CES717 – TEST # SOLUTIONS

①

a) The boundary integral must vanish because $\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = 0$; hence we have <u>homogeneous Neumann boundary value problem</u>

$$\begin{cases} -\Delta u = g & \text{in } \underline{\Omega} \\ \left.\frac{\partial u}{\partial n}\right|_{\partial \underline{\Omega}} = 0 \end{cases}$$

$$-\int_{\underline{\Omega}} \Delta u\, v\, d\underline{\Omega} = \int_{\underline{\Omega}} \nabla u \cdot \nabla v\, d\underline{\Omega} + \cancel{\oint_{\partial \underline{\Omega}} \frac{\partial u}{\partial n} v\, d\sigma}$$

$$= \int_{\underline{\Omega}} g v\, d\underline{\Omega}$$

b) The boundary condition is build into the conformation of the space

$$H^1_0 = \{ v \in H^1(\underline{\Omega}),\ v|_{\partial \underline{\Omega}} = 0 \}$$

Thus we have homogeneous Dirichlet boundary value problem

$$\begin{cases} -\Delta u = g & \text{in } \underline{\Omega} \\ u|_{\partial \underline{\Omega}} = 0 \end{cases}$$

2) $$\begin{cases} \dfrac{d^4u}{dx^4} = g & \text{in } [a,b] \\ u|_a = u|_b = 0 & (\text{"essential" BC}) \\ \dfrac{d^2u}{dx^2}\Big|_a = \dfrac{d^2u}{dx^2}\Big|_b = 0 & (\text{"natural" BC}) \end{cases}$$

Multiplying by a test function v and integrating by parts twice

$$\int_a^b \frac{d^4u}{dx^4} v\, dx = -\int_a^b \frac{d^3u}{dx^3}\frac{dv}{dx}\, dx + \left[\frac{d^3u}{dx^3} v\right]_a^b$$

$$= \int_a^b \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\, dx - \underbrace{\left[\frac{d^2u}{dx^2}\frac{dv}{dx}\right]_a^b}_{\text{vanishes because of the second BC}} + \underbrace{\left[\frac{d^3u}{dx^3} v\right]_a^b}_{\text{will vanish if we require } v|_a = v|_b = 0}$$

Thus, the appropriate function space is $H^2_0(a,b)$

$$H^2_0(a,b) = \{ v \in H^2(a,b),\ v|_a = v|_b = 0 \}$$

and the weak formulation is

find $u \in H^2_0(a,b)$ ST $\displaystyle\int_a^b \frac{d^2u}{dx^2}\frac{d^2v}{dx^2}\, dx = \int_a^b g v\, dx \quad \forall\, v \in H^2_0(a,b)$

3) Using the projection theorem

$$(u-w, v) = 0 \quad \text{for all } v \in W$$

We will take $w_1, \dots, w_n$ in terms of v. Then

$$\left(u - \sum_{j=1}^{n} \alpha_j w_j, w_i\right) = 0 \;, \quad i = 1, \dots, n$$

$$\sum_{j=1}^{n} (w_j, w_i)\alpha_j = (u, w_i)$$

$w_1, \dots, w_n$ - orthonormal $\Rightarrow (w_j, w_i) = \delta_{ji}$

$$\sum_{j=1}^{n} \delta_{ji}\alpha_j = \alpha_i = \underline{\underline{(u, w_i)}}$$

Thus, the best approximation $\underline{w = \sum_{i=1}^{n} (u, w_i) w_i}$

If $u \in W$, then it can be represented as
$$u = \sum_{i=1}^{n} \beta_i w_i$$
and it follows from the above that $\beta_i = \alpha_i$, $i = 1, \dots, n$, so that $u \equiv w$ and the $\underline{\text{error} = u - w = 0}$.

Free nodes:

$N_f = 3$

$f_1 = 4,\ f_2 = 5,\ f_3 = 6$

Constrained nodes:

$N_c = 6$

$c_1 = 1,\ c_2 = 2,\ c_3 = 3$

$c_4 = 7,\ c_5 = 8,\ c_6 = 9$

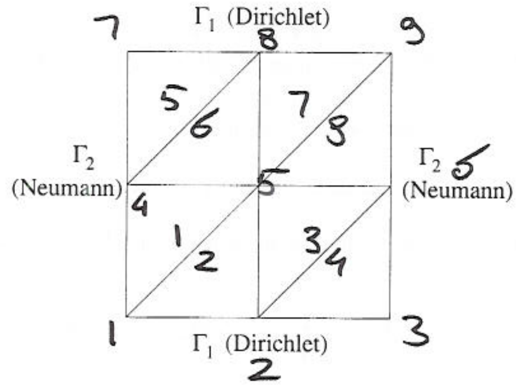

Figure 1: Grid and boundary conditions.

(a) determine the number N_f and the indices $f_1, \ldots, f_{N_f}$ of the *free nodes*,

(b) determine the number N_c and the indices $c_1, \ldots, c_{N_c}$ of the *constrained nodes*,

(c) using the table below determine which entries of the stiffness matrix $\mathbb{K}$ will vanish (○) and which will be nonzero ($\surd$); leave the rows and columns corresponding to the constrained nodes blank.

	1	2	3	4	5	6	7	8	9
1									
2									
3									
4				✓	✓	○			
5				✓	✓	✓			
6				○	✓	✓			
7									
8									
9									

[2 points]