FINAL EXAMINATION

Due: October 26 (Friday) by midnight

Instructions:

- The assignment consists of *four* questions, each worth 5 points.
- Submit your assignment *electronically* (via Email) to the instructor; hardcopy submissions will not be accepted.
- It is obligatory to use the MATLAB template file available at `http://www.math.mcmaster.ca/~bprotas/CES717/template.m` (see also the link in the "Computer Programs" section on the left); submissions non compliant with this template will not be accepted.
- Make sure to enter your name and student I.D. number in the appropriate section of the template.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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1. You are given the following boundary value problem

$$\begin{cases} -\dfrac{d^2u}{dx^2} + u = -e^x[\sinh(x) + 2\cosh(x)] & \text{for } x \in \Omega = [0,1] \\ u(0) = 0, & \\ \dfrac{du}{dx}(1) = e[\cosh(1) + \sinh(1)], & \end{cases}$$

   Solve this problem using the Galerkin Finite Element Method with $N = 10$ linear elements distributed uniformly in the domain Ω. Then plot on three separate figures:

   (a) all the basis functions,

   (b) the obtained solution, and

   (c) the error $e(x) = u(x) - u_{ex}(x)$ with respect to the analytical solution given by $u_{ex} = e^x \sinh(x)$,

   (5 points)
   HINT — It is probably easier to write the code from scratch, rather than adapt one of programs from the textbook. Given the simplicity of the domain geometry, the stiffness matrix and the load vector can be easily assembled without having to construct a sophisticated mesh structure.

2. Consider a boundary value problem defined in the circular disc $\Omega = \{(x,y),\ x^2 + y^2 < 1\}$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = \dfrac{1}{2}\sin(2\varphi), & \end{cases} \tag{1}$$

   where (r,φ) are the polar coordinates of the point (x,y). Using a linear Finite Element method, find a mesh such the the *relative* L_2 error with respect to the exact solution (which you can find employing analytical techniques) will be less than 0.1%. Then:

   (a) plot this mesh and the solution to problem (1) obtained on this mesh,

(b) print the numbers of vertices and triangles in this mesh.

(5 points)
HINT — Convert the problem to the Cartesian coordinates and then suitably adapt one of the codes from the textbook.

3. Using the grid determined in point 2, solve the Neumann version of problem (1), i.e.,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = \sin(2\varphi). \end{cases} \tag{2}$$

Note that the solution of problem (1) is also one of solutions of problem (2). Compute and print the relative L_2 error norm of the solution of problem (2). What needs to be done in order to solve this problem *correctly*?
(5 points)

4. Use the method of the Conjugate Gradients to solve the algebraic problems resulting from discretization of systems (1) and (2) at the finest grid and with the following levels of tolerance: $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}$. Plot the number of iterations requires to solve the problem in the two cases as a function of the tolerance. What explains the difference in the behavior of the Conjugate Gradients method in the two cases?
(5 points)