versus local symmetry of the kinematic fields after kinking. Both criteria must be satisfied simultaneously. Since they contradict each other [2], the only possible outcome is that kinks do not exist. Griffith's theory, spawned from local stability, actually disavows the addition of the ad-hoc kinking criteria meant to supplement it [5]. On the computational front, phase-field approximations have repeatedly demonstrated their ability to quantitatively predict fracture evolution [4].

Global minimality of the crack state at each time was key to mathematization of the evolution problem. In physics, however, any minimality postulate is technical convenience rather than the manifestation of a known physical imperative. In this case, global minimality in concert with free crack path has two serious drawbacks. First, it behaves hideously when the sample is subject to a force load — say $f(x,t)$ in its interior. Indeed, one should account for the work of $f(x,t)$ in such a case, thereby adding $-\int_\Omega f.u\,dx$ to $\mathcal{E}$. However, doing so prevents the realization of minimality because cutting off the support of $f(x,t)$ and moving it away from Ω drives that linear term to $-\infty$ while requiring only the price of that cut in surface energy and canceling the elastic energy altogether.

Because global minimality scans the entire energetic landscape, it enables situations wherein a steadily-growing crack would leverage information on distant materials properties or geometric features of which it has no possible knowledge. Therefore, abandoning global minimality in favor of a more local criterion would seem reasonable. Unfortunately, doing so would not cure either ailment: force loads would still favor immediate removal of their support, while initiation would be impossible for any kind of decent topology [6]. In addition, the link between phase-field and variational models would be severed, as Γ-convergence does not generally imply convergence of local minimizers.

Phase Field as an Autonomous Agent

In recent years, researchers have used phase-field models for an increasingly wide range of applications, including coupled problems like thermal cracking or hydraulic fracturing. They have also contemplated extensions to ductile and dynamic fracture propagation. These models are mostly ad-hoc, and mathematical understanding is partial at best. Yet conclusive quantitative comparisons with experiments, such as that in Figure 1, cannot be ignored.

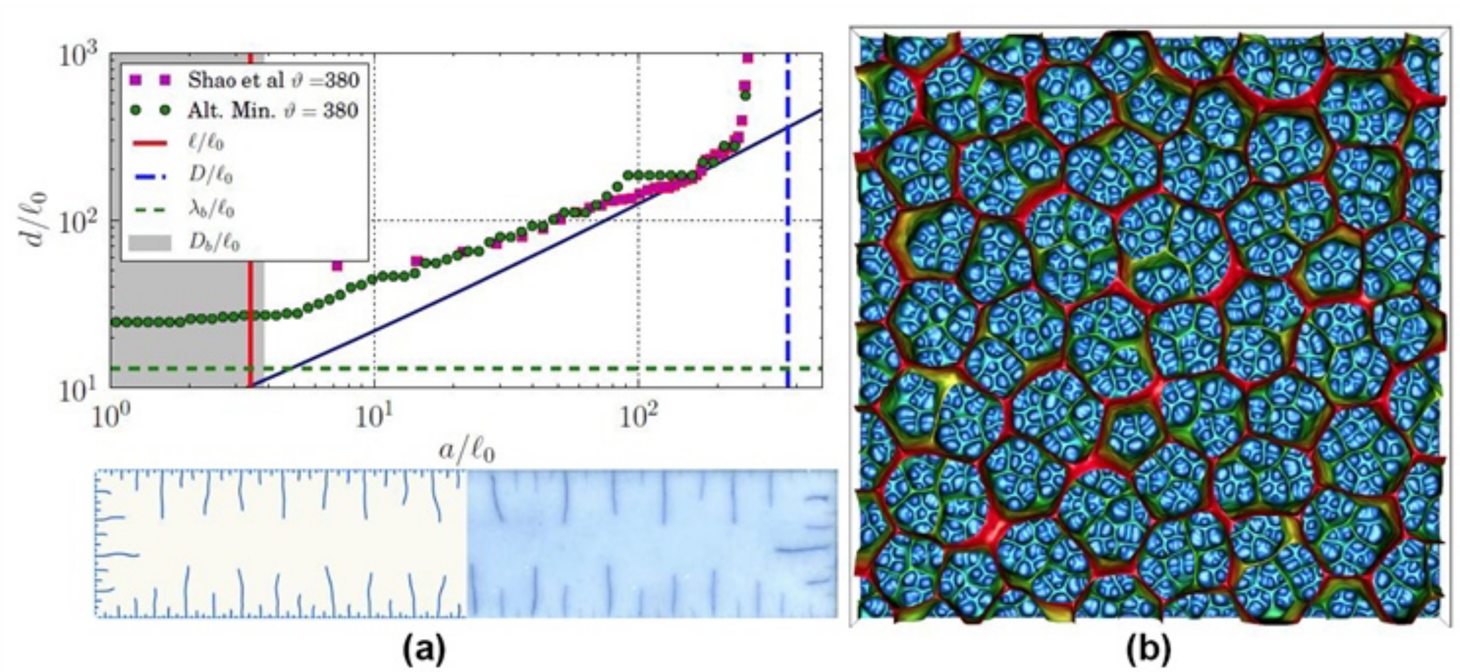

Figure 1. Phase-field simulation of a thermal shock [4]. **1a.** Crack spacing versus distance to edge, and crack geometry in numerical simulation versus experiments. **1b.** Complex crazing-like fracture pattern growing from the edge toward the inside of the domain in a large-scale, three-dimensional numerical simulation. The level line $\alpha = 0.95$ of the phase field is colored by the distance to the exposed surface; blue is toward the exposed face and red is inside the material. Figure courtesy of [4].

As mentioned above, the phase-field model finds its justification as an approximation of fracture through Γ-convergence, which is only concerned with global minimization. But global minimization is unrealistic. Moreover, there are no foolproof algorithms for computing global minima because $\mathcal{E}_\varepsilon$ is non-convex. Over the past 20 years, researchers have strived to incorporate a modicum of minimality into the computation of the phase-field evolution, with the understanding that doing so may lead them far from the fracture model [12]. The alternative is to view the phase-field model as the physical parent, thereby equating brittle fracture with a nonlocal damage gradient model — a radical step from a mechanician's standpoint.

To make things more complex, it is possible within the context of the phase-field approximation to devise a scheme that—by judiciously fixing the length scale—delivers the correct initiation sequence when quantitatively tested against experiments (see Figure 2). It is both remarkable and utterly baffling that well-devised phase-field evolutions provide such accurate approximations of fracture evolutions, from crack nucleation to sample failure.

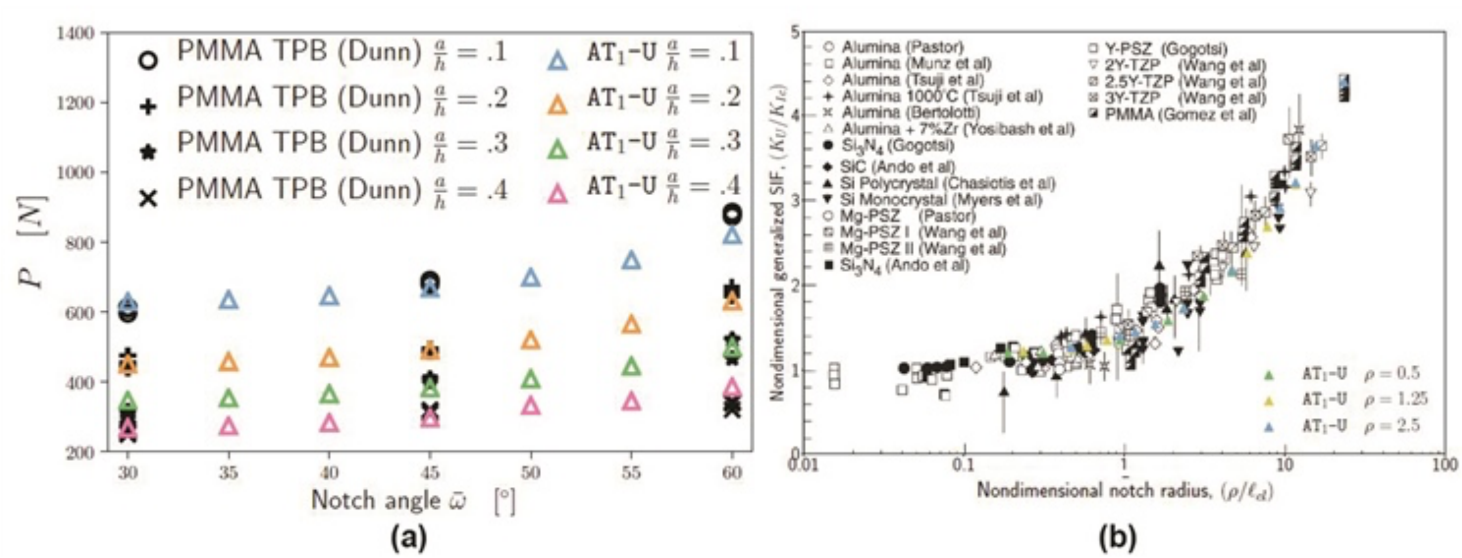

Figure 2. Crack nucleation in phase-field simulations (colored triangles) and experiments [12]. **2a.** Critical fracture load in a three-point bending experiment, with a V-notch as a function of the notch-relative depth a/h and opening half angle (black symbols) $\bar{\omega}$. **2b.** Phase-field computation of the generalized stress intensity factor near a U-notch, compared to experiments in a wide range of materials. Figure courtesy of [12].

The variational view of fracture has delivered a mathematically-consistent theory that quantitatively reproduces many experimental results. Yet its ability to do so is shrouded in mystery. The initial impetus to recast brittle fracture in a variational framework was motivated by the relevant advances in mathematics, with mechanics arguably lagging behind. The situation has since reversed; the past few years have witnessed the resurgence of mechanical considerations, especially in terms of phase-field models. The mathematical tools required for rigorous investigation of topics like initiation, local minimality, or inertial effects are still in their infancy.

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[1] The generalization to elasticity proved challenging and the three-dimensional case is still pending.

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