HOMEWORK #1

Due: October 11 (Wednesday) by 5pm

Instructions:

- The assignment consists of *three* questions worth, respectively, 5, 2 and 3 points.
- Submit your assignment *in hardcopy* to the 2E03 locker in the basement of Hamilton Hall (outside the room HH 105).
- The pages of your assignment should be stapled together with solutions to problems appearing in the correct order.
- Your name, student I.D. number and the course number must be clearly written on the first page of the assignment.
- Late submissions and submissions which do not comply with these guidelines will not be accepted.

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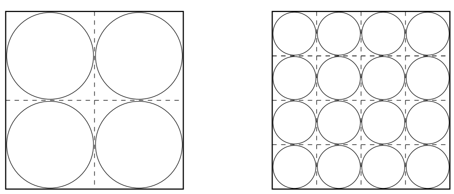
1. You are given a square two–dimensional box with the side L. The box is filled with circles in such a way that the centers of the circles are located at the nodes of a regular rectangular lattice with n nodes in each direction (see Figure).



$n = 2$ $n = 4$ . . .

Consider now the limit $n \to \infty$ of infinitely many infinitesimal circles stacked in the box. Answer the following questions:

(a) what is the probability that a point randomly placed inside the box will end up in one of the circles?

(b) how will this probability change when instead of circles in a 2D box we consider spheres in a 3D box?

(c) how will this probability change when we consider k–dimensional spheres in an k–dimensional box, where k is an *even* number? (note that in such case the volume of a sphere in an k–dimensional is given by $V_k = \frac{\pi^{\frac{k}{2}}}{(\frac{k}{2})!} R^k$, where R is the radius)

(2+1+2=5 points)

2. Regarding Moodelling Problem # 2 from the textbook, consider the situation when the path of the tunnel is given by the monotonically decreasing function $y = \eta(x)$ with $\eta(0) = h$, the depth of the lake at the tunnel entrance. What is the potential energy E_p of the fluid in the tunnel? Calculate also $\frac{d}{dt}(E_p + E_k)$, where $E_k = \frac{1}{2}\rho A x \dot{x}^2$ is the kinematic energy discussed at the lecture.
(2 points)

3. You are given the following ordinary differential equation modelling the growth of a population

$$\begin{cases} \dfrac{dx}{dt} = x^2, & t > 0, \\ x(0) = x_0, & t = 0. \end{cases}$$

Does this problem admit a solution existing for all times $0 < t \leq \infty$? If not, estimate the time when the solution ceases to exist.
Hint — integrate this equation to obtain its solution and see at what time it becomes singular.
(3 points)