Contents lists available at ScienceDirect



Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps



Anisotropy of the effective toughness of layered media

S. Brach^a, M.Z. Hossain^b, B. Bourdin^c, K. Bhattacharya^{a,*}

^a Division of Engineering and Applied Science, California Institute of Technology, Pasadena, CA 91125, USA

^b Department of Mechanical Engineering, University of Delaware, Newark, DE 19716, USA

^c Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

ARTICLE INFO

Article history: Received 16 April 2019 Revised 25 June 2019 Accepted 28 June 2019 Available online 29 June 2019

Keywords: Effective properties Effective toughness Layered media Toughening mechanisms Crack pinning Crack deflection

ABSTRACT

This continues the study of the effective toughness of layered materials started in Hossain et al. (2014) and Hsueh et al. (2018), with a focus on anisotropy. We use the phase-field model and the surfing boundary condition to propagate a crack macroscopically at various angles to the layers. We study two idealized situations, the first where the elastic modulus is uniform while the toughness alternates and a second where the toughness is uniform and the elastic modulus alternates. We find that in the first case of toughness heterogeneity the effective toughness displays 'anomalous isotropy' in that it is independent of the propagation direction and equal to that of the tougher material except when the crack propagation is parallel to the layers. In the second case of elastic heterogeneity, we find the behavior more anisotropic and consistent with the toughening effects of stress fluctuation and need for crack renucleation at the compliant-to-stiff interface. In both cases, the effective toughness is not convex in the sense of interfacial energy or Wulff shape reflecting the fact that crack propagation follows a critical path. Further, in both cases the crack path is not straight and consistent with a maximal dissipation principle. Finally, the effective toughness depends on the contrast and pinning, rather than on the extent of crack fluctuation.

© 2019 Elsevier Ltd. All rights reserved.

1. Introduction

Layered media composed of alternating layers of distinct materials are commonly found in nature and used in engineering. They have also served as an useful example in the study of the overall or effective behavior of heterogeneous media Milton (2002). The interest in the current work concerns the effective toughness, as it is well known that heterogeneities are a toughening mechanism Faber and Evans (1983a,b); He and Hutchinson (1989); Hutchinson and Suo (1992).

Hossain et al. (2014) use phase-field simulations to show that the overall toughness of a layered medium can be significantly larger than that of the constituent materials, a fact that was confirmed experimentally by Wang and Xia (2017). Hsueh et al. (2018) show through both phase-field simulations and experiments that the enhanced toughening in elastically heterogeneous materials is due to two reasons – the fluctuations in stress and the need for the crack to renucleate as it passes from the compliant to the stiff material. These works considered the situation where a Mode I crack propagates normal to the layers, and the crack path is straight.

* Corresponding author. E-mail address: bhatta@caltech.edu (K. Bhattacharya).

https://doi.org/10.1016/j.jmps.2019.06.021 0022-5096/© 2019 Elsevier Ltd. All rights reserved. The current work studies the situation where the overall direction of crack propagation is at an arbitrary angle to the layers. This overall direction of propagation is enforced using a surfing boundary condition (Hossain et al., 2014). We show that the crack path meanders and the overall toughness may depend on the direction of propagation. We consider two idealized situations.

In the first, the elastic modulus is uniform while the pointwise toughness alternates between two values. The interfacial toughness is taken equal to the lower of the two values. In this situation, we find that the computed crack meanders – with the deflection depending on the angle of overall propagation – though the uniform elasticity and principle of local symmetry predicts a straight path. We show that the computed crack path can be predicted using a maximum dissipation principle. Even though the crack path fluctuates, the effective or overall toughness is surprisingly isotropic – it is equal to that of the tougher material for all angles of overall propagation except the case when it propagates along the layer where the effective toughness is equal to the value of the brittle material. We call this *anomalous isotropy*. Finally, the overall toughness is not convex in the sense of interfacial energy or Wulff shapes (Herring, 1951). In other words, it is not stable against faceting. This reflects the fact that fracture follows a critical state rather than energy minimization.

We then consider the case where the pointwise and interfacial toughness are uniform, but the elastic modulus alternates between two values. The computed crack deflects as it approaches the stiffer material as anticipated by He and Hutchinson (1989). The overall toughness is still higher than that of the uniform pointwise value, though the amount of toughening decreases as the direction of overall propagation deviates from the normal to the layers. We show that this behavior is consistent with the decreasing fluctuation in the crack opening stress. Again, the overall toughness is not convex.

2. Computational approach

2.1. Variational phase-field method

We follow the variational phase-field method introduced by Bourdin et al. (2000, 2008). In this framework, sharp cracks or displacement discontinuities are regularized via a spatially-smooth continuum formulation, by introducing a scalar regularized phase-field $\alpha \in [0, 1]$ such that $\alpha = 0$ corresponds to the intact material and $\alpha = 1$ to a complete fracture. Given a heterogeneous material Ω , which exhibits a linearly-elastic isotropic behavior, phase-field methods minimize the regularized total energy functional

$$\mathcal{E}_{\ell}(\underline{u},\alpha) = \int_{\Omega} \frac{1}{2} \underline{\epsilon} : (\eta + (1-\alpha)^2)\mathbb{C} : \underline{\epsilon} d\Omega + \int_{\Omega} \frac{3G_{\mathsf{c}}}{8} \Big[\frac{\alpha}{\ell} + \ell |\nabla\alpha|^2\Big] d\Omega$$
(1)

where $\underline{\epsilon} = (\nabla \underline{u} + \nabla \underline{u}^t)/2$ is the symmetrized gradient of the displacement field \underline{u} , ℓ is a regularization parameter, \mathbb{C} (with *E* the Young's modulus and ν the Poisson's ratio) is the local energy release rate, G_c is the local toughness (critical energy release rate) and $\eta = o(\ell)$ is a small residual stiffness. The first term approximates the elastic energy stored in unfractured regions while the second term approximates the work of fracture.

We use non-dimensional units in the rest of the paper. Let L_0 be a characteristic lengthscale, so that $\tilde{\Omega} = \Omega/L_0$ is the non-dimensional domain, $\underline{\tilde{z}} = \underline{z}/L_0$ is a point in $\tilde{\Omega}$, $\underline{\tilde{u}} = \underline{u}/L_0$ is the non-dimensional displacement and $\tilde{\ell} = \ell/L_0$ is the non-dimensional regularization parameter. We take a characteristic Young's modulus E_0 and set $\mathbb{C} = \mathbb{C}/E_0$ and $\tilde{G}_c = G_c/(E_0L_0)$ to be the non-dimensional modulus and toughness. Note that the strain $\underline{\tilde{\xi}} = \underline{\epsilon}$, the phase field variable $\tilde{\alpha} = \alpha$ and $\tilde{\eta} = \eta$ since these are non-dimensional. The non-dimensional energy is now

$$\widetilde{\mathcal{E}}_{\ell}(\underline{\widetilde{u}},\widetilde{\alpha}) = \int_{\widetilde{\Omega}} \frac{1}{2} \, \underline{\widetilde{\epsilon}} : \left(\widetilde{\eta} + (1-\widetilde{\alpha})^2 \right) \widetilde{\mathbb{C}} : \underline{\widetilde{\epsilon}} \, d\widetilde{\Omega} + \int_{\widetilde{\Omega}} \frac{3\widetilde{G}_{\mathsf{c}}}{8} \Big[\frac{\widetilde{\alpha}}{\widetilde{\ell}} + \widetilde{\ell} |\widetilde{\nabla}\widetilde{\alpha}|^2 \Big] d\widetilde{\Omega}.$$

$$\tag{2}$$

For the sake of compactness, the tilde appearing above the non-dimensional parameters is dropped in what follows. Further, $\eta = 10^{-6}$, $\ell = 0.25$, E = 1, $\nu = 0.2$ and $G_c = 1$ if it is not specified otherwise. Note that the choice $E = G_c = 1$ is consistent with $L_0 \sim 1 \mu$ m for stiff polymers and metals, and $L_0 \sim 1$ nm for brittle ceramics. This means that our results hold for layers and domains that are larger than these scales.

2.2. Effective toughness

Following Hossain et al. (2014), the effective fracture toughness is determined by using a *surfing boundary condition*. Accordingly, a steadily-translating opening displacement $\underline{u}(\underline{z}, t) = \underline{U}(\underline{z} - t\underline{e}_x)$ is applied at the exterior boundary of the domain, with *t* denoting the time variable and where \underline{U} is the far-field Mode I crack Zehnder (2012),

$$\underline{U} = \frac{\sqrt{EG_{c}(1+\nu)}}{E} \left(\frac{3-\nu}{1+\nu} - \cos\phi\right) \sqrt{\frac{r}{2\pi}} \left(\cos\frac{\phi}{2}\underline{e}_{x} + \sin\frac{\phi}{2}\underline{e}_{y}\right)$$
(3)

As such, the crack is driven to propagate steadily macroscopically in the \underline{e}_x direction, while it can freely interact with the material heterogeneities at the local lengthscale.

At each time-step, the far-field energy-release rate is determined by computing the *J*-integral (Cherepanov, 1967; Rice, 1968) at the boundary of the computational domain. This quantity identifies the macroscopic driving force necessary to sustain crack propagation throughout the microstructure. This macroscopic driving force oscillates as the crack interacts with



Fig. 1. Computational domain.

the material heterogeneities. Thus, we define the effective fracture toughness G_c^{eff} as the smallest driving force necessary to propagate the crack through macroscopic distances. This corresponds to the *maximum* value attained by the far-field *J*-integral during the entire propagation: $G_c^{\text{eff}} = \max_t J(t)$.

This boundary condition was introduced and studied extensively in Hossain et al. (2014). In the case of propagation normal to the layers (and other examples), it was shown that the effective property is independent of (i) the spatial and temporal discretization; (ii) the amplitude of the surfing loading (which is kept constant in this paper); (iii) the particular expression of the opening displacement. Further, the approach was experimentally validated in Hsueh et al. (2018).

2.3. Computational domain

The computational domain Ω is a two-dimensional rectangular domain of (non-dimensional) length *L* and width *H*, comprising two materials (marked as 1 and 2 in Fig. 1) with alternating elastic/toughness properties. Let the Cartesian reference system ($\underline{e}_x, \underline{e}_y$) be introduced as in Fig. 1: so the crack is driven macroscopically in the horizontal direction in the figure. The layers are oriented at an angle θ with respect to \underline{e}_x : so the layers are aligned horizontally for $\theta = 0$ and vertically for $\theta = \pi/2$ in the figure. The (non-dimensional) thickness τ of the layers is defined normal to the layer and thus independent of θ . The domain width is set equal to H = 40 unless specified otherwise.

2.4. Numerical implementation and data sets

The fracture problem is solved by alternatively minimizing the total energy functional in Eq. (2) with respect the two state variables \underline{u} and α . The constrained minimization with respect to the fracture field α is implemented using the variational inequality solvers provided by PETSc Balay et al. (2013a, 2013b, 1997), whereas the minimization with respect to displacement field \underline{u} is a linear problem, solved by using preconditioned conjugated gradients. All computations are performed by means of the open source code mef90 Bourdin (2019). Input and output files can be found at the repository Brach et al. (2019). After discretization in finite elements with mesh size $\delta = 0.1$, the numerical toughness results equal to $G_c^{num} = G_c(1 + 3\delta/8\ell)$ Bourdin et al. (2008).

3. Toughness heterogeneity

3.1. Computational results

Consider the layered material comprised of alternating brittle (material 1) and tough (material 2) materials with identical elastic moduli. The toughness is $G_c^1 = G_c$ and $G_c^2 = 2G_c$ in material 1 and 2 respectively, while the elastic moduli are uniform with $E_1 = E_2 = E$ and $v_1 = v_2 = v$.

Fig. 2 shows the computed crack path for various values of the layer angle θ . The corresponding evolution of the *J*-integral and the work of fracture (proportional to the weighted crack length) as a function of time are shown in Fig. 3.

For $\theta = \pi/2$, the crack advances straight through the microstructure. Nevertheless, its propagation is not continuous in time. When heading from the brittle to the tough material, the crack gets arrested at the interface between the two layers. The far-field energy-release rate increases up to the highest value of the point-wise toughness (Fig. 3a). Subsequently, the propagation resumes and the crack grows steadily through the material till it reaches the interface with the brittle material (Fig. 3b). At that point, the propagation becomes unstable, the crack jumps and the macroscopic energy-release rate instantaneously drops. This is because the macroscopic *J* is higher than the toughness of the brittle material as the crack tip first arrives at the brittle-to-tough interface.

As the layer angle θ decreases, the crack does not propagate straight throughout the microstructure but it deviates as soon as it reaches the interface with the tough material (Fig. 2). The fracture process then continues along the interface by diverting from the macroscopically-imposed direction. The deviation and the distance along the interface increase with



Fig. 2. Layered material with toughness heterogeneity. Crack propagation for different values of the layer angle θ . The thickness of the layers is $\tau = 16$.



Fig. 3. Layered material with toughness heterogeneity. (a) Far-field *J*-integral normalized with respect to the numerical toughness G_c^{num} . (b) Work or fracture proportional to the weighted crack length. The thickness of the layers is $\tau = 16$.



Fig. 4. Layered material with toughness heterogeneity. Normalized effective toughness $G_c^{\text{eff}}/G_c^{\text{num}}$ as a function of the angle θ for different values of the layers thickness τ .

Table 1

Layered material with toughness heterogeneity. Normalized effective toughness $G_c^{\text{eff}}/G_c^{\text{num}}$ for $\theta = \pi/2$ for various values of width *H* and two different values of layers thickness τ . The domain length *L* is constant in these calculations.

Width H	Thickness $\tau = 4$	Thickness $\tau = 16$
10	2.046028	2.059314
20	2.046928	2.058361
40	2.045154	2.065222
60	2.050661	2.074267
80	2.051068	2.068809

decreasing angle. At some point, the surfing boundary condition manages to drive the propagation back towards the original path, the crack then kinks and penetrates the tough material with a small jump. The crack then propagates through the tough material till it reaches the interface with the brittle material which it penetrates with a jump accompanied by a drop in macroscopic *J*. The final crack path then consists of a zig-zag pattern of alternating regimes of propagation: the first along the brittle-to-tough interface that deviates the crack from the macroscopic path and the second through the tough and brittle regions that brings it back to the macroscopic path.

Interestingly, the layer angle θ does not influence the maximum value of the macroscopic *J*-integral (Fig. 3a). The amount of time that the crack needs to reach the interface with the upcoming tough layer is of course different, but the maximum value of the *J*-integral is always equal to the toughness of the tough constituent as long as the crack interacts with both materials (that is, for $\theta \neq 0$). Therefore, according to our definition, the macroscopic toughness G_c^{eff} is equal to the largest pointwise value.

Fig. 4 collects the results above in a polar plot of the effective toughness G_c^{eff} as a function of the layer angle θ . The effective toughness is equal to the toughness of the tough material for any angle $\theta \neq 0$, but falls to that of the brittle material for $\theta = 0$. In the latter case, the layers are parallel to the macroscopic direction of propagation, thus the crack only evolves through the brittle material. Fig. 4 also shows that the effective toughness is independent of the thickness τ of the layers (as long as it is sufficiently larger than the internal length ℓ). Further, the result is independent of the width H of the computational domain as documented in Table 1.

3.2. Analysis

The classical Griffith criterion (Griffith, 1921) states that a crack advances along a prescribed smooth crack path when the energy-release rate is equal to the toughness. However, it is silent about the crack path itself. The determination of crack paths has been the subject of a large literature and a number of criteria have been proposed (Chambolle et al., 2009). Three are widely used: (i) the principle of local symmetry, (ii) the principle of maximum energy-release rate and (iii) the principle of maximum energy dissipation. In homogeneous materials under far-field loading, the path predicted by the three

is quite close to each other, thus it is difficult to distinguish between them. However, the three criteria can lead to different predictions in heterogeneous media, including in the current situation.

The principle of local symmetry postulates that the crack propagates along a path where it locally experiences a strictly crack-opening local elastic field. In other words, the crack is always in Mode I propagation and the second stress-intensity factor is equal to zero, i.e. $K_{II} = 0$. In the current example, the elastic moduli are uniform and the surfing boundary condition at any instant of time imposes a purely Mode I far-field displacement. Therefore, the principle of local symmetry implies that the local crack path will follow the macroscopic one and consequently be straight. This is different from our computational results.

The principle of maximum energy-release rate states that the crack propagates or kinks in a direction that maximizes the energy-release rate. Denoting as \underline{t} the vector tangent to the crack path at the tip, this propagation criterion can thus be rephrased as $\underline{t} = \operatorname{argmax}_{\underline{t}} J(\underline{t})$. Again, this would predict a straight path and this is different from our computational results.

Finally, the principle of maximum energy dissipation states that the crack propagates or kinks in a direction that maximizes the rate of energy dissipation, $\underline{t} = \operatorname{argmax}_{\underline{t}} (J(\underline{t}) - G_c(\underline{t}))$. We now show that the computed crack path broadly follows this criterion.

For this analysis, consider a layer angle θ away from $\pi/2$. It is convenient to define the vector j as follows:

$$\underline{j} = \oint_{\mathcal{C}} \left(W \underline{l} - \nabla \underline{u}^T \underline{\sigma} \right) \underline{n} \, ds \tag{4}$$

where C is any contour around the crack tip, W is the elastic energy density and \underline{n} the outward normal to the contour. Note that the *J*-integral is obtained as the component of j that is tangential to the crack at the crack tip, that is

$$J = \underline{j} \cdot \underline{t} \tag{5}$$

Consider a straight crack advancing along the prescribed macroscopic path in the brittle material 1 approaching an interface with the tough material 2 as shown in Fig. 5(a) at the point marked 1. The crack tip is sufficiently distant from the interface for the toughness to be equal to that of the brittle material (namely, G_c^1) in all directions, as shown by the bold red circle on the left of the figure. Since the crack is straight, the vector \underline{j} is horizontal as shown in the figure. Further, the crack propagates exactly when $J = |\underline{j}| = G_c^2$ also as shown. This continues till the crack propagates all the way to the interface and the crack tip reaches the point marked 2.

The crack would see now a different toughness if it were to propagate in different directions as shown by the solid red curve in Fig. 5(b). This curve is discontinuous: If the crack were to propagate in any direction that takes it to the tough material, it would see a toughness G_c^2 while it would see a toughness G_c^1 if it were to propagate in any direction that retains it in the brittle layer. However, since j is still at the previous value (denoted as 2^- in the figure), the crack is unable to propagate and it arrests. As the loading proceeds, the vector j gradually increases in magnitude while maintaining the direction until it reaches the magnitude marked as 2^+ in the figure. At this point, the component of the vector along the interface or the *J*-integral is exactly equal to G_c^1 . So the crack begins to propagate along the interface.

As it propagates along the interface, it is no longer straight and the direction of the vector <u>j</u> changes as shown in Fig. 5(c). However, for the crack to continue propagating along the interface, it goes along the dashed line where the component along the interface is equal to G_c^1 as shown. As the crack continues to propagate along the interface and deviates from the macroscopic path, the vector <u>j</u> rotates away from the interface; however the magnitude increases to keep it along the dashed line so that $I = G_c^1$.

This proceeds till the vector j has magnitude G_c^2 as shown in the point marked 4 in Fig. 5(d). At this point, the crack can penetrate the tough material 2 at the same angle made by the vector j at the point marked 4. Consequently, the crack moves towards the horizontal line determined by macroscopic loading and the vector j rotates along the circle of radius G_c^2 till it reaches the horizontal when the crack is completely restored to the macroscopic horizontal trajectory.

This crack path is consistent with the computational results. Further, as the angle θ increases, so does the deviation of the crack from the macroscopically-imposed path. This is consistent with results in Fig. 2. Thus, the computational results follow the principle of maximum dissipation.

Finally, the toughness domain $G_c^{\text{eff}}(\theta) : \theta \mapsto G_c^{\text{eff}}$ is not convex in the sense of interfacial energy and Wulff construction Zangwill (1988). Consider a straight crack where the crack face normal is at an angle $\bar{\theta}$ to the horizontal. Now suppose the crack is broken up into a zig-zag crack where the normal to one segment makes an angle θ_1 while the normal to the other segment makes an angle θ_2 . A short trigonometric calculations shows that the straight crack would have less energy if and only if

$$G_{\rm c}^{\rm eff}(\bar{\theta}) \le G_{\rm c}^{\rm eff}(\theta_1) \frac{1}{\cos(\theta_1 - \bar{\theta}) + \sin(\theta_1 - \bar{\theta})\cot(\bar{\theta} - \theta_2)} + G_{\rm c}^{\rm eff}(\theta_2) \frac{1}{\cot(\theta_1 - \bar{\theta})\sin(\bar{\theta} - \theta_2) + \cos(\bar{\theta} - \theta_2)} \tag{6}$$

This is the stability condition for interfaces. The failure of this condition means that the crack can lower its energy by faceting. Taking $\theta_1 = 2\bar{\theta}, \theta_2 = 0$, this inequality for the computed G_c^{eff} reduces to

$$2 \le \frac{3}{2} \frac{1}{\cos \bar{\theta}} \tag{7}$$



Fig. 5. Layered material with toughness heterogeneity. Analysis of crack paths. Material 1 and 2 are respectively brittle and tough.

which fails for $\bar{\theta}$ small enough. Thus, the effective energy is not stable in the sense of interfaces. The relaxation of this is given by the Wulff shape: the set of points *p* that satisfy

$$p \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \le G_{\rm c}^{\rm eff}(\theta). \tag{8}$$

The Wulff shape for this is shown in Fig. A1 (a).



Fig. 6. Layered material with elastic heterogeneity. Crack propagation for different values of the layer angle θ . The thickness of the layers is $\tau = 32$.

The fact that the resulting effective toughness is not stable against faceting is not surprising: it indicates that crack propagation in this simulation follows a critically stable path, and not energy minimization.

4. Elastic heterogeneity

4.1. Computational results

The layered material is now assumed to be comprised of alternating compliant (material 1) and stiff (material 2) phases with uniform toughness. The elastic moduli are equal to $E_1 = E$, $E_2 = 2E$ and $\nu_1 = \nu_2 = \nu$, whereas the local toughness is uniform, $G_c^1 = G_c^2 = G_c$.

The fracture path is shown in Fig. 6 for various layer angles θ . The crack proceeds along the macroscopic direction perpendicular to the layers at $\theta = \pi/2$. However, the propagation is not steady: it gets pinned at the compliant-to-stiff interface as the *J*-integral increases, and breaks through with a jump when the macroscopic *J*-integral reaches a critical value that is significantly higher than the uniform toughness. The current results are consistent with the systematic previous studies Hossain et al. (2014); Hsueh et al. (2018) and Wang and Xia (2017).

As the layer angle θ decreases from $\pi/2$, the crack begins to meander away from the macroscopically imposed direction of propagation. However, there are two regimes. When the deviation from $\pi/2$ is relatively small (see the case $\theta = 3\pi/8$, $\theta = 5\pi/24$ in Fig. 6), the crack meanders as it approaches the compliant-to-stiff interface, but penetrates the interface at a single point. However, when the deviation from $\pi/2$ becomes large (see the cases $\theta = \pi/6$, $\pi/8$ in Fig. 6), the crack propagates along the compliant-to-stiff interface before penetrating into the stiff material.

Fig. 7 (a) shows further details of the first regime focussing on $\theta = 3\pi/8$. While in the compliant material, the crack begins to deviate from the horizontal and attempts to become tangential to the interface as it approaches the upcoming stiff layer (just prior to the time marked A). The progress also considerably slows down and stalls at the interface as it is evident from the work of fracture vs. time. The *J*-integral correspondingly increases up to a value which is strictly larger



Fig. 7. Layered material with elastic heterogeneity. Far-field *J*-integral/ G_c^{num} and work of fracture for a layer angle equal to (a) $\theta = 3\pi/8$ and (b) $\pi/8$. The thickness of the layers is $\tau = 32$.



Fig. 8. Layered material with elastic heterogeneity. (a) Normalized effective toughness J_c/G_c^{num} and (b) crack angle γ as a function of θ . The thickness of the layers is $\tau = 32$. The angle γ has been computed when the crack is at the compliant-to-rigid interface.

than the local toughness. Finally, at the point marked A, the *J*-integral reaches a critical value, when it breaks into the stiff material with a kink and a jump accompanied by a drop in the *J*-integral to the value of the local toughness (step B). The crack propagates and returns to the macroscopic path accompanied with a drop in the macroscopic *J*-integral and acceleration of the crack tip as it approaches the stiff-to-compliant interface (step C). It also reaches the macroscopically imposed path by this time. The crack is briefly arrested as it enters the stiff material and the *J*-integral recovers to the toughness of the material. It then begins a steady propagation and the entire cycle repeats (steps D and E).

The details of the second regime are shown in Fig. 7(b) focussing on the case $\theta = \pi/8$. The crack again deviates from the horizontal as it approaches the compliant-to-stiff interface. It is almost tangential as it reaches it and propagates along the



Fig. 9. Layered material with elastic heterogeneity. (a) Effective toughness G_c^{eff} as a function of the angle θ for different values of the thickness τ of the layers. (b) Influence of the width *H* of the computational domain on the computed effective toughness for $\theta = \pi/2$.

interface as the *J*-integral increases. As the *J*-integral reaches a critical value (step A), it penetrates the stiff material with a jump, a kink and a drop in the *J*-integral (step B). Notice that the peak value of the *J*-integral is smaller than that observed in the case $\theta = 3\pi/8$. The crack subsequently propagates smoothly as it returns to the macroscopically imposed axis with an attendant drop in the *J*-integral till reaches the stiff-to-compliant interface (step C). The crack is briefly arrested as it enters the stiff material and the *J*-integral recovers to the toughness of the material. It then begins a steady propagation and the entire cycle repeats (steps D and E).

Fig. 8 shows a polar plot of the effective toughness G_c^{eff} as a function of the layer angle θ . The macroscopic toughness G_c^{eff} decreases as the layers become more and more aligned with the prescribed crack path, and it finally equals the local toughness for $\theta = 0$. The same figure also reports the computed values of the angle γ , which the crack makes with the compliant-to-stiff interface. As the layers are rotated from $\theta = \pi/2$ towards $\theta = 0$, the crack angle γ monotonically increases from $\pi/2$ (perpendicular to the interface) to values close to π (tangential to the interface).

Finally, Fig. 9 shows the effect of layer thickness τ and domain width *H* on the effective toughness The macroscopic toughness G_c^{eff} decreases as the layers become more and more aligned with the prescribed crack path, and it finally equals the local toughness for $\theta = 0$. It increases with both τ and *H* but eventually saturates.



Fig. 10. Layered material with elastic heterogeneity. Intensity of the stress singularity $(1 - \lambda)$ as a function of the layer angle θ . The thickness of the layers is $\tau = 32$.

4.2. Analysis

He and Hutchinson (1989) studied the problem of a Mode I crack approaching an interface separating two semi-infinite solids. As the crack approaches the interface at an angle different from perpendicular, it begins to experience shear stress. They argued on the basis of the principle of local symmetry that the crack would deflect and that the deflection would depend on the elastic moduli of the two materials. The computationally observed tendency of the crack to turn towards the tangent to the interface as it approaches the stiff layer is consistent with their argument (see the case $\alpha > 0$ in Fig. 8 of He and Hutchinson (1989)).

The computations also show that the effective toughness decreases with the layer angle θ . Hsuch et al. (2018) argued that there are two contributions to the change of toughness. The first comes from the fact that the nominal stress fluctuates through the layered material. It is an easy calculation to use compatibility and equilibrium to compute the nominal stresses in a layered material, and show that the crack opening stress in the two layers is given by

$$\sigma_{yy}^{1,2} = \left((1+\nu) \left(\frac{E_{1,2}}{\langle E \rangle} - 1 \right) \cos^4 \theta - (2+\nu) \left(\frac{E_{1,2}}{\langle E \rangle} - 1 \right) \cos^2 \theta + \frac{E_{1,2}}{\langle E \rangle} \right) \Sigma_{yy} \tag{9}$$

where Σ_{yy} is the macroscopic applied stress, $E_1 = E$, $E_2 = 2E$ and $\langle E \rangle = 3E/2$ is the average elastic modulus. The stress is smaller in the compliant material and higher in the stiff material. So, if the crack were propagating along the macroscopic direction, the macroscopic applied stress would have to be higher to sustain crack propagation. This would result in an effective toughness of

$$G_{c}^{\text{eff,fluc}} = G_{c} \left(-\frac{1+\nu}{3} \cos^{4}\theta + \frac{2+\nu}{3} \cos^{2}\theta + \frac{2}{3} \right)^{-1}.$$
 (10)

The second effect comes from re-nucleation. When the crack-tip is at the compliant-to-stiff interface, the stress singularity is no longer 1/2, but depends on the angle γ that the tangent to the crack tip makes to the interface and on the elastic properties of both layers. This singularity $\lambda(\theta)$ can be computed using the results of Bogy (1971a,b) reported in the Appendix and the observed angle γ for each layer angle. This is shown in Fig. 10. Note that this λ is greater than 1/2 for each θ so that the driving force on the crack tip is zero. This means that the crack has to renucleate at the interface. Tanné et al. (2018) studied nucleation at a wedge and proposed a criterion for nucleation of a crack when the stress is less singular than 1/2. Hsuch et al. (2018) adopted this argument to interfaces. Following their argument,



Fig. 11. Layered material with elastic heterogeneity. (a) Toughening due to stress fluctuations and crack renucleation. (b) Comparison between the numerically-computed (for $\tau = 32$) and theoretical-predicted (Eq. 12 effective toughness.

$$\Delta G_c^{\text{nuc}} = \frac{G_c}{1 - \nu^2} \rho^{2\lambda(\theta) - 1} \sin^4 \theta \tag{11}$$

where ρ is a dimensional parameter.

Putting these effects together,

$$G_{c}^{\text{eff,th}} = G_{c} \left(\left(-\frac{1+\nu}{3} \cos^{4}\theta + \frac{2+\nu}{3} \cos^{2}\theta + \frac{2}{3} \right)^{-1} + \rho^{2\lambda(\theta)-1} \sin^{4}\theta \right)$$
(12)

We use the value at $\theta = \pi/2$ to fit ρ and find 0.2630. Fig. 11(a) shows the two terms individually, while Fig. 11(b) compares the result (12) with the computed value. We see that the trends are similar, though this analysis predicts less toughening than the computations as the overall direction of propagation deviates from the layer normal.



Fig. A1. Layered material with toughness (a) and elastic (b) heterogeneity, with layer thickness respectively equal to $\tau = 16$ and $\tau = 32$. Wulff shapes and computed results.

Finally, it is possible to verify as in the previous section that the resulting effective energy in Fig. 9 is not stable in the sense of interfaces. The Wulff shape for this is shown in Fig. A1 (b). Again, the fact that the resulting effective toughness is not stable against faceting is not surprising: it indicates that crack propagation in this simulation follows a critically stable path, and not energy minimization.

5. Concluding remarks

This continues the study of the effective toughness of layered materials started in Hossain et al. (2014) and Hsueh et al. (2018), with a focus on anisotropy. We use the phase-field model and the surfing boundary condition to propagate a crack macroscopically at various angles to the layers. We study two idealized situations, the first where the elastic

modulus is uniform while the toughness alternates and a second where the toughness is uniform and elastic modulus alternates.

In the first situation, we find that the computed crack meanders – with the deflection depending on the angle of overall propagation – though the uniform elasticity and principle of local symmetry predicts a straight path. We show that the computed crack path can be predicted using a maximum dissipation principle. Even though the crack path fluctuates, the effective or overall toughness displays anomalous isotropy – it is equal to that of the tougher material for all angles of overall propagation except the case when it propagates along the layer where the effective toughness is equal to the value of the brittle material. An important observation here is that the overall or effective toughness is unaffected by the amount of crack deflection – the overall toughness remains the same independent of the angle, but the amount of crack deflection changes with it. Thus, the effective toughness is independent from the integrated pointwise work of fracture, but governed by crack arrest instead.

In the second situation, the computed crack deflects as it approaches the stiffer material as anticipated by He and Hutchinson (1989). The overall toughness is higher than that of the uniform pointwise value, the amount of toughening decreases as the direction of overall propagation deviates from the normal to the layers. We show that the toughening is a result of fluctuation in the nominal stress as well as the need to renucleate at the compliant-to-stiff boundary.

Finally, the resulting anisotropic overall toughness is not convex in the sense of interfacial energy. This reflects the fact that crack propagation follows a critically stable path, and not energy minimization.

Acknowledgement

We gratefully acknowledge the support of the US National Science Foundation through Award no. DMS-1535083 and DMS-1535076 under the Designing Materials to Revolutionize and Engineer our Future (DMREF) Program. Some numerical simulations were performed using the Extreme Science and Engineering Discovery Environment (XSEDE) stampede2 at the Texas Advanced Computing Center (TACC) through the allocation TG-DMS060014, supported by National Science Foundation grant number ACI-1548562. The rest of simulations were performed at the Caltech high performance cluster supported in part by the Moore Foundation, USA.

Appendix

Elastic singularity for a crack impinging on an interface

Accordingly to Bogy Bogy (1971a,b), the order λ of the singularity for a crack with a tip at an elastic interface (see Fig. 8(b)) is computed by solving

$$K_1(\gamma,\beta,\lambda)\alpha^2 + K_2(\gamma,\beta,\lambda)\alpha + K_3(\gamma,\beta,\lambda) = 0$$
(13)

where $\alpha \in [-1, 1]$ and $\beta \in [-0.5, 0.5]$ are the Dunders parameters Dunders (1969)

$$\alpha = \frac{\mu_1 m_2 - \mu_2 m_1}{\mu_1 m_2 + \mu_2 m_1} \qquad \beta = \frac{\mu_1 (m_2 - 2) - \mu_2 (m_1 - 2)}{\mu_1 m_2 + \mu_2 m_1} \tag{14}$$

with $m_1 = 4(1 - \nu_1)$ and $m_2 = 4(1 - \nu_2)$ and where the indexes 1 and 2 respectively refer to the material containing the crack and to its adjacent intact counterpart. Coefficients in Eq. (13) are equal to

$$K_{1}(\gamma, \beta, \lambda) = A\beta^{2} + (2A - B)\beta + A - B + 1$$

$$K_{2}(\gamma, \beta, \lambda) = (-2A + B + C)\beta^{3} + (-4A + 2B + C - D - 2)\beta^{2}$$

$$+ (-2A + B - C)\beta - C + D$$

$$K_{3}(\gamma, \beta, \lambda) = (A - B - C + D + E + 1)\beta^{4} + (2A - B - C)\beta^{3}$$

$$+ (A + C - D - 2E)\beta^{2} + C\beta + E$$
(15)

with

$$A(\lambda, \gamma) = 4\lambda^{4} \sin^{4}(\gamma) + \sin^{2}[\lambda(2\gamma - \pi)]$$

$$B(\lambda, \gamma) = 4\lambda^{2} \sin^{2}(\gamma) + 2\sin^{2}[\lambda(2\gamma - \pi)]$$

$$C(\lambda, \gamma) = 4\lambda^{2} \sin^{2}(\gamma) \{\sin^{2}(\lambda\gamma) + \sin^{2}[\lambda(\gamma - \pi)] - 1\}$$

$$D(\lambda, \gamma) = 2\{\sin^{2}(\lambda\gamma) + \sin^{2}[\lambda(\gamma - \pi)] - 1\}$$

$$E(\lambda) = 1 - \sin^{2}(\lambda\pi)$$

(16)

Notice that for a normal crack when $\gamma = \pi/2$, the roots of Eq. (13) are real for any choice of Dunder's parameters α and β . Further, for an interfacial crack when $\gamma = \pi$, the roots of Eq. (13) are complex for any choice of Dunder's parameters α and β . For intermediate values of the angle γ the roots may be real or complex depending on the Dunder's parameters. For our elastic (Dunder's) parameters and our computed γ , the root is real for $\theta \in (\pi/8, \pi/2)$ and shown in Fig. 10.

Wulff shape The Wulff shape for the effective toughness in both cases is shown in Fig. A1.

References

Balay, S., Abhyankar, S., Adams, M., Brown, J., Brune, P., Buschelman, K., Dalcin, L.D., Eijkhout, V., Gropp, W., Kaushik, D., Knepley, M., May, D., McInnes, L.C., Munson, T., Rupp, K., Sanan, P., Smith, B., Zampini, S., Zhang, H., Zhang, H., 2013. PETSc Users Manual Revision 3.4. Technical Report. Argonne National Laboratory (ANL), Argonne, IL (United States).

Balay, S., Brown, J., Buschelman, K., Gropp, W., Kaushik, D., Knepley, M., McInnes, L., Smith, B., Zhang, H., 2013b. PETSc/Tao: Home Page.

Balay, S., Gropp, W.D., McInnes, L.C., Smith, B.F., 1997. Efficient management of parallelism in object-oriented numerical software libraries. In: Modern Software Tools for Scientific Computing. Birkhäuser Boston, Boston, MA, pp. 163–202. doi:10.1007/978-1-4612-1986-6_8.

Bogy, D.B., 1971. On the plane elastostatic problem of a loaded crack terminating at a material interface. J. Appl. Mech. 38 (4), 911-918.

Bogy, D.B., 1971. Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. J. Appl. Mech. 38 (2), 377-386.

Bourdin, B., 2019. mef90/vDef: variational models of defect mechanics. Available at https://github.com/bourdin/mef90.

Bourdin, B., Francfort, G., Marigo, J.J., 2000. Numerical experiments in revisited brittle fracture. J. Mech. Phys. Solids 48 (4), 797-826. doi:10.1016/ S0022-5096(99)00028-9.

Bourdin, B., Francfort, G.A., Marigo, J.J., 2008. The variational approach to fracture. J. Elast. 91 (1-3), 5-148. doi:10.1007/s10659-007-9107-3.

Brach, S., Hossain, Z., Bourdin, B., Bhattacharya, K., 2019. Data repository of: Anisotropy of the effective toughness of layered media. Available at https://data.caltech.edu/records/1244. doi: 10.22002/d1.1244.

Chambolle, A., Francfort, G.A., Marigo, J.-J., 2009. When and how do cracks propagate? J. Mech. Phys. Solids 57 (9), 1614-1622.

Cherepanov, G., 1967. Crack propagation in continuous media: PMM vol. 31, no. 3, 1967, pp. 476-488. J. Appl. Math. Mech. 31 (3), 503-512. doi:10.1016/ 0021-8928(67)90034-2.

Dunders, J., 1969. Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. J. Appl. Mech. 36, 650-652.

Faber, K., Evans, A., 1983. Crack deflection processes. 1. theory. Acta Metall. Mater. 31 (4), 565-576.

Faber, K., Evans, A., 1983. Crack deflection processes. 2. experiment. Acta Metall. Mater. 31 (4), 577-584.

Griffith, A.A., 1921. The phenomena of rupture and flow in solids. Philosoph. Trans. R. Soc. A 221 (582-593), 163-198. doi:10.1098/rsta.1921.0006.

He, M.Y., Hutchinson, J.W., 1989. Crack deflection at an interface between dissimilar elastic-Materials. Int. J. Solids Struct. 25 (9), 1053-1067.

Herring, C., 1951. Some theorems on the free energies of crystal surfaces. Phys. Rev. 82, 87-93.

- Hossain, M.Z., Hsueh, C.J., Bourdin, B., Bhattacharya, K., 2014. Effective toughness of heterogeneous media. J. Mech. Phys. Solids 71 (1), 15–32. doi:10.1016/j. jmps.2014.06.002.
- Hsueh, C.J., Avellar, L., Bourdin, B., Ravichandran, G., Bhattacharya, K., 2018. Stress fluctuation, crack renucleation and toughening in layered materials. J. Mech. Phys. Solids 120, 68–78.

Hutchinson, J., Suo, Z., 1992. Mixed mode cracking in layered materials. Adv. Appl. Mech. 29, 63-191.

Milton, G., 2002. The Theory of Composites. Cambridge University Press.

Rice, J.R., 1968. Mathematical analysis in the mechanics of fracture. Technical Report.

- Tanné, E., Li, T., Bourdin, B., Marigo, J.J., Maurini, C., 2018. Crack nucleation in variational phase-field models of brittle fracture. J. Mech. Phys. Solids 110, 80–99.
- Wang, N., Xia, S., 2017. Cohesive fracture of elastically heterogeneous materials: an integrative modeling and experimental study. J. Mech. Phys. Solids 98, 87–105.
- Zangwill, A., 1988. Physics at surfaces. Cambridge University Press.
- Zehnder, A.T., 2012. Fracture mechanics. Lecture Notes in Applied and Computational Mechanics, 62. Springer Netherlands, Dordrecht doi:10.1007/ 978-94-007-2595-9.