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EXTREMAL LIGHT-WEIGHT MICROSTRUCTURES

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ABSTRACT

We introduce a new class of high-porosity microstructures, called "single-scale laminates," made from arrays of parallel walls. They are extremal, in the sense that no stiffer structure exists with the same total mass. They are simple, in the sense of being easy to describe, and perhaps to manufacture. And they are universal: for any high-porosity microstructure there is a single-scale laminate using at most as much material which is at least as stiff. Moreover, any nondegenerate high-porosity Hooke's law can be bounded both above and below by a single-scale laminate of the same weight.

We give a simple formula for the effective Hooke's law of such a structure. It reduces the task of minimum-weight design in the high-porosity regime to a problem of linear programming.

Keywords: Topology optimization, extremal microstructures, Michell truss.

INTRODUCTION

High-porosity materials, like "cellular materials" and "metallic foams," have attracted increasing attention over the past decade, see e.g. (Evans 2001). Such materials display remarkable structural efficiency; for example, they can achieve high stiffness with very low mass per unit volume. They also display considerable flexibility, since the mechanical properties vary widely with changes in the microstructure.

Most work in this area has addressed the analysis of specific high-porosity materials. Many examples have been considered — of biological, physical, and synthetic origin — including bone, wood, periodic lattices, and random foams. The well-known book (Gibson and Ashby 1997) discusses a wide variety of structures and gives extensive references; see also (Christensen 2000) and (Cioranescu and Saint Jean Paulin 1999) for further developments.

The present work is different: our goal is not analysis but *design*. Our specific contribution is the introduction of a new class of high-porosity microstructures, which we call *single-scale laminates*. These structures are *extremal*, in the sense that no stiffer structure is possible using the same total mass. They are also *simple*, in the sense of being easy to describe (and, we hope, to manufacture). Finally, they are *universal*, in the sense that given any Hooke's law — no matter how anisotropic — there exists a single-scale laminate of the same weight that is stiffer. In the non degenerate case, one can also bound any high-porosity composite from below, using a single-scale laminate.

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We are, evidently, contributing to the analysis problem. Indeed, our single-scale laminates comprise a new class of high-porosity structures (though special cases have been analyzed before). Moreover our simple and intuitive formula (9) for the Hooke's law of such a composite seems to be new. But the real advantage of our contribution lies in the optimality and universality of this class of structures. In view of these properties, we have essentially solved the optimal design problem of finding the *minimum-weight* high-porosity structure with a given Hooke's law. Indeed, it suffices to minimize the weight among single-scale laminates; this amounts to solving a linear programming problem.

Homogenization-based optimal design

This achievement may seem like a radical development. After all, we are reducing an apparently intractable geometric optimization (over all microstructural geometries) to an entirely tractable linear program (solvable by the simplex method). How is such a thing possible? The answer lies in recent developments concerning structural optimization and bounds on effective moduli. Expositions of this topic can be found in the recent monographs by Allaire (2002), Cherkaev (2000), and Cherkaev and Kohn (1997); see also (Murat and Tartar 1985), (Lurie and Cherkaev 1986), (Kohn and Strang 1986a and 1986b) for selected antecedents. Rather than concentrate on the high-porosity regime, this work has considered mixtures of two elastic materials with *arbitrary* volume fractions. It includes a proof that maximal stiffness is achieved, at given volume fraction, within the special class of microstructures known as *sequential laminates* (Avellaneda 1987) (see also (Allaire and Kohn 1993a) and (Francfort et al. 1995)). It also provides a simple formula for the effective Hooke's law of a sequential laminate (Francfort and Murat 1986) and also (Tartar 1985). Some papers (those based on G-convergence) apply only to mixtures of two nondegenerate materials, but others (those based on quasiconvexification) apply also to porous composites, see e.g. (Kohn and Strang 1986a) and (Allaire and Francfort 1998). This theory has led to a considerable body of literature on homogenization-based structural optimization — where the goal is to optimize a loaded structure by permitting the local volume fraction and microstructure to vary from point to point; representative examples include e.g. (Bendsøe and Kikuchi 1988), (Bendsøe 1995), and (Allaire et al. 1997).

It is interesting to ask, in connection with the homogenization-based theory, whether geometrical complexity is required for optimality. Can extremal microstructures be achieved using a single length scale, for example by a periodic array of appropriately shaped inclusions or holes? Or do they require greater complexity, such as multiple length scales? We do not know the answer in complete generality, but we know a lot. Let us call a microstructure with Hooke's law C "extremal" if one cannot obtain a stiffer composite (one whose Hooke's law C' satisfies $C' \geq C$) by mixing the same components with the same volume fractions in a different microgeometry. In some cases, there are simple single-length-scale extremal microstructures, like the Vigdergauz construction in two space dimensions (Vigdergauz and Cherkaev 1986) or (Grabovsky and Kohn 1995)). In other cases, however, one can prove there is *no* single-scale microstructure achieving the extremal Hooke's law. A two-dimensional example is the second-rank laminate minimizing $\langle C^{-1}\sigma, \sigma\rangle$ when σ is a pure shear (Allaire and Aubry 1999). This situation is troublesome, since microstructures with multiple length scales are clearly unmanufacturable.

The high–porosity limit

The present paper explores the consequences of this homogenization-based viewpoint, for the analysis and design of high-porosity microstructures. The analysis, presented in (Bourdin and Kohn 2002), is far from trivial, since we are considering a very singular limit: the Hooke's

law of one component is being set to 0, and the volume fraction of the other component is tending to 0 as well. We concentrate here on the results, which reveal interesting and unexpected simplifications relative to the case of arbitrary volume fraction. In fact, we show that:

1. There is no need for multiple length scales in the high-porosity limit.
2. "Single-scale laminates" provide a convenient class of extremal microstructures. These relatively simple structures consist of families of parallel walls in 3D (parallel beams in 2D).
3. The Hooke's law of a single-scale laminate has a simple, intuitive formula. Indeed, it is given by linear superposition of the (degenerate) Hooke's laws associated with the distinct families of parallel walls.
4. *Any* nondegenerate Hooke's law achievable by a high–porosity microstructure can be bounded above and below by a pair of single-scale laminates with at most the same density of material.

Our work is nearly the first examination of homogenization-based structural optimization in the high-porosity regime. The only prior work of this type was restricted to compliance optimization with a single load (Allaire and Kohn 1993b), (Bendsøe and Haber 1993), (Allaire et al. 1997). In 2D the high-porosity limit is the classical Michell truss problem (Michell 1904) — an early indication that the need for multiple length scales should disappear in this regime. In 3D the high-porosity limit is known *not* to be a Michell truss problem. According to the present analysis, the proper picture in 3D is not a truss (consisting of one-dimensional members) but rather a single-scale laminate (made of two-dimensional walls). The intuitive reason is shown by an example in (Bourdin and Kohn 2002): 2D walls are more efficient than 1D trusses because each point in the wall does "double-duty," supporting stresses in two principal directions rather than just one.

SINGLE-SCALE LAMINATES

We begin by describing what we mean by a single-scale laminate. This microstructure consists of p families of parallel walls beams in 2D). The ith family of walls has unit normal k_i and thickness θt_i; here $t_i \geq 0$ and θ is a scaling parameter, which will tend to 0. In mathematical terms the characteristic function of the microstructure (the function χ which equals 1 where there is material and 0 where there is void) has the form

$$\chi(y) = \sum_{i=1\ldots p} \chi^{(i)}(y) - f^{(ov)}(y), \tag{1}$$

where $\chi^{(i)}$ is the characteristic function of the ith family of walls (a periodic function of $y \cdot k_i$ with average value θt_i) and $f^{(ov)}$ is supported in the region where different families of walls overlap.

Note that the parameters $\{t_i, k_i\}_{i=1\ldots p}$ do not fully determine the microgeometry; they determine the thicknesses and directions of the walls, but not their relative positions. In two dimensions for example, both the triangular and hexagonal lattices can be achieved this way, using the same thicknesses and directions but different translations. Additional examples of a similar character are shown in Figure 1. This ambiguity doesn't matter: we show in (Bourdin and Kohn 2002) that, to principal order as $\theta \to 0$, the effective behavior depends only on the parameters $\{t_i, k_i\}_{i=1\ldots p}$. This isn't obvious, but it should seem reasonable. Indeed, as $\theta \to 0$ the walls overlap on a very small volume, whose effect should be negligible. So the walls act more or less independently, and their relative positions don't matter.

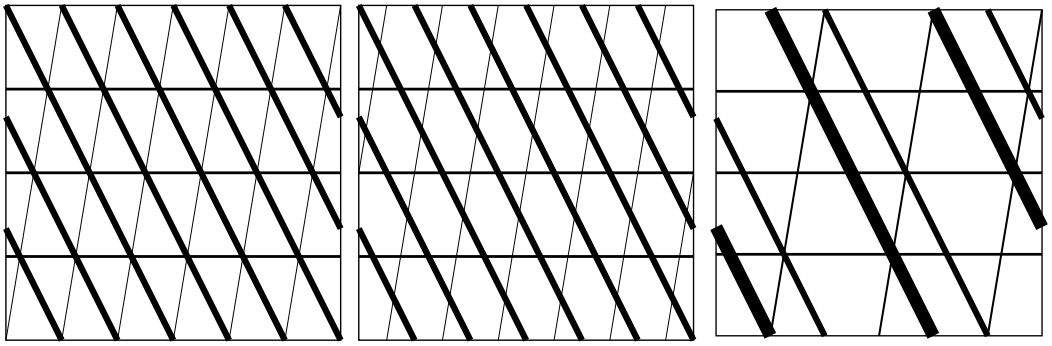

FIG. 1. Different geometries with the same directions and thicknesses.

The effective Hooke's law of a single-scale laminate

What is the effective Hooke's law of such a microstructure? We have argued that overlaps are unimportant and the walls act more or less independently. Therefore the presence or absence of a separation of scales should be irrelevant. In other words, we expect the Hooke's law to be the same (to principal order as $\theta \to 0$) as that of the sequential laminate with the same parameters. So we can guess the formula by considering a sequential laminate mixing two isotropic materials with Hooke's laws A and B, then (i) setting $B = 0$ and (ii) sending the volume fraction of A to 0. Readers not not familiar sequential lamination can skip the following calculation, going directly to its simple and intuitive outcome, Eqn. (8).

We review briefly the analysis of sequential lamination, using the notation of (Allaire 2002). First consider the composite obtained by layering A and B in proportions θt_1 and $1-\theta t_1$, with layers normal to k_1. The effective Hooke's law $C_1^{(l)}$ of the resulting "rank–1 laminate" is determined by

$$(1-\theta t_1)\left(C_1^{(l)^{-1}} - A^{-1}\right)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} + \theta t_1 f_A^c(k_1). \tag{2}$$

Here $f_A^c(k_1)$ is a projection operator (a degenerate Hooke's law), defined by

$$f_A^c(k)\xi = A^{1/2}\Pi_{A^{-1/2}W(k)}A^{1/2}\xi, \tag{3}$$

where

$$W(k) = \{\eta \in \mathcal{M}_N\,;\, \eta \cdot k = 0\}\,, \tag{4}$$

$\mathcal{M}_N$ is the space of all $N \times N$ symmetric matrices, and Π_X denotes orthogonal projection on the vector space X.

The degenerate Hooke's law $f_A^c(k)$ has a simple mechanical meaning. Indeed, it follows from (8) below that $f_A^c(k)$ is the Hooke's law (per unit mass) of a single family of thin walls made from material A with unit normal k. Of course there is an explicit formula in the isotropic case: when $A\xi = 2\mu_A\xi + \lambda_A\,(\mathrm{tr}\xi)\,\mathbf{I}$ we have

$$\langle f_A^c(k)\xi,\xi\rangle = \langle A\xi,\xi\rangle - \frac{1}{\mu_A}|A\xi k|^2 + \frac{\lambda_A+\mu_A}{\mu_A(\lambda_A+2\mu_A)}\langle A\xi k,k\rangle^2$$

The Hooke's law $C_2^{(l)}$ of a rank–2 laminate is obtained by a similar calculation, with B replaced by $C_1^{(l)}$. Sequential laminates of higher order are defined by the obvious inductive

procedure. This iterative procedure leads to the following formula for the effective Hooke's law $C_p^{(l)}$ of a rank–p laminate:

$$\left[\prod_{i=1}^{p}(1-\theta t_i)\right]\left(C_p^{(l)^{-1}}-A^{-1}\right)^{-1}=(B^{-1}-A^{-1})^{-1}+\theta\sum_{i=1}^{p}l_i f_A^c(k_i), \tag{5}$$

where the lamination parameters l_i are defined by

$$l_i=t_i\prod_{j=1}^{i-1}(1-\theta t_j). \tag{6}$$

Now, let us consider the limit of (5) when material B is replaced with void. Of course, the resulting composite might be degenerate (in 2D, for example a rank–2 sequential laminate with the axes as layering directions cannot sustain a shear). Formally, however, for any stress field sustainable by the composite (in other words for any $\sigma^{(p)}\in\sum_{i=1}^{p}W(k_i)$), one has

$$\left[\prod_{i=1}^{p}(1-\theta t_i)\right]\left(C_p^{(l)^{-1}}-A^{-1}\right)^{-1}=\theta\sum_{i=1}^{p}l_i f_A^c(k_i). \tag{7}$$

Finally let us consider the high–porosity limit obtained by sending $\theta\to 0$ while the other parameters remain fixed. The Hooke's law $C_p^{(l)}$ must also tend to 0 (it is at most of order θ) so one easily obtains

$$C_p^{(l)}=\theta\sum_{i=1}^{p}t_i f_A^c(k_i)+\mathcal{O}(\theta^2). \tag{8}$$

This Hooke's law can of course be degenerate: it is invertible on $\sum_{i=1}^{p}W(k_i)$. It is however nondegenerate if the construction uses a sufficient number of independent directions (i.e. if $\sum_{i=1}^{p}W(k_i)=\mathcal{M}_N$).

We did this calculation because we expected the Hooke's law of a single-scale laminate to agree, in the high-porosity limit, with that of the corresponding sequential laminate. Thus the leading-order term of (8) gives the Hooke's law of a high-porosity single-scale laminate:

$$C_p^{(s)}=\sum_{i=1}^{p}t_i f_A^c(k_i). \tag{9}$$

This formula is reasonable and intuitive. Indeed, $tf_A^c(k)$ is the Hooke's law of a family of walls orthogonal to k with volume fraction t. Our high-porosity single-scale laminates are obtained by geometric superposition of such walls, and their Hooke's law are additive. The walls operate, in effect, "in parallel," because in the high-porosity limit their intersections are neglible.

Comments about the rigorous analysis

It remains to justify our intuition, by showing that single-scale laminates and sequential laminates have the same effective behavior to leading order in θ. This is the key to all the claims enunciated in the introduction. Indeed, Claim 1 simply states this intuition. Claim 2 (extremality) follows from the known optimality of sequential laminates, by passage to the high porosity limit. Claim 3 is a description of the formula. And Claim 4 (universality) follows

from the fact that every Hooke's law is less than or equal to that of a sequential laminate with the same weight (Avellaneda 1987).

We summarize briefly the justification of our intuition; details will be presented in (Bourdin and Kohn 2002). We show two bounds: a *lower bound*, demonstrating that the effective behavior of a single-scale laminate is no smaller (to principal order in the high-porosity limit) than that of the associated sequential laminate; and an *upper bound*, showing it is no larger.

The idea behind the lower bound is easy to explain. For any average stress, one can deduce from the sequential lamination calculation the associated stresses in the composite at every length scale. Use of the same stresses (despite the lack of a separation of scales) gives a statically admissible stress field for the single-scale laminate. The desired lower bound follows from the principle of minimum complementary energy applied to this stress field.

The argument for the upper bound is quite different. It uses the Hashin-Shtrikman variational principle, applied in roughly the same manner as in (Avellaneda 1987). For any composite, this principle gives a nonlocal but explicit upper bound for its Hooke's law. The bound depends only on the H-measure of the microstructure, and it is achieved by the associated sequential laminate. For a single-scale laminate in the high-porosity regime this H-measure is explicit; in fact it agrees (to principal order) with that of the associated sequential laminate. This gives the desired upper bound.

Achieving a specified Hooke's law with minimal mass

Our single-scale laminates are not just a convenient class of high-porosity composites. They are as explained in the Introduction, an *extremal* class. In view of (9), the minimum (scaled) weight required to achieve at least a given Hooke's law C is given by

$$\min\left\{\sum_i t_i \ : \ t_i \geq 0; |k_i| = 1; \sum_i t_i f_A^c(k_i) \geq C\right\}. \tag{10}$$

In a numerical implementation, it is natural to discretize the set of admissible layering directions. Considering a fixed (but possibly large) number p of unit vectors $(k_1, \ldots, k_p)$, (10) simplifies to a linear programming problem in the associated thicknesses $(t_1, \ldots, t_p)$.

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