TEST # 1 - SOLUTIONS (CES 712)

① $\{m_1, \dots, m_N\}$ - given, G - gravitational constant

a)

$$\begin{cases} m_i \ddot{\bar{x}}_i = G \sum_{\substack{j=1 \\ j \neq i}}^{N} m_i m_j \frac{\bar{x}_i - \bar{x}_j}{|x_i - x_j|^3}, & i = 1, \dots, N \\ \bar{x}_i(0) = \bar{x}_{i0} & \bar{x}_i = [x_{1i}, x_{2i}, x_{3i}] \\ \dot{\bar{x}}_i(0) = \bar{v}_{i0} \end{cases}$$

PP formulation

PM formulation

b)

$$\begin{cases} m_i \ddot{\bar{x}}_i = -m_i \nabla \phi(\bar{x}_i) \\ \begin{bmatrix} \Delta \phi(x) = -G \sum_{j=1}^{N} m_j \delta(\bar{x}_0 - \bar{x}_j) & i = 1, \dots, N \\ + \text{BCs (e.g. of Dirichlet type } \phi|_{\partial\Omega} = \phi_b) \end{bmatrix} \\ \bar{x}_i(0) = \bar{x}_{i0} \\ \dot{x}_i(0) = \bar{v}_{i0} \end{cases}$$

c) # of degrees of freedom

per particle - 3 velocity components
3 position components $\Big\} \Rightarrow \underline{6N}$

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2 $\{\Gamma_1, \dots, \Gamma_N\}$ - given

a) PP formalism

$$\begin{cases} \dot{x}_i = \sum\limits_{\substack{j=1 \\ j \neq i}}^{N} \Gamma_j \frac{\partial}{\partial y}\left[\frac{1}{2\pi} \ln|\bar{x}_i - \bar{x}_j|\right] \\ \dot{y}_j = -\sum\limits_{\substack{j=1 \\ j \neq i}}^{N} \Gamma_j \frac{\partial}{\partial x}\left[\frac{1}{2\pi} \ln|\bar{x}_i - \bar{x}_j|\right] \qquad i = 1, \dots, N \\ x_i(0) = x_{i0} \\ y_i(0) = y_{i0} \end{cases}$$

2 - cont.

b) PM formulation (ψ - streamfunction)

$$\begin{cases} \dot{x}_i = \dfrac{\partial}{\partial y}\psi(\bar{x}_i) & i=1,\dots,N \\ \dot{y}_i = -\dfrac{\partial}{\partial x}\psi(\bar{x}_i) \\ \left[\begin{array}{l} \Delta\psi(x) = -\displaystyle\sum_{j=1}^{N}\Gamma_j\,\delta(\bar{x}-\bar{x}_j) \\ + \text{BCs (e.g. Dirichlet } \underline{\text{type}}\ \psi|_{\partial\Omega} = \psi_b) \end{array}\right. \\ x_i(0) = x_{i0} \\ y_i(0) = y_{i0} \end{cases}$$

c) # of degrees of freedom:

per particle: - 2 components of position $\Big\} \Rightarrow \underline{2N}$

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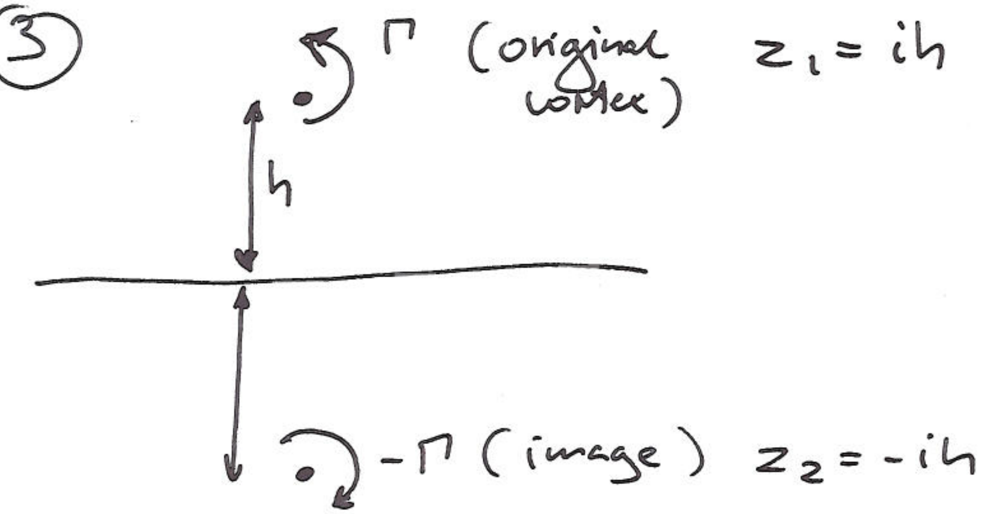
(3)



The original vortex moves in the velocity field induced by the image vortex

$$V_2(z_1) = -\frac{\Gamma}{2\pi i}\,\frac{1}{z_1 - z_2} = -\frac{\Gamma}{2\pi i}\,\frac{1}{ih + ih}$$

$$= (u - iv)(z_1)$$

$$= \frac{\Gamma}{4\pi h} \quad \text{(purely real)}$$

Thus: $\left. u \right|_{\substack{\text{original}\\ \text{vortex}}} = \mathrm{Re}\left[V_2(z_1)\right] = \frac{\Gamma}{4\pi h}$

The original vortex will move to the <u>right</u>.

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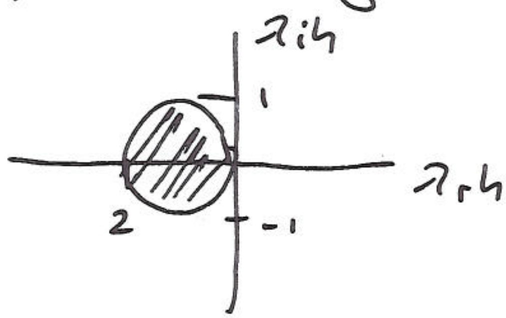
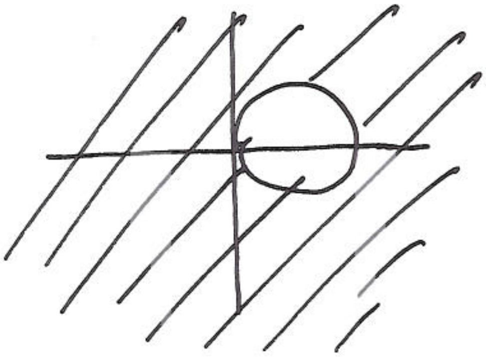
(4)

a) Euler explicit



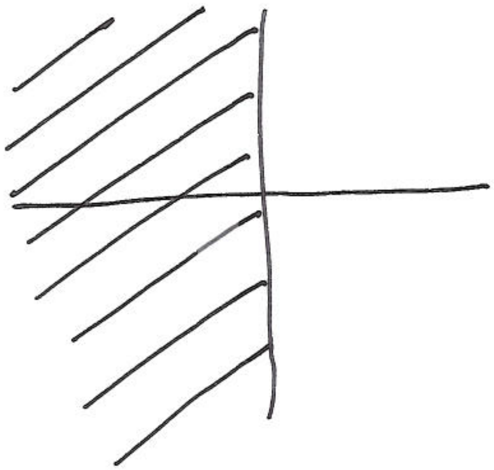
global error $O(h)$

b) Euler implicit



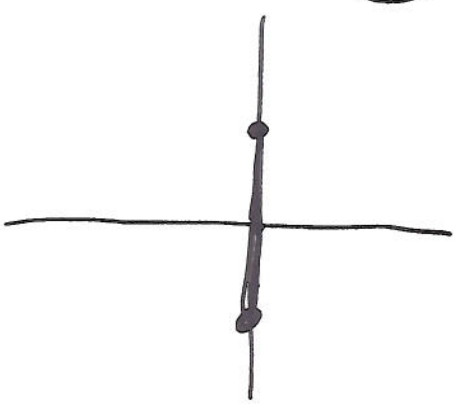
global error $O(h)$

c) Crank-Nicolson



global error $O(h^2)$

a) Leapfrog



global error $O(h^2)$

The scheme with the best conservation properties is Leapfrog $\Rightarrow |\sigma| \equiv 1$ if $\underline{h \leq \frac{1}{|\lambda_i|}}$