Q1

a) Probability $P = \frac{V_C}{V_B} = \frac{\text{volume of the circles}}{\text{volume of the box}}$

Volume of the box (independent of n) : $V_B = L^2$

Volume of the circle for n=1 $\quad V_{C(1)} = \pi R^2 = \pi \left(\frac{L}{2}\right)^2 = \frac{\pi L^2}{4}$

——— " ——— for n=2

$$V_{C(2)} = \underbrace{2^2}_{\#\text{ of circles}} \pi \left(\frac{L}{2\cdot 2}\right)^2 = 4\pi \frac{L^2}{16} = \frac{\pi L^2}{4}$$

Volume of the circle for n=4

$$V_{C(4)} = \underbrace{4^2}_{\#\text{ of circles}} \pi \left(\frac{L}{2\cdot 4}\right)^2 = 16\pi \frac{L^2}{64} = \pi\frac{L^2}{4}$$

and for an arbitrary n

$$V_{C(n)} = n^2 \cdot \pi \left(\frac{L}{2\cdot n}\right)^2 = \cancel{n^2} \pi \frac{L^2}{4\cdot \cancel{n^2}} = \frac{\pi L^2}{4}$$

Therefore, the corresponding probability

$$P_n = \frac{V_{C(n)}}{V_B} = \frac{\pi \frac{L^2}{4}}{L^2} = \frac{\pi}{4} = \lim_{n\to\infty} P_n$$

(independent of n!)

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b) Now a sphere in a 3D box

Volume of a sphere: $V_s = \frac{4}{3}\pi R^3$

Volume of a 3D box: $V_B = L^3$

radius of a sphere (when n sphere are packed in every direction) $R = \frac{L/2}{n} = \frac{L}{2n}$

# of spheres in a 3D box : n^3

Volume of spheres

$$V_{S(n)} = n^3 \cdot \frac{4}{3}\pi\left(\frac{L}{n\cdot 2}\right)^3 = \cancel{n^3}\,\frac{4}{3}\pi\,\frac{L^3}{\cancel{n^3}\,8} = \frac{\pi}{6}L^3$$

(again, independent of n)

$$P = \frac{V_{S(n)}}{V_B} = \frac{\frac{\pi}{6}L^3}{L^3} = \frac{\pi}{6}$$

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c) Volume of a K-dimensional sphere

$$V_K = \frac{\pi^{\frac{K}{2}}}{\left(\frac{K}{2}\right)!} R^K$$

Note that this is valid only for <u>K even</u>

(for K odd the expression for V is different)

Note that for $K=2$

$$V_2 = \frac{\pi}{1}R^2 = \pi R^2 \quad (\text{see a})$$

Radius of a K-sphere (with n spheres packed in every direction) $R = \frac{L/2}{n} = \frac{L}{2\cdot n}$

# of spheres in an K-dimensional box : n^K

Volume of the spheres:

$$V_{S(n)} = n^K \cdot \frac{\pi^{\frac{k}{2}}}{\left(\frac{k}{2}\right)!} \left(\frac{L}{2 \cdot n}\right)^K = \cancel{n^K} \frac{\pi^{\frac{k}{2}}}{\left(\frac{k}{2}\right)!} \frac{L^K}{2^K \cdot \cancel{n^K}}$$

again, independent of n

Volume of a k-dimensional box: $V_B = L^K$

Probability:

$$P = \frac{V_{S(n)}}{V_B} = \frac{\frac{\pi^{k/2}}{(k/2)!} \frac{L^K}{2^K}}{L^K} = \frac{\pi^{k/2}}{(k/2)!\, 2^K}$$

Note that the result of point (a) is obtained for ~~…~~ $k=2$

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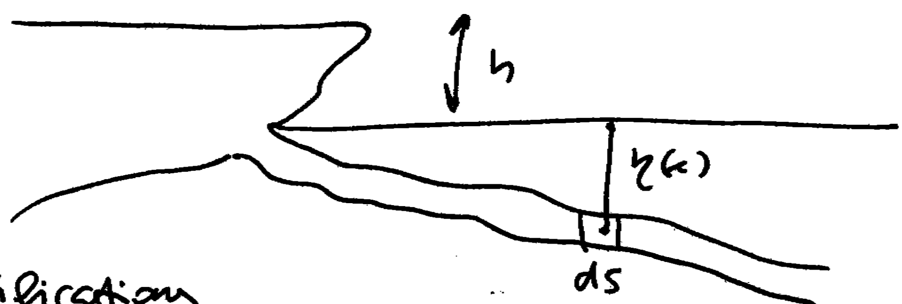
Q2



Modifications

– potential energy of the fluid must be taken into account

Potential energy:

New energy $E = E_K + E_P$, $E_K = \frac{1}{2} \rho A x(\dot{t})^2$ (old)

Potential energy of an infinitesimal element ds

$$dE_P = \rho g [\eta(s) - h] A\, ds$$

where h is an arbitrary reference level

Then

$$E_p = \int_0^{x(t)} dE_p = \int_0^{x(t)} \rho g[\eta(x')-h]A\,dx' = \rho g A \int_0^{x(t)} (\eta(x')-h)\,dx'$$

Now the time derivative:

$$\frac{d}{dt}(E_p + E_k) = \frac{d}{dt}E_p + \frac{d}{dt}E_k$$

$\frac{dE_k}{dt}$ was calculated at the lecture and is given by

$$\frac{dE_k}{dt} = \rho A \dot{x}\left(\frac{1}{2}\dot{x}^2 + x\ddot{x}\right)$$

Then using the chain rule

$$\frac{dE_p}{dt} = \frac{d}{dx}\left[\rho g A \int_0^{x} (\eta(x')-h)\,dx'\right]\frac{dx}{dt} = \rho g A[\eta(x(t))-h]\dot{x}$$

By the Fundamental Theorem of Calculus

Hence, finally

$$\frac{dE}{dt} = \rho A \dot{x}\left[\frac{1}{2}\dot{x}^2 + x\ddot{x} + g(\eta(x)-h)\right]$$

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Solve the equation using separation of variables

$$\frac{dx}{dt} = x^2 \Rightarrow \int x^{-2}\,dx = \int dt$$

$$-\frac{1}{x} = t + C \Rightarrow x(t) = -\frac{1}{t+C}$$

Using the initial condition $x(0) = x_0$

$$x(0) = -\frac{1}{C} = x_0 \Rightarrow C = -\frac{1}{x_0}$$

Hence: $x(t) = -\dfrac{1}{t - \frac{1}{x_0}}$

HW1/4

We note that

$$\lim_{t \to \left(\frac{1}{x_0}\right)^+} x(t) = \infty$$

Therefore the solution $x(t)$ is defined only for $t \in [0, \frac{1}{x_0})$ and ceases to exist at $\underline{\underline{t = \frac{1}{x_0}}}$