* Time-integration of the evolution equations

Regardless of how the potential ϕ (the streamfunction ψ) is calculated, i.e. via direct summation of particle-particle interactions (PP approach), or solution of the "field equation" for the potential (PM approach), we obtain the following system of ODEs which must be integrated in time:

- for the N-vortex problem (2D)

$$\begin{cases} \dot{x}_i = \frac{\partial}{\partial y}\psi(\bar{x}_1, \dots, \bar{x}_N) \\ \dot{y}_i = -\frac{\partial}{\partial x}\psi(\bar{x}_1, \dots, \bar{x}_N) \qquad i=1,\dots,N \\ \bar{x}_i(0) = \bar{x}_{0i} \end{cases}$$

- for the N-~~[illegible]~~ body problem

$$\begin{cases} \ddot{x}_i = -\frac{\partial}{\partial x}\phi(\bar{x}_1, \dots, \bar{x}_N) \\ \ddot{y}_i = -\frac{\partial}{\partial y}\phi(\bar{x}_1, \dots, \bar{x}_N) \qquad i=1,\dots,N \\ \ddot{z}_i = -\frac{\partial}{\partial z}\phi(\bar{x}_1, \dots, \bar{x}_N) \\ \bar{x}_i(0) = \bar{x}_{0i} \\ \dot{\bar{x}}_i(0) = \bar{v}_{0i} \end{cases}$$

Note that this is a system of second-order ODEs. In order to be consistent with existing ODE integrators, it must be converted to an (augmented) system of first-order ODEs

$$
\begin{cases}
\dot{x}_i = u_i \\
\dot{y}_i = v_i \\
\dot{z}_i = w_i \\
\dot{u}_i = -\frac{\partial}{\partial x}\phi(\bar{x}_1, \dots, \bar{x}_n) \\
\dot{v}_i = -\frac{\partial}{\partial y}\phi(\bar{x}_1, \dots, \bar{x}_n) \\
\dot{w}_i = -\frac{\partial}{\partial z}\phi(\bar{x}_1, \dots, \bar{x}_n) \\
\bar{x}_i(0) = \bar{x}_{0i} \\
\bar{V}_i(0) = \bar{V}_{i0}
\end{cases}
\qquad
\begin{array}{l}
i = 1, \dots, N \\
\bar{V} = [u, v, w]
\end{array}
$$

* See the lecture notes "Finite Difference Methods for Differential equations" (pages 34-51) for a review of relevant numerical techniques