FINAL EXAMINATION

Due: December 3 (Monday) by midnight

Instructions:

- The assignment consists of *three* questions, worth respectively 4, 4 and 2 points.
- Submit your assignment *electronically* (via Email) to the instructor; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/~bprotas/CES712/template.m` (see also the link in the "Computer Programs" section on the left); submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Submissions which do not comply with these guidelines will not be accepted.

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1. Using the Particle–Particle formulation of the 2D vortex dynamics problem (e.g., the code `vortex_02.m`), develop a code that will solve the corresponding problem in 2D "celestial mechanics" described by the equations

$$\begin{cases} m_i \ddot{\mathbf{x}}_i(t) = -G\nabla \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_i m_j}{2\pi} \ln|\mathbf{x}_i(t) - \mathbf{x}_j(t)|, & i = 1,\ldots,N, \\ \dot{\mathbf{x}}_i(0) = \mathbf{v}_i^0, \\ \mathbf{x}_i(0) = \mathbf{x}_i^0, \end{cases}$$

   where you can assume the 2D "gravitational constant" $G = 10$. Consider a system with $N = 4$ bodies and the following masses and initial conditions:

$$\begin{array}{llll} m_1 = 1, & m_2 = 1, & m_3 = 1, & m_4 = 0.1, \\ \mathbf{v}_1^0 = [0,0], & \mathbf{v}_2^0 = [0,0], & \mathbf{v}_3^0 = [0,0], & \mathbf{v}_4^0 = [0,0], \\ \mathbf{x}_1^0 = [0,1], & \mathbf{x}_2^0 = [0,-1], & \mathbf{x}_3^0 = [0.5,0.5], & \mathbf{x}_4^0 = [2,0]. \end{array}$$

   Integrate the system in time using the leapfrog scheme with $\Delta t = 1.0\cdot 10^{-3}$ and use $N_t = 2\cdot 10^3$ time steps. Plot the trajectories of all bodies as well as the time histories of the X and Y coordinates of the mass centroid of the system.
   [4 points]

2. Consider the Particle–Mesh formulation of the 2D vortex dynamics problems involving interaction of two same–sign vortex patches each containing 50 point vortices (i.e., the initial configuration used in the code `vortex_06.m`).
   (a) Implement the Dirichlet boundary conditions in the Poisson equation for the streamfunction Ψ that *correctly* represent an unbounded domain.
   (b) Same as in point (a) above, but assume that the domain is a half–plane bounded by a wall at $y = -3$.

Using the same integration parameters as employed in the code `vortex_06.m` and a time interval consisting of 25 time steps, generate plots of the streamfunction $\Psi(x,y)$ at the end of the simulation in both cases.
[4 points]

3. Using the concepts of multipole expansions, implement an efficient way to compute the potential (streamfunction) at some distant point $\mathbf{x}_f$. Perform computations for $\mathbf{x}_f = [10,5]$ and using the initial vortex distribution from point (2a) above. Assume $\mathbf{x}_c = [0,0]$ as the center of the multipole expansions. Plot the *relative* error with respect to the exact value (obtained by evaluating suitable Green's functions) as a function of the number $p \in \{1,2,\ldots,25\}$ of terms used in the multipole expansion.
[2 points]